

S. S. Podkorytov

**HOMOTOPY SIMILARITY OF MAPS.
MAPS OF THE CIRCLE**

ABSTRACT. We describe the r -similarity relation and finite-order invariants on the homotopy set $[S^1, Y] = \pi_1(Y)$.

§1. INTRODUCTION

This paper continues [4], and we adopt the notation and conventions thereof. Here we are mainly interested in the set $[S^1, Y] = \pi_1(Y)$; in Part I, however, we consider a more general case. Let X and Y be cellular spaces, with X compact. Let X be equipped with maps $\mu: X \rightarrow X \vee X$ (comultiplication) and $\nu: X \rightarrow X$ (coinversion). The set Y^X carries the operations

$$(a, b) \mapsto (a * b: X \xrightarrow{\mu} X \vee X \xrightarrow{a \vee b} Y)$$

and

$$a \mapsto (a^\dagger: X \xrightarrow{\nu} X \xrightarrow{a} Y).$$

We suppose that the set $[X, Y]$ is a group with the identity $1 = [\triangleleft_Y^X]$, the multiplication

$$[a][b] = [a * b],$$

and the inversion

$$[a]^{-1} = [a^\dagger].$$

Under these assumptions, we call $(X, \mu, \nu; Y)$ an *admissible couple*.

Put

$$[X, Y]^{(r+1)} = \{ \mathbf{a} \in [X, Y] \mid 1 \stackrel{r}{\sim} \mathbf{a} \}.$$

We get the filtration

$$[X, Y] = [X, Y]^{(1)} \supseteq [X, Y]^{(2)} \supseteq \dots$$

We prove that the subsets $[X, Y]^{(r+1)}$ are normal subgroups and form an N -series (Theorems 4.1 and 4.3), and the equivalence

$$\mathbf{a} \stackrel{r}{\sim} \mathbf{b} \iff \mathbf{a}^{-1} \mathbf{b} \in [X, Y]^{(r+1)}$$

holds (Theorem 4.2).

Key words and phrases: r -similar loops, equation in a nilpotent group.

In Part II, we concentrate on the case $X = S^1$ (with the standard μ and ν), when $[X, Y] = \pi_1(Y)$. We prove that

$$\pi_1(Y)^{(r+1)} = \gamma_{r+1}\pi_1(Y)$$

(Theorem 6.2). Here, as usual,

$$G = \gamma_1 G \supseteq \gamma_2 G \supseteq \dots$$

is the lower central series of a group G .

For a homotopy invariant (i. e., a function) $f: \pi_1(Y) \rightarrow L$, where L is an abelian group, its order $\text{ord } f \in \{-\infty, 0, 1, \dots, \infty\}$ is defined (see [4, §1]). We prove that $\text{ord } f = \text{deg } f$ (Theorem 7.2). Recall that, for a function $f: G \rightarrow L$, where G is a group, its degree $\text{deg } f$ is defined (see §7).

Do invariants of order at most r distinguish elements of $\pi_1(Y)$ that are not r -similar? In general, the answer is negative. For $r \geq 3$, there is a group G and an element $g \in G \setminus \gamma_{r+1}G$ such that, for any abelian group L and function $f: G \rightarrow L$ of degree at most r , one has $f(1) = f(g)$ (see [5] for $r = 3$ and [3, Ch. 2]). Take a cellular space Y with $\pi_1(Y) = G$. Then, by Theorems 6.2 and 7.2, the homotopy classes 1 and g in $\pi_1(Y)$ are not r -similar, but cannot be distinguished by invariants of order at most r .

In Appendix, which does not depend on the rest of the paper, we prove group-theoretic Theorem 12.1, which we need for the proof of the above-mentioned Theorem 6.2.

PART I

In this part, we discuss operations over coherent ensembles of maps between arbitrary spaces (§§ 2 and 3) and give our results concerning an arbitrary admissible couple (§4).

§2. COMPOSITIONS

Let X, Y, X' , and Y' be spaces and $k: X' \rightarrow X$ and $h: Y \rightarrow Y'$ be maps. Introduce the homomorphisms

$$k^\# : \langle Y^X \rangle \rightarrow \langle Y^{X'} \rangle, \quad \langle a \rangle \mapsto \langle a \circ k \rangle,$$

and

$$h_\# : \langle Y^X \rangle \rightarrow \langle Y'^X \rangle, \quad \langle a \rangle \mapsto \langle h \circ a \rangle.$$

2.1. Lemma. *We have*

$$k^\#(\langle Y^X \rangle^{(r+1)}) \subseteq \langle Y^{X'} \rangle^{(r+1)} \quad \text{and} \quad h_\#(\langle Y^X \rangle^{(r+1)}) \subseteq \langle Y'^X \rangle^{(r+1)}.$$

Proof. Take an ensemble $A \in \langle Y^X \rangle^{(r+1)}$.

To show that $k^\#(A) \in \langle Y^{X'} \rangle^{(r+1)}$, we take $T' \in \text{Sub}_r(X')$ and check that $k^\#(A)|_{T'} = 0$. We have the commutative diagram

$$\begin{array}{ccc} k^\#(A) & \langle Y^{X'} \rangle & \xleftarrow{k^\#} \langle Y^X \rangle & A \\ & \downarrow ?|_{T'} & & \downarrow ?|_{k(T')} \\ k^\#(A)|_{T'} & \langle Y^{T'} \rangle & \xleftarrow{q^\#} \langle Y^{k(T')} \rangle, & A|_{k(T')=0} \end{array}$$

where $q = k|_{T' \rightarrow k(T')}$. Since $k(T') \in \text{Sub}_r(X)$, we have $A|_{k(T')} = 0$. By the diagram, $k^\#(A)|_{T'} = 0$.

To show that $h_\#(A) \in \langle Y'^X \rangle^{(r+1)}$, we take $T \in \text{Sub}_r(X)$ and check that $h_\#(A)|_T = 0$. We have the commutative diagram

$$\begin{array}{ccc} A & \langle Y^X \rangle & \xrightarrow{h_\#} \langle Y'^X \rangle & h_\#(A) \\ & \downarrow ?|_T & & \downarrow ?|_T \\ 0=A|_T & \langle Y^T \rangle & \xrightarrow{h_\#} \langle Y'^T \rangle. & h_\#(A)|_T \end{array}$$

We have $A|_T = 0$. By the diagram, $h_\#(A)|_T = 0$. \square

2.2. Corollary. *Let $a, b \in Y^X$ satisfy $a \stackrel{r}{\sim} b$. Then the relation $a \circ k \stackrel{r}{\sim} b \circ k$ holds in $Y^{X'}$, and the relation $h \circ a \stackrel{r}{\sim} h \circ b$ holds in Y'^X .*

Proof. There is an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i \langle a_i \rangle,$$

where $a_i \sim a$, such that $A \stackrel{r}{=} \langle b \rangle$. By Lemma 2.1, $k^\#(A) \stackrel{r}{=} \langle b \circ k \rangle$ and $h_\#(A) \stackrel{r}{=} \langle h \circ b \rangle$. Since all the maps of $k^\#(A)$ are homotopic to $a \circ k$, we get $a \circ k \stackrel{r}{\sim} b \circ k$. Since all the maps of $h_\#(A)$ are homotopic to $h \circ a$, we get $h \circ a \stackrel{r}{\sim} h \circ b$. \square

§3. JOINING COHERENT ENSEMBLES

Let X_1, X_2 , and Y be spaces. Introduce the homomorphism

$$(\nabla): \langle Y^{X_1} \rangle \otimes \langle Y^{X_2} \rangle \rightarrow \langle Y^{X_1 \vee X_2} \rangle, \quad \langle a \rangle \otimes \langle b \rangle \mapsto \langle a \nabla b \rangle.$$

3.1. Lemma. *For $p, q \geq 0$, we have*

$$(\nabla)(\langle Y^{X_1} \rangle^{(p)} \otimes \langle Y^{X_2} \rangle^{(q)}) \subseteq \langle Y^{X_1 \vee X_2} \rangle^{(p+q)}.$$

Proof. Take $A \in \langle Y^{X_1} \rangle^{(p)}$ and $B \in \langle Y^{X_2} \rangle^{(q)}$. We show that

$$(\overline{\nabla})(A \otimes B) \in \langle Y^{X_1 \vee X_2} \rangle^{(p+q)}.$$

Take $T \in \text{Sub}_{p+q-1}(X_1 \vee X_2)$. We check that $(\overline{\nabla})(A \otimes B)|_T = 0$. We have $T = T_1 \vee T_2$ for some finite subspaces $T_i \subseteq X_i$, where $i = 1, 2$. We obtain the commutative diagram

$$\begin{array}{ccc} A \otimes B & \langle Y^{X_1} \rangle \otimes \langle Y^{X_2} \rangle & \xrightarrow{(\overline{\nabla})} \langle Y^{X_1 \vee X_2} \rangle & (\overline{\nabla})(A \otimes B) \\ & \downarrow \text{?!}_{T_1} \otimes \text{?!}_{T_2} & & \downarrow \text{?!}_T \\ A|_{T_1} \otimes B|_{T_2} & \langle Y^{T_1} \rangle \otimes \langle Y^{T_2} \rangle & \xrightarrow{(\overline{\nabla})} \langle Y^T \rangle & (\overline{\nabla})(A \otimes B)|_T \end{array}$$

We have

$$T_1 \in \text{Sub}_{p-1}(X_1) \text{ or } T_2 \in \text{Sub}_{q-1}(X_2).$$

Thus $A|_{T_1} = 0$ or $B|_{T_2} = 0$. By the diagram, $(\overline{\nabla})(A \otimes B)|_T = 0$. \square

§4. SIMILARITY FOR AN ADMISSIBLE COUPLE

Let $(X, \mu, \nu; Y)$ be an admissible couple.

4.1. Theorem. *The subset $[X, Y]^{(r+1)} \subseteq [X, Y]$ is a subgroup.*

Proof. To show that the subset $[X, Y]^{(r+1)}$ is closed under multiplication, we take $a, b \in Y^X$ such that $\frown \overset{r}{\sim} a$ and $\frown \overset{r}{\sim} b$ and check that $\frown \overset{r}{\sim} a * b$. There are ensembles $D, E \in \langle Y^X \rangle$,

$$D = \sum_i u_i \langle d_i \rangle \quad \text{and} \quad E = \sum_j v_j \langle e_j \rangle,$$

where $d_i \sim \frown$ and $e_j \sim \frown$, such that $D \overset{r}{=} \langle a \rangle$ and $E \overset{r}{=} \langle b \rangle$. Consider the maps

$$a \overline{\nabla} b, d_i \overline{\nabla} e_j: X \vee X \rightarrow Y$$

and the ensemble $F \in \langle Y^{X \vee X} \rangle$,

$$F = \sum_{i,j} u_i v_j \langle d_i \overline{\nabla} e_j \rangle.$$

Then

$$\begin{aligned} \langle a \overline{\nabla} b \rangle - F &= (\overline{\nabla})(\langle a \rangle \otimes \langle b \rangle) - (\overline{\nabla})(D \otimes E) \\ &= (\overline{\nabla})(\langle a \rangle - D) \otimes \langle b \rangle + (\overline{\nabla})(D \otimes (\langle b \rangle - E)) \in \langle Y^{X \vee X} \rangle^{(r+1)}, \end{aligned}$$

where the membership in $\langle Y^{X \vee X} \rangle^{(r+1)}$ holds by Lemma 3.1. Since all the maps of F are null-homotopic, we get $\lrcorner \stackrel{r}{\sim} a \bar{\vee} b$. Since $a * b = (a \bar{\vee} b) \circ \mu$, Corollary 2.2 yields $\lrcorner \stackrel{r}{\sim} a * b$.

Take $a \in Y^X$ such that $\lrcorner \stackrel{r}{\sim} a$. Since $a^\dagger = a \circ \nu$, Corollary 2.2 then yields $\lrcorner \stackrel{r}{\sim} a^\dagger$. Thus $[X, Y]^{(r+1)}$ is closed under inversion. \square

4.2. Theorem. *For $a, b \in [X, Y]$, we have*

$$\mathbf{a} \stackrel{r}{\sim} \mathbf{b} \iff \mathbf{a}^{-1} \mathbf{b} \in [X, Y]^{(r+1)}. \quad (1)$$

Proof. It suffices to check the implication

$$a \stackrel{r}{\sim} b \implies c * a \stackrel{r}{\sim} c * b$$

for $a, b, c \in Y^X$. Given an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i \langle a_i \rangle,$$

where $a_i \sim a$, such that $A \stackrel{r}{=} \langle b \rangle$, consider the ensemble $F \in \langle Y^{X \vee X} \rangle$,

$$F = \sum_i u_i \langle c \bar{\vee} a_i \rangle.$$

We have

$$\langle c \bar{\vee} b \rangle - F = (\bar{\vee})(\langle c \rangle \otimes (\langle b \rangle - A)) \in \langle Y^{X \vee X} \rangle^{(r+1)},$$

where the membership in $\langle Y^{X \vee X} \rangle^{(r+1)}$ holds by Lemma 3.1. We thus obtain $F \stackrel{r}{=} \langle c \bar{\vee} b \rangle$. Since $c \bar{\vee} a_i \sim c \bar{\vee} a$, we get $c \bar{\vee} a \stackrel{r}{\sim} c \bar{\vee} b$. Taking composition with μ , we get $c * a \stackrel{r}{\sim} c * b$ by Corollary 2.2. \square

Theorems 4.1 and 4.2 imply that the relation $\stackrel{r}{\sim}$ on $[X, Y]$ is an equivalence, which is a special case of [4, Theorem 8.1] (note that we did not use it here).

One can prove similarly that

$$\mathbf{a} \stackrel{r}{\sim} \mathbf{b} \iff \mathbf{b} \mathbf{a}^{-1} \in [X, Y]^{(r+1)}. \quad (2)$$

It follows from (1) and (2) that the subgroup $[X, Y]^{(r+1)} \subseteq [X, Y]$ is normal. This is a special case of the following theorem.

Let $[[,]]$ denote the group commutator.

4.3. Theorem. *Put $M^s = [X, Y]^{(s)} \subseteq [X, Y]$. Then $[[M^p, M^q]] \subseteq M^{p+q}$.*

Proof. Introduce the map

$$\zeta: X \xrightarrow{\mu^{(3)}} X \vee X \vee X \vee X \xrightarrow{(\text{in}_1 \circ \nu) \bar{\vee} (\text{in}_2 \circ \nu) \bar{\vee} \text{in}_1 \bar{\vee} \text{in}_2} X \vee X,$$

where

$$\mu^{(3)}: X \xrightarrow{\mu} X \vee X \xrightarrow{\mu \vee \text{id}_X} X \vee X \vee X \xrightarrow{\mu \vee \text{id}_X \vee \text{id}_X} X \vee X \vee X \vee X$$

(4-fold comultiplication). For $a, b \in Y^X$, we have

$$[(a \underline{\vee} b) \circ \zeta] = [[a], [b]] \quad (3)$$

in the group $[X, Y]$.

Take $a, b \in Y^X$ such that $\frown \overset{p-1}{\sim} a$ and $\frown \overset{q-1}{\sim} b$. We show that

$$\frown \overset{p+q-1}{\sim} (a \underline{\vee} b) \circ \zeta.$$

There are ensembles $D, E \in \langle Y^X \rangle$,

$$D = \sum_i u_i \langle d_i \rangle \quad \text{and} \quad E = \sum_j v_j \langle e_j \rangle,$$

where $d_i \sim \frown$ and $e_j \sim \frown$, such that $D \overset{p-1}{=} \langle a \rangle$ and $E \overset{q-1}{=} \langle b \rangle$. Consider the ensemble $F \in \langle Y^{X \vee X} \rangle$,

$$F = \sum_i u_i \langle d_i \underline{\vee} b \rangle + \sum_j v_j \langle a \underline{\vee} e_j \rangle - \sum_{i,j} u_i v_j \langle d_i \underline{\vee} e_j \rangle.$$

We have

$$\langle a \underline{\vee} b \rangle - F = (\underline{\vee})((\langle a \rangle - D) \otimes (\langle b \rangle - E)) \in \langle Y^{X \vee X} \rangle^{(p+q)},$$

where the membership in $\langle Y^{X \vee X} \rangle^{(p+q)}$ holds by Lemma 3.1. Thus

$$F \overset{p+q-1}{=} \langle a \underline{\vee} b \rangle.$$

By Lemma 2.1,

$$\zeta^\#(F) \overset{p+q-1}{=} \langle (a \underline{\vee} b) \circ \zeta \rangle.$$

By (3), all the maps of $\zeta^\#(F)$ are null-homotopic. Thus we get

$$\frown \overset{p+q-1}{\sim} (a \underline{\vee} b) \circ \zeta. \quad \square$$

PART II

In this part, we consider the group $[S^1, Y] = \pi_1(Y)$.

§5. MANAGING AN ENSEMBLE OF MAPS $S^1 \rightarrow Y$

For $n \geq 0$ and a group G , introduce the function

$$M: G^n \rightarrow G, \quad (g_1, \dots, g_n) \mapsto g_1 \dots g_n.$$

For $K \subseteq (n)$, let $\omega_K: G^n \rightarrow G^K$ be the projection.

5.1. Lemma. *Consider an ensemble $A \in \langle Y^{S^1} \rangle$,*

$$A = \sum_{i \in I} u_i \langle a_i \rangle,$$

such that $A \stackrel{r}{=} 0$. Then, for some $n \geq 1$, there exist elements $z_i \in \pi_1(Y)^n$, for $i \in I$, such that $[a_i] = M(z_i)$ and the element

$$Z = \sum_{i \in I} u_i \langle z_i \rangle \in \langle \pi_1(Y)^n \rangle \quad (4)$$

satisfies $\langle \omega_K \rangle(Z) = 0$ in $\langle \pi_1(Y)^K \rangle$ for all $K \subseteq (n)$ with $|K| \leq r$.

Proof. Take a finite subspace $D \subseteq S^1$ consisting of $n \geq 2$ points. It cuts S^1 into closed arcs B_k , $k \in (n)$. A continuous function $v: B_k \rightarrow Y$ with $v(\partial B_k) = \{\frown_Y\}$ has the (relative to ∂B_k) homotopy class $[v] \in \pi_1(Y)$. For a map $w: S^1 \rightarrow Y$ with $w(D) = \{\frown_Y\}$, we have

$$[w] = \prod_{k=1}^n [w|_{B_k}] \quad (5)$$

in $\pi_1(Y)$ (we assume that B_k are oriented and numbered properly).

By [4, Corollary 6.2], we may assume that $A \stackrel{r}{\Gamma} 0$ for some open cover Γ of S^1 . We suppose that D is chosen dense enough so that each B_k is contained in some $G_k \in \Gamma$. Put

$$V = \bigvee_{i \in I} S^1.$$

Let U be the quotient of V by the identifications $\text{in}_i(x) \approx \text{in}_j(x)$ for $x \in B_k$ and $i, j \in I$ such that $a_i =|_{G_k} a_j$. The space U is a graph. Let $h: V \rightarrow U$ be the projection. Introduce the maps

$$e_i: S^1 \xrightarrow{\text{in}_i} V \xrightarrow{h} U.$$

There is a map $q: U \rightarrow S^1$ such that $q \circ e_i = \text{id}_{S^1}$. Put

$$b = \overline{\bigvee_{i \in I} a_i}: V \rightarrow Y.$$

There is a map $a: U \rightarrow Y$ such that $b = a \circ h$. Clearly, $a_i = a \circ e_i$. Put $\tilde{D} = q^{-1}(D) \subseteq U$. The subspace \tilde{D} is finite. The map $a|_{\tilde{D}}$ is null-homotopic because the inclusion $\tilde{D} \rightarrow U$ is. Extending the homotopy, we get a map $\hat{a}: U \rightarrow Y$ such that $\hat{a} \sim a$ and $\hat{a}(\tilde{D}) = \{\frown_Y\}$. Put

$$\hat{a}_i = \hat{a} \circ e_i: S^1 \rightarrow Y.$$

Clearly, $\hat{a}_i \sim a_i$ and $\hat{a}_i(D) = \{\frown_Y\}$. Put

$$z_i = ([\hat{a}_i|_{B_k}]_{k \in (n)}) \in \pi_1(Y)^n.$$

We have

$$[a_i] = [\hat{a}_i] \stackrel{(*)}{=} \prod_{k=1}^n [\hat{a}_i|_{B_k}] = M(z_i),$$

where $(*)$ follows from (5). For $k \in (n)$ and $i, j \in I$, we have the implication

$$a_i =|_{G_k} a_j \implies [\hat{a}_i|_{B_k}] = [\hat{a}_j|_{B_k}] \quad (6)$$

because the premise implies that $e_i =|_{B_k} e_j$ and thus $\hat{a}_i =|_{B_k} \hat{a}_j$.

Consider the element $Z \in \langle \pi_1(Y)^n \rangle$ given by (4). Take $K \subseteq (n)$. Put

$$G(K) = \{\frown_Y\} \cup \bigcup_{k \in K} G_k \subseteq S^1.$$

By (6), we have the implication

$$a_i =|_{G(K)} a_j \implies \omega_K(z_i) = \omega_K(z_j).$$

Suppose that $|K| \leq r$. Then $A|_{G(K)} = 0$ because $A \stackrel{r}{\Gamma} 0$. Thus

$$\langle \omega_K \rangle(Z) = 0. \quad \square$$

§6. SIMILARITY ON $\pi_1(Y)$

6.1. Lemma. *Let G be a group. Consider an element $Z \in \langle G^n \rangle$,*

$$Z = \sum_{i \in I} u_i \langle z_i \rangle,$$

where I has a distinguished element 0 and $u_0 = 1$. Suppose that

$$\langle \omega_K \rangle(Z) = 0 \text{ for all } K \subseteq (n) \text{ with } |K| \leq r$$

and $M(z_i) \in \gamma_{r+1}G$ for all $i \neq 0$. Then $M(z_0) \in \gamma_{r+1}G$.

Proof. We have $z_i = (z_{i1}, \dots, z_{in})$, where $z_{ik} \in G$. Take distinct

$$g_1, \dots, g_m \in G$$

that include all the z_{ik} . We have

$$z_{ik} = \prod_{l=1}^m g_l^{[z_{ik}=g_l]}$$

and thus

$$M(z_i) = \prod_{k=1}^n \prod_{l=1}^m g_l^{[z_{ik}=g_l]}.$$

Hereafter, given a condition C , the integer $[C]$ is 1 under C and 0 otherwise. By Theorem 12.1, for some $q \geq 0$, there are rational polynomials $P_j(X)$,

$$X = (X_{kl})_{k \in (n), l \in (m)}, \quad j \in (q),$$

of degree at most r and integers $d_j \geq 0$ such that, for any collection

$$x = (x_{kl})_{k \in (n), l \in (m)}, \quad x_{kl} \in \mathbb{Z},$$

we have the equivalence

$$\prod_{k=1}^n \prod_{l=1}^m g_l^{x_{kl}} \in \gamma_{r+1}G \iff (P_j(x) \in d_j \mathbb{Z}, j \in (q)).$$

Order the set $(n) \times (m)$ totally. We have

$$P_j(X) = \sum_{\substack{0 \leq s \leq r, \\ (k_1, l_1) \leq \dots \leq (k_s, l_s)}} P_{jk_1 l_1 \dots k_s l_s}^{(s)} X_{k_1 l_1} \dots X_{k_s l_s}$$

for some $P_{jk_1 l_1 \dots k_s l_s}^{(s)} \in \mathbb{Q}$. Then

$$\begin{aligned} & \sum_{i \in I} u_i P_j \left(([z_{ik} = g_l])_{k \in (n), l \in (m)} \right) \\ &= \sum_{\substack{0 \leq s \leq r, \\ (k_1, l_1) \leq \dots \leq (k_s, l_s)}} P_{jk_1 l_1 \dots k_s l_s}^{(s)} \sum_{i \in I} u_i [z_{ik_t} = g_{l_t}, t \in (s)] \stackrel{(*)}{=} 0, \end{aligned} \quad (7)$$

where $(*)$ holds because the inner sum is zero, which is because

$$\langle \omega_K \rangle(Z) = 0 \quad \text{for } K = \{k_1, \dots, k_s\}.$$

Since $M(z_i) \in \gamma_{r+1}(G)$ for $i \neq 0$, we have

$$P_j \left(([z_{ik} = g_l])_{k \in (n), l \in (m)} \right) \in d_j \mathbb{Z} \quad (8)$$

for $i \neq 0$. Since $u_0 = 1$, it follows from (7) that (8) holds for $i = 0$ too. Thus $M(z_0) \in \gamma_{r+1}(G)$. \square

6.2. Theorem. *Let Y be a cellular space. Then the equality*

$$\pi_1(Y)^{(r+1)} = \gamma_{r+1}\pi_1(Y) \quad (9)$$

holds.

Proof. The inclusion \supseteq in (9) follows from Theorem 4.3. To prove the inclusion \subseteq , we take $a \in Y^{S^1}$ such that $\lrcorner \sim^r a$ and check that

$$[a] \in \gamma_{r+1}\pi_1(Y).$$

There is an ensemble $D \in \langle Y^{S^1} \rangle$,

$$D = \sum_i u_i \langle d_i \rangle,$$

where $d_i \sim \lrcorner$, such that $D \stackrel{r}{=} \langle a \rangle$. By Lemma 5.1 applied to the ensemble $\langle a \rangle - D$, there are, for some $n \geq 1$, elements $z, w_i \in \pi_1(Y)^n$ such that $M(z) = [a]$ and $M(w_i) = 1$ in $\pi_1(Y)$ and, putting

$$W = \sum_i u_i \langle w_i \rangle \in \langle \pi_1(Y)^n \rangle,$$

we have $\langle \omega_K \rangle (\langle z \rangle - W) = 0$ for all $K \subseteq (n)$ with $|K| \leq r$. By Lemma 6.1,

$$M(z) \in \gamma_{r+1}\pi_1(Y),$$

which is what we need. \square

§7. FINITE-ORDER INVARIANTS ON $\pi_1(Y)$

For a group G , the abelian group $\langle G \rangle$ equipped with the standard ring structure is the group ring of G . Let $\Delta_G \subseteq \langle G \rangle$ be the augmentation ideal, i.e., the kernel of the ring homomorphism (called the augmentation)

$$\langle G \rangle \rightarrow \mathbb{Z}, \quad \langle g \rangle \mapsto 1.$$

7.1. Lemma. *Let G be a group and take $Z \in \langle G^n \rangle$ such that $\langle \omega_K \rangle (Z) = 0$ in $\langle G^K \rangle$ for all $K \subseteq (n)$ with $|K| \leq r$. Then the element $\langle M \rangle (Z) \in \langle G \rangle$ belongs to Δ_G^{r+1} .*

Proof. For $K \subseteq (n)$, consider the function

$$\epsilon_K: G^K \rightarrow G^n, \quad (g_k)_{k \in K} \mapsto (\tilde{g}_k)_{k \in (n)},$$

where \tilde{g}_k equals g_k if $k \in K$ and 1 otherwise, the composition

$$\rho_K: G^n \xrightarrow{\omega_K} G^K \xrightarrow{\epsilon_K} G^n$$

and the homomorphism $S_K: \langle G^n \rangle \rightarrow \langle G^n \rangle$,

$$S_K = \sum_{L \subseteq K} (-1)^{|L|} \langle \rho_L \rangle.$$

If $K = \{k_1, \dots, k_t\}$, $k_1 < \dots < k_t$, then

$$(\langle M \rangle \circ S_K)(\langle (g_k)_{k \in (n)} \rangle) = (1 - \langle g_{k_1} \rangle) \dots (1 - \langle g_{k_t} \rangle)$$

in $\langle G \rangle$. Thus

$$\text{Im}(\langle M \rangle \circ S_K) \subseteq \Delta_G^{|K|}. \quad (10)$$

We have

$$\begin{aligned} \sum_{K \subseteq (n)} (-1)^{|K|} S_K &= \sum_{K \subseteq (n)} (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} \langle \rho_L \rangle = \\ &= \sum_{L \subseteq (n)} (-1)^{|L|} \left(\sum_{K \subseteq (n): K \supseteq L} (-1)^{|K|} \right) \langle \rho_L \rangle. \end{aligned}$$

The inner sum equals $(-1)^n [L = (n)]$. Thus

$$\sum_{K \subseteq (n)} (-1)^{|K|} S_K = \langle \rho_{(n)} \rangle = \text{id}_{\langle G^n \rangle}.$$

For $L \subseteq (n)$, $|L| \leq r$, we have $\langle \rho_L \rangle(Z) = \langle \epsilon_L \rangle(\langle \omega_L \rangle(Z)) = 0$. Therefore $S_K(Z) = 0$ if $|K| \leq r$. We get

$$Z = \sum_{K \subseteq (n)} (-1)^{|K|} S_K(Z) = \sum_{K \subseteq (n): |K| \geq r+1} (-1)^{|K|} S_K(Z).$$

Thus

$$\langle M \rangle(Z) = \sum_{K \subseteq (n): |K| \geq r+1} (-1)^{|K|} (\langle M \rangle \circ S_K)(Z).$$

By (10), $\langle M \rangle(Z) \in \Delta_G^{r+1}$. \square

A function $f: G \rightarrow L$, where L is an abelian group, gives rise to the homomorphism

$${}^+f: \langle G \rangle \rightarrow L, \quad \langle g \rangle \mapsto f(g).$$

We define $\text{deg } f \in \{-\infty, 0, 1, \dots, \infty\}$, the *degree* of f , as the infimum of $r \in \mathbb{Z}$ such that ${}^+f|_{\Delta_G^{r+1}} = 0$ (adopting $\Delta_G^s = \langle G \rangle$ for $s \leq 0$).

7.2. Theorem *Let Y be a cellular space, L be an abelian group and*

$$f: \pi_1(Y) \rightarrow L$$

be a homotopy invariant (i.e., a function). Then

$$\text{ord } f = \text{deg } f.$$

Proof. We suppose $f \neq 0$ omitting the zero case.

First, suppose that $\text{ord } f \leq r$ ($r \geq 0$). We show that $\text{deg } f \leq r$. It suffices to check that

$${}^+f((1 - \langle a_1 \rangle) \dots (1 - \langle a_{r+1} \rangle)) = 0$$

for any $a_1, \dots, a_{r+1} \in Y^{S^1}$. Put $W = S^1 \vee \dots \vee S^1$ ($r+1$ summands) and

$$q = a_1 \bar{\vee} \dots \bar{\vee} a_{r+1}: W \rightarrow Y.$$

Let $p: S^1 \rightarrow W$ be the $(r+1)$ -fold comultiplication and $\Lambda_d: W \rightarrow W$, for $d \in \{0, 1\}^{r+1} \subseteq \mathbb{Z}^{r+1}$, be as in [4, §3]. Consider the ensemble $A \in \langle Y^{S^1} \rangle$,

$$A = \sum_{d \in \{0, 1\}^{r+1}} (-1)^{|d|} \langle a(d) \rangle,$$

where

$$a(d): S^1 \xrightarrow{p} W \xrightarrow{\Lambda(d)} W \xrightarrow{q} Y.$$

Clearly,

$$[a(d)] = [a_1]^{d_1} \dots [a_{r+1}]^{d_{r+1}}$$

in $\pi_1(Y)$. By [4, Lemma 3.1], $A \stackrel{r}{=} 0$. We have

$$\begin{aligned} {}^+f((1 - \langle a_1 \rangle) \dots (1 - \langle a_{r+1} \rangle)) &= \sum_{d \in \{0, 1\}^{r+1}} (-1)^{|d|} f([a_1]^{d_1} \dots [a_{r+1}]^{d_{r+1}}) \\ &= \sum_{d \in \{0, 1\}^{r+1}} (-1)^{|d|} f([a(d)]) \stackrel{(*)}{=} 0, \end{aligned}$$

where $(*)$ holds because $\text{ord } f \leq r$.

Second, suppose that $\text{deg } f \leq r$ ($r \geq 0$). We show that $\text{ord } f \leq r$. Take an ensemble $A \in \langle Y^{S^1} \rangle$,

$$A = \sum_{i \in I} u_i \langle a_i \rangle,$$

such that $A \stackrel{r}{=} 0$. We should show that

$$\sum_{i \in I} u_i f([a_i]) = 0.$$

By Lemma 5.1, for some $n \geq 1$, there exist elements

$$z_i \in \pi_1(Y)^n, \quad i \in I,$$

such that $[a_i] = M(z_i)$ and the element $Z \in \langle \pi_1(Y)^n \rangle$ given by (4) satisfies $\langle \omega_K \rangle(Z) = 0$ in $\langle \pi_1(Y)^K \rangle$ for all $K \subseteq (n)$ with $|K| \leq r$. We have

$$\sum_{i \in I} u_i f([a_i]) = {}^+ f(\langle M \rangle(Z)).$$

By Lemma 7.1, $\langle M \rangle(Z) \in \Delta_{\pi_1(Y)}^{r+1}$. Since $\deg f \leq r$, ${}^+ f(\langle M \rangle(Z)) = 0$. \square

APPENDIX

In this Appendix, which is algebraic and does not depend on the rest of the paper, we prove Theorem 12.1.

§8. CULTURED SETS

Let E be a set. Consider the \mathbb{Q} -algebra \mathbb{Q}^E of functions $E \rightarrow \mathbb{Q}$. A *culture* on E is a filtration $\Phi = (\Phi_s)_{s \geq 0}$ of \mathbb{Q}^E by \mathbb{Q} -submodules

$$\Phi_0 \subseteq \Phi_1 \subseteq \dots \subseteq \mathbb{Q}^E$$

such that

$$1 \in \Phi_0 \quad \text{and} \quad \Phi_s \Phi_t \subseteq \Phi_{s+t}.$$

A set equipped with a culture is called a *cultured set*. The culture of a cultured set E is denoted by Φ^E .

A way to define a culture on a set E is to choose a collection of pairs (u_i, s_i) , where $u_i \in \mathbb{Q}^E$ is a function and $s_i \geq 1$ is a number called the *weight*, and to let Φ_s be spanned by all products $u_{i_1} \dots u_{i_p}$ ($p \geq 0$) with $s_{i_1} + \dots + s_{i_p} \leq s$. We define the cultured set

$$\mathbb{Q}_{s_1 \dots s_m}^m \tag{11}$$

as \mathbb{Q}^m with the culture given by the collection (ξ_i, s_i) , $i \in (m)$, where

$$\xi_i: \mathbb{Q}^m \rightarrow \mathbb{Q}$$

is the i th coordinate. Hereafter, we put $(m) = \{1, \dots, m\}$. The cultured set

$$\mathbb{Z}_{s_1 \dots s_m}^m$$

is defined similarly. We put $\mathbb{Q}_s = \mathbb{Q}_s^1$ and $\mathbb{Z}_s = \mathbb{Z}_s^1$.

A function $g: E \rightarrow F$ between cultured sets is called a *cultural morphism* if the induced algebra homomorphism $g^\#: \mathbb{Q}^F \rightarrow \mathbb{Q}^E$ satisfies

$$g^\#(\Phi_s^F) \subseteq \Phi_s^E$$

for all s . A function

$$g: \mathbb{Q}_{s_1 \dots s_m}^m \rightarrow \mathbb{Q}_{t_1 \dots t_n}^n$$

is a cultural morphism if and only if it has the form

$$g(x_1, \dots, x_m) = (P_j(x_1, \dots, x_m))_{j \in (n)},$$

where P_j is a rational polynomial of degree at most t_j with respect to its arguments having weights s_1, \dots, s_m . Cultural maps

$$\mathbb{Z}_{s_1 \dots s_m}^m \rightarrow \mathbb{Z}_{t_1 \dots t_n}^n$$

are characterized similarly (their coordinate polynomials need not have integer coefficients).

Cultured sets and cultural morphisms form a category with products. We have

$$\mathbb{Q}_{s_1 \dots s_m}^m \times \mathbb{Q}_{t_1 \dots t_n}^n = \mathbb{Q}_{s_1 \dots s_m t_1 \dots t_n}^{m+n} \quad \text{and} \quad \mathbb{Z}_{s_1 \dots s_m}^m \times \mathbb{Z}_{t_1 \dots t_n}^n = \mathbb{Z}_{s_1 \dots s_m t_1 \dots t_n}^{m+n}.$$

A cultural morphism $g: E \rightarrow F$ is called a *cultural immersion* if

$$g^\#(\Phi_s^F) = \Phi_s^E \text{ for all } s.$$

Then a function $f: D \rightarrow E$, where D is a cultured set, is a cultural morphism if the composition

$$D \xrightarrow{f} E \xrightarrow{g} F$$

is. If the composition

$$E \xrightarrow{g} F \xrightarrow{h} G$$

of two cultural morphisms is a cultural immersion, then g is.

§9. THE TRUNCATED FREE ALGEBRA $\mathbf{A}/\mathbf{A}^{(r+1)}$

Consider the algebra \mathbf{A} of rational polynomials in non-commuting variables T_1, \dots, T_n . It is graded in the standard way,

$$\mathbf{A} = \bigoplus_{s \geq 0} \mathbf{A}_s.$$

Introduce the ideals $\mathbf{A}^{(s)} \subseteq \mathbf{A}$,

$$\mathbf{A}^{(s)} = \bigoplus_{t \geq s} \mathbf{A}_t.$$

We fix $r \geq 0$ and consider the algebra $\mathbf{A}/\mathbf{A}^{(r+1)}$. Let $\bar{T}_i \in \mathbf{A}/\mathbf{A}^{(r+1)}$ be the image of T_i . An element $w \in \mathbf{A}/\mathbf{A}^{(r+1)}$ has the form

$$w = \sum_{s \geq 0, i_1, \dots, i_s} w_{i_1 \dots i_s}^{(s)} \bar{T}_{i_1} \dots \bar{T}_{i_s},$$

where $w_{i_1 \dots i_s}^{(s)} \in \mathbb{Q}$ are uniquely defined for $s \leq r$ and arbitrary for greater s . Introduce the group

$$\mathcal{U} = 1 + \mathbf{A}^{(1)} / \mathbf{A}^{(r+1)} \subseteq (\mathbf{A} / \mathbf{A}^{(r+1)})^\times$$

with the filtration by subgroups

$$\mathcal{U} = \mathcal{U}^{(1)} \supseteq \mathcal{U}^{(2)} \supseteq \dots$$

where

$$\mathcal{U}^{(s)} = 1 + \mathbf{A}^{(s)} / \mathbf{A}^{(r+1)}, \quad s \leq r + 1,$$

and $\mathcal{U}^{(s)} = 1$, $s \geq r + 1$. We have $[\mathcal{U}^{(s)}, \mathcal{U}^{(t)}] \subseteq \mathcal{U}^{(s+t)}$. In particular, the inclusion $\gamma_s \mathcal{U} \subseteq \mathcal{U}^{(s)}$ holds.

We equip the set \mathcal{U} with the culture given by the collection of pairs

$$(\xi_{i_1 \dots i_s}^{(s)}, s), \quad 1 \leq s \leq r, \quad i_1, \dots, i_s \in (n),$$

where

$$\xi_{i_1 \dots i_s}^{(s)} : \mathcal{U} \rightarrow \mathbb{Q}, \quad u \mapsto u_{i_1 \dots i_s}^{(s)} : \mathcal{U} \rightarrow \mathbb{Q}$$

for

$$u = \sum_{t \geq 0, j_1, \dots, j_t} u_{j_1 \dots j_t}^{(t)} \bar{T}_{j_1} \dots \bar{T}_{j_t}$$

with $u^{(0)} = 1$. Clearly, the cultured set \mathcal{U} is a special case of (11).

For a group G , let $M_G : G \times G \rightarrow G$ be the multiplication.

9.1. Lemma. *The function*

$$M_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{U}$$

is a cultural morphism.

Proof. Given $u, v \in \mathcal{U}$,

$$u = \sum_{s \geq 0, i_1, \dots, i_s} u_{i_1 \dots i_s}^{(s)} \bar{T}_{i_1} \dots \bar{T}_{i_s} \quad \text{and} \quad v = \sum_{t \geq 0, j_1, \dots, j_t} v_{j_1 \dots j_t}^{(t)} \bar{T}_{j_1} \dots \bar{T}_{j_t}$$

with $u^{(0)} = v^{(0)} = 1$, we have

$$uv = \sum_{\substack{s \geq 0, i_1, \dots, i_s, \\ t \geq 0, j_1, \dots, j_t}} u_{i_1 \dots i_s}^{(s)} v_{j_1 \dots j_t}^{(t)} \bar{T}_{i_1} \dots \bar{T}_{i_s} \bar{T}_{j_1} \dots \bar{T}_{j_t}.$$

We consider $u_{i_1 \dots i_s}^{(s)}$ and $v_{j_1 \dots j_t}^{(t)}$ with $s, t > 0$ here as variables of weights s and t , respectively. In the last expression, the monomial in \bar{T}_i has degree $s + t$, and its coefficient is $u_{i_1 \dots i_s}^{(s)} v_{j_1 \dots j_t}^{(t)}$ and thus has degree $s + t$.

Thus the total coefficient of each monomial in \bar{T}_i of some degree $z > 0$ in this series is a polynomial in $u_{i_1 \dots i_s}^{(s)}$ and $v_{j_1 \dots j_t}^{(t)}$ of degree at most z . \square

9.2. Lemma. For $u \in \mathcal{U}^{(s)}$, the function

$$E_u: \mathbb{Z}_s \rightarrow \mathcal{U}, \quad x \rightarrow u^x,$$

is a cultural morphism. Moreover, it extends to a cultural morphism

$$\widehat{E}_u: \mathbb{Q}_s \rightarrow \mathcal{U}.$$

Proof. We have $u = 1 + w$ in $\mathbf{A}/\mathbf{A}^{(r+1)}$ for some $w \in \mathbf{A}^{(s)}/\mathbf{A}^{(r+1)}$,

$$w = \sum_{t \geq s, i_1, \dots, i_t} w_{i_1 \dots i_t}^{(t)} \bar{T}_{i_1} \dots \bar{T}_{i_t}.$$

For $x \in \mathbb{Z}$, we have

$$\begin{aligned} E_u(x) &= u^x = (1 + w)^x = \sum_{p \geq 0} \binom{x}{p} w^p \\ &= \sum_{p \geq 0} \sum_{\substack{t_1 \geq s, i_{11}, \dots, i_{1t_1}, \\ \dots, \\ t_p \geq s, i_{p1}, \dots, i_{pt_p}}} w_{i_{11}, \dots, i_{1t_1}}^{(t_1)} \dots w_{i_{p1}, \dots, i_{pt_p}}^{(t_p)} \binom{x}{p} \cdot T, \end{aligned}$$

where

$$T = \bar{T}_{i_{11}} \dots \bar{T}_{i_{1t_1}} \dots \bar{T}_{i_{p1}} \dots \bar{T}_{i_{pt_p}}.$$

Consider x here as a variable of weight s . In the last expression, the monomial in \bar{T}_i has degree $t_1 + \dots + t_p$, which is at least ps . Its coefficient is a rational multiple of $\binom{x}{p}$ and thus a polynomial in x of degree at most ps . Thus the total coefficient of each monomial in \bar{T}_i of some degree $z > 0$ in this series is a polynomial in x of degree at most z .

The extension \widehat{E}_u exists automatically. \square

§10. THE FREE NILPOTENT GROUP \mathcal{N}

Recall that we fix numbers n and r . Let \mathbf{F} be the free group on the generators Z_1, \dots, Z_n . Consider the free nilpotent group $\mathcal{N} = \mathbf{F}/\gamma_{r+1}\mathbf{F}$. Put $\mathcal{N}^{(s)} = \gamma_s\mathcal{N} \subseteq \mathcal{N}$, $s \geq 1$.

Following Magnus, consider the homomorphism

$$\rho: \mathcal{N} \rightarrow \mathcal{U}, \quad \bar{Z}_i \mapsto 1 + \bar{T}_i.$$

Hereafter, the bar denotes the projection to the proper quotient group. The homomorphism ρ exists because $\gamma_{r+1}\mathcal{U} = 1$. The quotient $\mathcal{N}^{(s)}/\mathcal{N}^{(s+1)}$ is

abelian and finitely generated. Since $\rho(\mathcal{N}^{(s)}) \subseteq \gamma_s \mathcal{U} \subseteq \mathcal{U}^{(s)}$, there is a homomorphism

$$\sigma^{(s)}: \mathcal{N}^{(s)}/\mathcal{N}^{(s+1)} \rightarrow \mathbf{A}_s$$

such that

$$\rho(h) \equiv 1 + \sigma^{(s)}(\bar{h}) \pmod{\mathbf{A}^{(s+1)}}, \quad h \in \mathcal{N}^{(s)}.$$

By Magnus [2] (see also [6, Part I, Ch. IV, Theorem 6.3]), $\mathcal{N}^{(s)} = \rho^{-1}(\mathcal{U}^{(s)})$. It follows that $\sigma^{(s)}$ are injective and $\mathcal{N}^{(s)}/\mathcal{N}^{(s+1)}$ are torsion-free and thus free abelian. It follows that there is a filtration

$$\mathcal{N} = \mathcal{N}^1 \supseteq \mathcal{N}^2 \supseteq \dots \supseteq \mathcal{N}^q \supseteq \mathcal{N}^{q+1} = 1$$

such that $\mathcal{N}^{(s)} = \mathcal{N}^{j_s}$ for some $1 = j_1 \leq \dots \leq j_{r+1} = q + 1$ and $\mathcal{N}^j/\mathcal{N}^{j+1}$ are infinite cyclic. For $j \leq n+1$, we choose \mathcal{N}^j to be the subgroup generated by $\bar{Z}_j, \dots, \bar{Z}_n$ and $\mathcal{N}^{(2)}$. Put

$$s_j = \max\{s \mid j_s \leq j\}, \quad j \in (q).$$

Clearly, $1 \leq s_1 \leq \dots \leq s_q \leq r$, $s_1 = \dots = s_n = 1$, and $\mathcal{N}^j \subseteq \mathcal{N}^{(s_j)}$. The subgroups $\mathcal{N}^j \subseteq \mathcal{N}$ are normal.

For each $j \in (q)$, choose an element $b_j \in \mathcal{N}^j$ such that \bar{b}_j generates $\mathcal{N}^j/\mathcal{N}^{j+1}$. In doing so, we put

$$b_j = \bar{Z}_j, \quad j \in (n).$$

The collection (b_1, \dots, b_q) is a ‘‘Mal’cev basis’’ [1, 4.2.2]. For $j \in (q + 1)$, the function

$$\beta^j: \mathbb{Z}^{q-j+1} \rightarrow \mathcal{N}^j, \quad (x_j, \dots, x_q) \mapsto b_j^{x_j} \dots b_q^{x_q},$$

is bijective. We put

$$\beta = \beta^1: \mathbb{Z}^q \rightarrow \mathcal{N}.$$

The elements $\sigma^{(s_j)}(\bar{b}_j) \in \mathbf{A}$ are linearly independent.

Any group G carries the *immanent* culture Φ with Φ_s consisting of all functions $G \rightarrow \mathbb{Q}$ of degree at most s (see §7). If \mathcal{N} is equipped with its immanent culture,

$$\beta: \mathbb{Z}_{s_1 \dots s_q}^q \rightarrow \mathcal{N}$$

becomes a culture isomorphism. The proof is omitted.

10.1. Lemma. *The composition*

$$\eta^j: \mathbb{Z}_{s_j \dots s_q}^{q-j+1} \xrightarrow{\beta^j} \mathcal{N}^j \xrightarrow{\rho^j} \mathcal{U},$$

where $\rho^j = \rho|_{\mathcal{N}^j}$, is a cultural immersion.

Introduce the projections

$$\mathbf{p}: \mathbb{Q}^{q-j+1} \rightarrow \mathbb{Q}, \quad (x_j, \dots, x_q) \mapsto x_j,$$

and

$$\mathbf{R}: \mathbb{Q}^{q-j+1} \rightarrow \mathbb{Q}^{q-j}, \quad (x_j, \dots, x_q) \mapsto (x_{j+1}, \dots, x_q).$$

Proof. We show that η^j is a cultural morphism by backward induction on j . For $j = q + 1$, the assertion is trivial. Take $j \leq q$. Since $b^j \in \mathcal{N}^{(s_j)}$, we have $\rho(b^j) \in \mathcal{U}^{(s_j)}$. We have the decomposition

$$\eta^j: \mathbb{Z}_{s_j \dots s_q}^{q-j+1} = \mathbb{Z}_{s_j} \times \mathbb{Z}_{s_{j+1} \dots s_q}^{q-j} \xrightarrow{E_{\rho(b_j)} \times \eta^{j+1}} \mathcal{U} \times \mathcal{U} \xrightarrow{M_{\mathcal{U}}} \mathcal{U},$$

where $E_{\rho(b_j)}: x \mapsto \rho(b_j)^x$. The following functions are cultural morphisms: $E_{\rho(b_j)}$ by Lemma 9.2, η^{j+1} by the induction hypothesis, and $M_{\mathcal{U}}$ by Lemma 9.1. Thus the function η^j is a cultural morphism.

For each $j \in (q)$, choose a linear functional $\phi_j: \mathbf{A}_{s_j} \rightarrow \mathbb{Q}$ such that the value $\phi_j(\sigma^{(s_j)}(b_k))$ equals 1 for $k = j$ and 0 for all other k with $s_k = s_j$. Given $j \leq q + 1$ and $x = (x_j, \dots, x_q) \in \mathbb{Z}^{q-j+1}$, we have

$$\eta^j(x) = \rho(\beta^j(x)) = \rho(b_j^{x_j} \dots b_q^{x_q}) = \rho(b_j^{x_j}) \dots \rho(b_q^{x_q}).$$

Assume $j \leq q$. Then

$$\eta^j(x) \equiv 1 + \sum_{k \geq j: s_k = s_j} x_k \sigma^{(s_j)}(b_k) \pmod{\mathbf{A}^{(s_j+1)}/\mathbf{A}^{(r+1)}}$$

in $\mathbf{A}/\mathbf{A}^{(r+1)}$ and

$$\eta^j(x) = \rho(b_j)^{x_j} \eta^{j+1}(\mathbf{R}(x)).$$

Note that, for any linear functional $F: \mathbf{A}_{s_j} \rightarrow \mathbb{Q}$, the composition

$$F!: \mathcal{U} \xrightarrow{\text{in}} \mathbf{A}/\mathbf{A}^{(r+1)} \xrightarrow{\text{pr}} \mathbf{A}_{s_j} \xrightarrow{F} \mathbb{Q}_{s_j}$$

is a cultural morphism. For $c \in \mathbb{Q}$, we have

$$(c\phi_j)!(\eta^j(x)) = cx_j, \quad x = (x_j, \dots, x_q) \in \mathbb{Z}^{q-j+1}.$$

We show that the function η^j is a cultural immersion by constructing a cultural morphism

$$\theta^j: \mathcal{U} \rightarrow \mathbb{Q}_{s_j \dots s_q}^{q-j+1}$$

such that $\theta^j \circ \eta^j$ is the inclusion

$$\mathbb{Z}_{s_j \dots s_q}^{q-j+1} \rightarrow \mathbb{Q}_{s_j \dots s_q}^{q-j+1}.$$

Backward induction on j . Let θ^{q+1} be the unique function $\mathcal{U} \rightarrow \mathbb{Q}^0$. Take $j \leq q$. Introduce the cultural morphism

$$L: \mathcal{U} \xrightarrow{(-\phi_j)!} \mathbb{Q}_{s_j} \xrightarrow{\widehat{E}_{\rho(b_j)}} \mathcal{U}$$

where $\widehat{E}_{\rho(b_j)}$ is given by Lemma 9.2. Given $x = (x_j, \dots, x_q) \in \mathbb{Z}^{q-j+1}$, we have $L(\eta^j(x)) = \rho(b_j)^{-x_j}$. Introduce the cultural morphism

$$\theta': \mathcal{U} \xrightarrow{L\overline{\times}\text{id}} \mathcal{U} \times \mathcal{U} \xrightarrow{M_{\mathcal{U}}} \mathcal{U} \xrightarrow{\theta^{j+1}} \mathbb{Q}_{s_{j+1}\dots s_q}^{q-j}.$$

Hereafter, $\overline{\times}$ combines two morphisms with one source into a morphism to the product of their targets. We have

$$\begin{aligned} \theta'(\eta^j(x)) &= \theta^{j+1}(L(\eta^j(x))\eta^j(x)) = \theta^{j+1}(\rho(b_j)^{-x_j}\eta^j(x)) = \\ &= \theta^{j+1}(\eta^{j+1}(\mathbf{R}(x))) = \mathbf{R}(x). \end{aligned}$$

Put

$$\theta^j: \mathcal{U} \xrightarrow{\phi_j!\overline{\times}\theta'} \mathbb{Q}_{s_j} \times \mathbb{Q}_{s_{j+1}\dots s_q}^{q-j} = \mathbb{Q}_{s_j\dots s_q}^{q-j+1}.$$

We get

$$\theta^j(\eta^j(x)) = (\phi_j!(\eta^j(x)), \theta'(\eta^j(x))) = (x_j, \mathbf{R}(x)) = x. \quad \square$$

10.2. Lemma. *Define a function m^j by the commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}_{s_j\dots s_q}^{q-j+1} \times \mathbb{Z}_{s_j\dots s_q}^{q-j+1} & \xrightarrow{m^j} & \mathbb{Z}_{s_j\dots s_q}^{q-j+1} \\ \beta^j \times \beta^j \downarrow & & \downarrow \beta^j \\ \mathcal{N}^j \times \mathcal{N}^j & \xrightarrow{M_{\mathcal{N}^j}} & \mathcal{N}^j. \end{array}$$

Then the function m^j is a cultural morphism. It extends to a cultural morphism

$$\widehat{m}^j: \mathbb{Q}_{s_j\dots s_q}^{q-j+1} \times \mathbb{Q}_{s_j\dots s_q}^{q-j+1} \rightarrow \mathbb{Q}_{s_j\dots s_q}^{q-j+1}.$$

The coordinate polynomials of m^j are known as the ‘‘multiplication polynomials’’ [1, 4.2.2].

Proof. We have the commutative diagram

$$\begin{array}{ccc}
 \mathbb{Z}_{s_j \dots s_q}^{q-j+1} \times \mathbb{Z}_{s_j \dots s_q}^{q-j+1} & \xrightarrow{m^j} & \mathbb{Z}_{s_j \dots s_q}^{q-j+1} \\
 \beta^j \times \beta^j \downarrow & & \downarrow \beta^j \\
 \mathcal{N}^j \times \mathcal{N}^j & \xrightarrow{M_{\mathcal{N}^j}} & \mathcal{N}^j \\
 \rho^j \times \rho^j \downarrow & & \downarrow \rho^j \\
 \mathcal{U} \times \mathcal{U} & \xrightarrow{M_{\mathcal{U}}} & \mathcal{U}.
 \end{array}$$

Here $M_{\mathcal{U}}$ is a cultural morphism by Lemma 9.1. It follows from Lemma 10.1 that the composition in the left column is a cultural morphism. Since the composition in the right column is a cultural immersion by Lemma 10.1, m^j is a cultural morphism.

The extension \widehat{m}^j exists automatically. \square

10.3. Lemma. *Given an element $h \in \mathcal{N}^j$, define a function e_h^j by the commutative diagram*

$$\begin{array}{ccc}
 & & \mathbb{Z}_{s_j \dots s_q}^{q-j+1} \\
 & \nearrow e_h^j & \downarrow \beta^j \\
 \mathbb{Z}_{s_j} & \xrightarrow{x \mapsto h^x} & \mathcal{N}^j.
 \end{array}$$

Then the function e_h^j is a cultural morphism. It extends to a cultural morphism

$$\widetilde{e}_h^j : \mathbb{Q}_{s_j} \rightarrow \mathbb{Q}_{s_j \dots s_q}^{q-j+1}.$$

The coordinate polynomials of e_h^j are a specialization of the ‘‘exponentiation polynomials’’ [1, 4.2.2].

Proof. The composition

$$\mathbb{Z}_{s_j} \xrightarrow{e_h^j} \mathbb{Z}_{s_j \dots s_q}^{q-j+1} \xrightarrow{\beta^j} \mathcal{N}^j \xrightarrow{\rho^j} \mathcal{U}$$

sends x to $\rho(h)^x$ and thus coincides with $E_{\rho(h)}$, which is a cultural morphism by Lemma 9.2. Since $\rho^j \circ \beta^j$ here is a cultural immersion by Lemma 10.1, e_h^j is a cultural morphism.

The extension \widetilde{e}_h^j exists automatically. \square

§11. CULTURAL VIEW OF A SUBGROUP OF \mathcal{N}

Let $K \subseteq \mathcal{N}$ be a subgroup. For each $j \in (q)$, the image of $\mathcal{N}^j \cap K$ in the quotient $\mathcal{N}^j/\mathcal{N}^{j+1}$ is generated by $\bar{b}_j^{d_j}$ for some $d_j \geq 0$.

11.1. Lemma. *There exists a cultural morphism*

$$f: \mathbb{Q}_{s_1 \dots s_q}^q \rightarrow \mathbb{Q}_{s_1 \dots s_q}^q$$

such that

$$\beta^{-1}(K) = f^{-1}(d_1\mathbb{Z} \times \dots \times d_q\mathbb{Z})$$

as subsets of \mathbb{Q}^q .

The morphism f constructed in the proof is a cultural isomorphism, its j th coordinate f_j depends on the first j coordinates of the argument only, and, for $x \in \mathbb{Z}^q$, $f_j(x) \in \mathbb{Z}$ if $f_k(x) \in d_k\mathbb{Z}$ for all $k < j$. The proof of these properties is omitted.

Proof. For each $j \in (q+1)$, we construct a cultural morphism

$$f^j: \mathbb{Q}_{s_j \dots s_q}^{q-j+1} \rightarrow \mathbb{Q}_{s_j \dots s_q}^{q-j+1}$$

such that

$$(\beta^j)^{-1}(\mathcal{N}^j \cap K) = (f^j)^{-1}(d_j\mathbb{Z} \times \dots \times d_q\mathbb{Z})$$

as subsets of \mathbb{Q}^{q-j+1} . Backward induction on j . Let $f^{q+1}: \mathbb{Q}^0 \rightarrow \mathbb{Q}^0$ be the unique function. Take $j \leq q$.

Case $d_j = 0$. Then $\mathcal{N}^j \cap K \subseteq \mathcal{N}^{j+1}$. Put

$$f^j: \mathbb{Q}_{s_j \dots s_q}^{q-j+1} = \mathbb{Q}_{s_j} \times \mathbb{Q}_{s_{j+1} \dots s_q}^{q-j} \xrightarrow{\text{id} \times f^{j+1}} \mathbb{Q}_{s_j} \times \mathbb{Q}_{s_{j+1} \dots s_q}^{q-j} = \mathbb{Q}_{s_j \dots s_q}^{q-j+1}.$$

Take $x = (x_j, \dots, x_q) \in \mathbb{Q}^{q-j+1}$. We have

$$\begin{aligned} & x \in \mathbb{Z}^{q-j+1} \text{ and } \beta^j(x) \in \mathcal{N}^j \cap K \\ \iff & x \in 0 \times \mathbb{Z}^{q-j} \text{ and } \beta^{j+1}(\mathbf{R}(x)) \in \mathcal{N}^{j+1} \cap K \\ \iff & x_j = 0 \text{ and } f^{j+1}(\mathbf{R}(x)) \in d_{j+1}\mathbb{Z} \times \dots \times d_q\mathbb{Z} \\ \iff & f^j(x) \in 0 \times d_{j+1}\mathbb{Z} \times \dots \times d_q\mathbb{Z}. \end{aligned}$$

Case $d_j \neq 0$. Choose an element $k \in \mathcal{N}^j \cap K$ such that $\bar{k} = \bar{b}_j^{d_j}$ in $\mathcal{N}^j/\mathcal{N}^{j+1}$. Consider the cultural morphisms

$$l: \mathbb{Q}_{s_j \dots s_q}^{q-j+1} \xrightarrow{-\mathbf{P}/d_j} \mathbb{Q}_{s_j} \xrightarrow{\hat{e}_k^j} \mathbb{Q}_{s_j \dots s_q}^{q-j+1},$$

where $-\mathbf{p}/d_j : (x_j, \dots, x_q) \mapsto -x_j/d_j$ and \widehat{e}_k^j is given by Lemma 10.3, and

$$\begin{aligned} f' &= (\mathbb{Q}_{s_j \dots s_q}^{q-j+1} \xrightarrow{l\overline{\text{id}}} \mathbb{Q}_{s_j \dots s_q}^{q-j+1} \times \mathbb{Q}_{s_j \dots s_q}^{q-j+1} \\ &\quad \xrightarrow{\widehat{m}^j} \mathbb{Q}_{s_j \dots s_q}^{q-j+1} \\ &\quad \xrightarrow{\mathbf{R}} \mathbb{Q}_{s_{j+1} \dots s_q}^{q-j} \\ &\quad \xrightarrow{f^{j+1}} \mathbb{Q}_{s_{j+1} \dots s_q}^{q-j}), \end{aligned}$$

where \widehat{m}^j is given by Lemma 10.2. Put

$$f^j : \mathbb{Q}_{s_j \dots s_q}^{q-j+1} \xrightarrow{\mathbf{p}\overline{\Sigma}f'} \mathbb{Q}_{s_j} \times \mathbb{Q}_{s_{j+1} \dots s_q}^{q-j} = \mathbb{Q}_{s_j \dots s_q}^{q-j+1}.$$

Take $x = (x_j, \dots, x_q) \in d_j\mathbb{Z} \times \mathbb{Z}^{q-j}$. Then

$$k^{-x_j/d_j} \beta^j(x) \in \mathcal{N}^{j+1}$$

and

$$k^{-x_j/d_j} \beta^j(x) = \beta^j(y),$$

where $y \in \mathbb{Z}^{q-j+1}$,

$$y = \widehat{m}^j(\widehat{e}_k^j(-x_j/d_j), x).$$

Thus $\mathbf{p}(y) = 0$ and

$$k^{-x_j/d_j} \beta^j(x) = \beta^j(y) = \beta^{j+1}(\mathbf{R}(y)) = \beta^{j+1}(\mathbf{R}(\widehat{m}^j(\widehat{e}_k^j(-x_j/d_j), x))).$$

Take $x = (x_j, \dots, x_q) \in \mathbb{Q}^{q-j+1}$. Put

$$y = \widehat{m}^j(\widehat{e}_k^j(-x_j/d_j), x) \in \mathbb{Q}^{q-j+1}$$

and $y' = \mathbf{R}(y) \in \mathbb{Q}^{q-j}$. We show that

$$\mathbf{p}(y) = 0 \tag{12}$$

and

$$x = \widehat{m}^j(\widehat{e}_k^j(x_j/d_j), y). \tag{13}$$

Since y depends on x polynomially, (12) and (13) are equalities of certain rational polynomials in the coordinates of x . Thus it suffices to consider the case $x \in d_j\mathbb{Z} \times \mathbb{Z}^{q-j}$. Then, as shown above, $y \in \mathbb{Z}^{q-j+1}$, $\mathbf{p}(y) = 0$, and

$$k^{-x_j/d_j} \beta^j(x) = \beta^j(y).$$

Thus

$$\beta^j(x) = k^{x_j/d_j} \beta^j(y),$$

which implies (13). It follows from (12) and (13) that

$$(x_j \in d_j \mathbb{Z} \text{ and } y' \in \mathbb{Z}^{q-j}) \implies x \in d_j \mathbb{Z} \times \mathbb{Z}^{q-j}.$$

We have $f'(x) = f^{j+1}(y')$ and

$$\begin{aligned} & x \in \mathbb{Z}^{q-j+1} \text{ and } \beta^j(x) \in \mathcal{N}^j \cap K \\ \iff & x \in d_j \mathbb{Z} \times \mathbb{Z}^{q-j} \text{ and } k^{-x_j/d_j} \beta^j(x) \in \mathcal{N}^{j+1} \cap K \\ \iff & x_j \in d_j \mathbb{Z}, y' \in \mathbb{Z}^{q-j}, \text{ and } \beta^{j+1}(y') \in \mathcal{N}^{j+1} \cap K \\ \iff & x_j \in d_j \mathbb{Z} \text{ and } f^{j+1}(y') \in d_{j+1} \mathbb{Z} \times \dots \times d_q \mathbb{Z} \\ \iff & x_j \in d_j \mathbb{Z} \text{ and } f'(x) \in d_{j+1} \mathbb{Z} \times \dots \times d_q \mathbb{Z} \\ \iff & f^j(x) \in d_j \mathbb{Z} \times \dots \times d_q \mathbb{Z}. \quad \square \end{aligned}$$

§12. DEFINING $\gamma_{r+1}G$ BY EQUATIONS AND CONGRUENCES

12.1. Theorem. *Let G be a group and $g_1, \dots, g_n \in G$. Fix $r \geq 0$. Then, for some $q \geq 0$, there are rational polynomials*

$$P_j(X_1, \dots, X_n), \quad j \in (q),$$

of degree at most r and integers $d_1, \dots, d_q \geq 0$ such that

$$g_1^{x_1} \dots g_n^{x_n} \in \gamma_{r+1}G \iff P_j(x) \in d_j \mathbb{Z} \text{ for } j \in (q)$$

for any $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$.

The polynomials P_j constructed in the proof have the following integrality property: for $x \in \mathbb{Z}^n$, $P_j(x) \in \mathbb{Z}$ if $P_k(x) \in d_k \mathbb{Z}$ for $k < j$. The check is omitted.

Proof. We use the constructions of the previous sections of this Appendix for the given n and r . In particular, we let the required q be the size of the Mal'cev basis of \mathcal{N} , the free nilpotent group of rank n and class r . Consider the homomorphism

$$t: \mathcal{N} \rightarrow G/\gamma_{r+1}G, \quad \bar{Z}_i \mapsto \bar{g}_i.$$

Put $K = \text{Ker } t \subseteq \mathcal{N}$. By Lemma 11.1, there are integers $d_1, \dots, d_q \geq 0$ and a cultural morphism $f: \mathbb{Q}_{s_1 \dots s_q}^q \rightarrow \mathbb{Q}_{s_1 \dots s_q}^q$ such that

$$\beta^{-1}(K) = f^{-1}(d_1 \mathbb{Z} \times \dots \times d_q \mathbb{Z})$$

as subsets of \mathbb{Q}^q . Define the required polynomials P_j by the equality

$$f(x_1, \dots, x_n, 0, \dots, 0) = (P_j(x))_{j \in (q)}, \quad x = (x_1, \dots, x_n) \in \mathbb{Q}^n.$$

Since $s_1 = \dots = s_n = 1$, the degree of P_j is at most s_j , which does not exceed r . Given $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, we have

$$g_1^{x_1} \dots g_n^{x_n} \bmod \gamma_{r+1}G = t(\bar{Z}_1^{x_1} \dots \bar{Z}_n^{x_n}) = t(\beta(x_1, \dots, x_n, 0, \dots, 0))$$

in $G/\gamma_{r+1}G$ and thus

$$\begin{aligned} & g_1^{x_1} \dots g_n^{x_n} \in \gamma_{r+1}G \\ \iff & \beta(x_1, \dots, x_n, 0, \dots, 0) \in K \\ \iff & f(x_1, \dots, x_n, 0, \dots, 0) \in d_1\mathbb{Z} \times \dots \times d_q\mathbb{Z} \\ \iff & P_j(x) \in d_j\mathbb{Z} \text{ for } j \in (q). \quad \square \end{aligned}$$

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St. Petersburg Department
of Steklov Institute of Mathematics
Fontanka 27, St. Petersburg 191011, Russia
E-mail: ssp@pdmi.ras.ru

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