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# CLIQUE LADDERS IN THE GORDIAN GRAPH

Abstract. Drawing inspiration from results of S. Baader, we present a new geometric pattern in the Gordian graph of knots. We show that any two vertices at distance $m$ in the Gordian graph of knots can be connected by infinitely many internally vertex-disjoint paths of length $m$, such that for each $k \in \{1, 2, \ldots, m-1\}$, the $k$-th vertices on these paths form a clique. The proof exploits a sufficient geometric condition for the non-splittability of links.

## §1. Introduction

The *Gordian distance* $d_G(K, Q)$ between two tame knots $K$ and $Q$ is defined as the minimal number of crossing changes required to transform $K$ into $Q$. The *Gordian graph* is the graph whose vertices are ambient isotopy classes of unoriented tame knots in $S^3$, with two vertices connected by an edge if and only if the corresponding knots are related by a single crossing change (see [2]). We denote the Gordian graph by $G$.

In 2006, Baader [1] revealed a nontrivial geometric pattern in the Gordian graph: for every pair of knots $K$ and $Q$ with Gordian distance two, there exist infinitely many non-equivalent knots whose Gordian distance to both $K$ and $Q$ is one, see Fig. 1 (left). Stated differently, whenever two knots in the Gordian graph $G$ are at Gordian distance exactly two, the intersection of their unit neighborhoods contains infinitely many knots. In 2012, Horiuchi and Ohyama [4] gave a two-fold generalization of Baader's result: crossing changes become $C_k$-moves for arbitrary $k \geqslant 3$, and unit neighborhoods are extended to neighborhoods of arbitrary radius. This generalization also extends the theorem of Horiuchi [3] on the $\Delta$-move.

In this paper, following the approach in [5], we present a new pattern that extends Baader's result; see Fig. 1.

*Key words and phrases*: knot theory, Gordian graph, crossing change, non-splittability, satellite knots, incompressible tori.

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**Theorem 1.** *For any two vertices $K_0$ and $K_m$ at distance $m$ in $G$, there exist infinitely many internally vertex-disjoint paths of length $m$ connecting them, such that for each $k \in \{1, 2, \ldots, m-1\}$ the collection of $k$-th vertices on these paths forms a clique.*

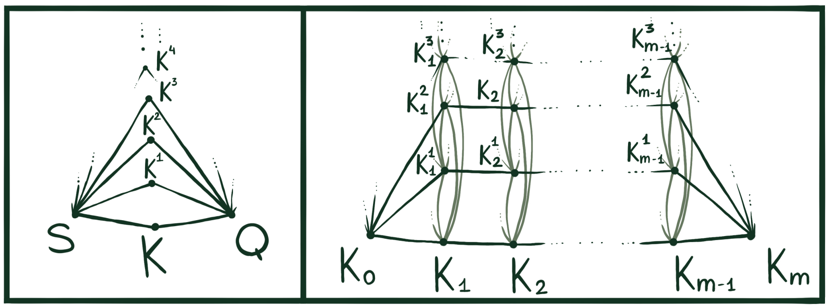


Figure 1. Baader's pattern and our pattern.

Before proving Theorem 1, we collect the necessary preliminaries in the next section.

## §2. Preliminaries

**Definition 1.** A knot $K^*$ is a *satellite* of a nontrivial knot $K$ if there exist a simple closed curve $q$ lying in some solid torus $V$, not contained in any 3-ball inside $V$, and not isotopic to the core $c$ of $V$, and an embedding $\varphi\colon V \to S^3$ such that the simple closed curve $\varphi(c) \subset S^3$ represents the ambient isotopy class of the knot $K$, and the simple closed curve $\varphi(q) \subset S^3$ represents the ambient isotopy class of the knot $K^*$. If $K^*$ is a satellite of $K$, then $K$ is called a *companion* of $K^*$.

**Remark 1.** Let $k$ be a simple closed curve in $S^3$ representing the ambient isotopy class of the knot $K$, let $N(k)$ be a closed tubular neighborhood of $k$, and let $T$ be an incompressible non-boundary-parallel torus in $S^3 \setminus \mathrm{Int}(N(k))$. Suppose there exists an embedding $\varphi\colon V \to S^3$ of some solid torus $V$ into $S^3$ such that $\varphi(\partial V) = T$ and the simple closed curve $\varphi(c) \subset S^3$, where $c$ is the core of $V$, represents the ambient isotopy class of a nontrivial knot $Q$. In this case, $Q$ is a companion of $K$.

**Lemma 1** (see [5])**.** *Let $W$ be a non-trivially knotted solid torus in $S^3$, and let $K$ be a knot in* $\mathrm{Int}(W)$*. Then the following conditions are equivalent*:

(1) *a meridian*[1] *curve of the solid torus $W$ is linked*[2] *to $K$*;
(2) *the torus $\partial W$ is incompressible in $S^3 \setminus K$.*

**Definition 2.** We say that a surface $H \subset S^3$ is an *unknotted torus with a hole* (or *unknotted one-holed torus*) if there exists an unknotted solid torus $V \subset S^3$ such that $H \subset \partial V$ and $\partial V \setminus H$ is an open disk.

A pair of simple closed curves lying in an unknotted one-holed torus $H$ is called a *meridian-longitude pair* of $H$ if it is a meridian-longitude pair of some unknotted torus that contains $H$.

**Lemma 2** (see [5])**.** *Let $K$ be a knot in $S^3$, let $H$ be an unknotted one-holed torus in $S^3 \setminus K$, and let $\{m_H, l_H\}$ be a meridian-longitude pair of $H$. If each of $m_H$ and $l_H$ is linked to $K$, then $\partial H$ is linked to $K$.*

**Lemma 3.** *Let $K_0, K_1, \ldots, K_m$ be knots such that each $K_i$ is obtained from $K_{i-1}$ by a single crossing change for $i = 1, 2, \ldots, m$. Then $K_0$ admits a special diagram locally structured as shown in Fig.* 2*. The simultaneous changes of the crossings labeled* $\boxed{c_1}, \boxed{c_2}, \ldots, \boxed{c_i}$ *result in a diagram of $K_i$* (*see Fig.* 3)*.*

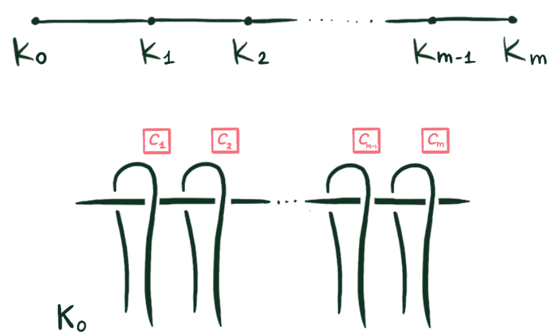


Figure 2. A part of a special diagram of $K_0$.

[1]Recall that an essential simple closed curve in the torus $\partial(D^2 \times S^1)$ is called a *meridian curve* if it bounds a disk in the solid torus $D^2 \times S^1$.

[2]For a link $K \cup L$ in $S^3$, we say $K$ is *unlinked to* $L$ if there exists a 2-sphere in $S^3 \setminus (K \cup L)$ that separates $K$ and $L$, and otherwise $K$ is *linked to* $L$.

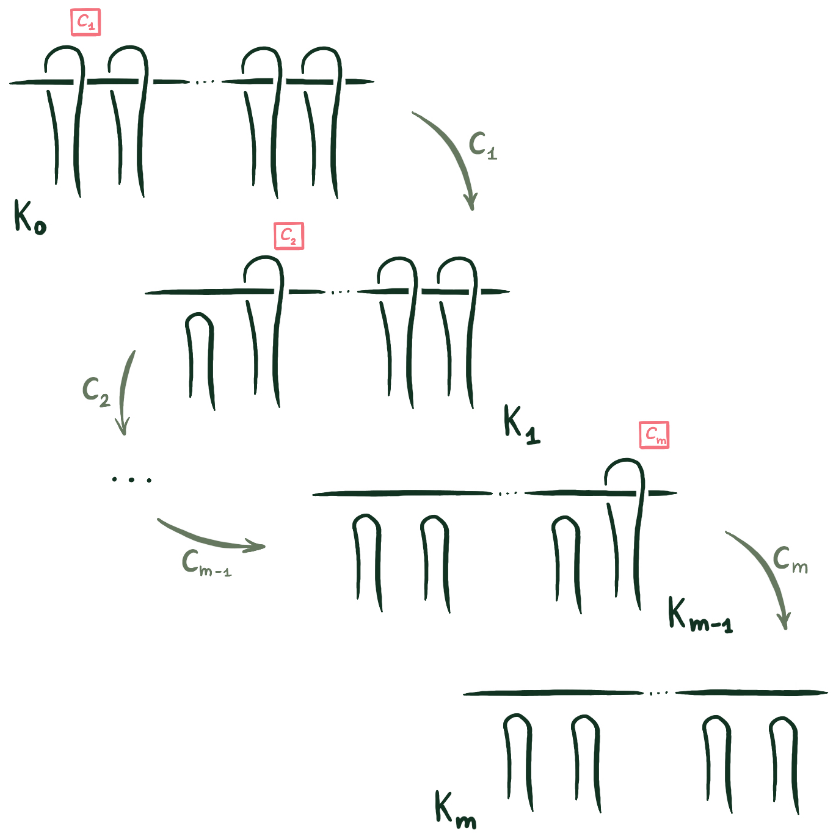


Figure 3. The simultaneous changes of crossings.

**Proof of Lemma 3.** We introduce an auxiliary operation on knots, which we call a surgery twist. Let $k \subset S^3$ be a knot, and let $D \subset S^3$ be an embedded disk such that $k$ intersects $D$ transversely in exactly two points, both lying in the interior of $D$. The knot obtained by performing $+1$-framed surgery along the trivial knot $\partial D$ is referred to as the result of a *surgery twist* of $k$ along $D$. As shown in Fig. 4, two knots are related by a single crossing change if and only if they are related by a single surgery twist along some disk.

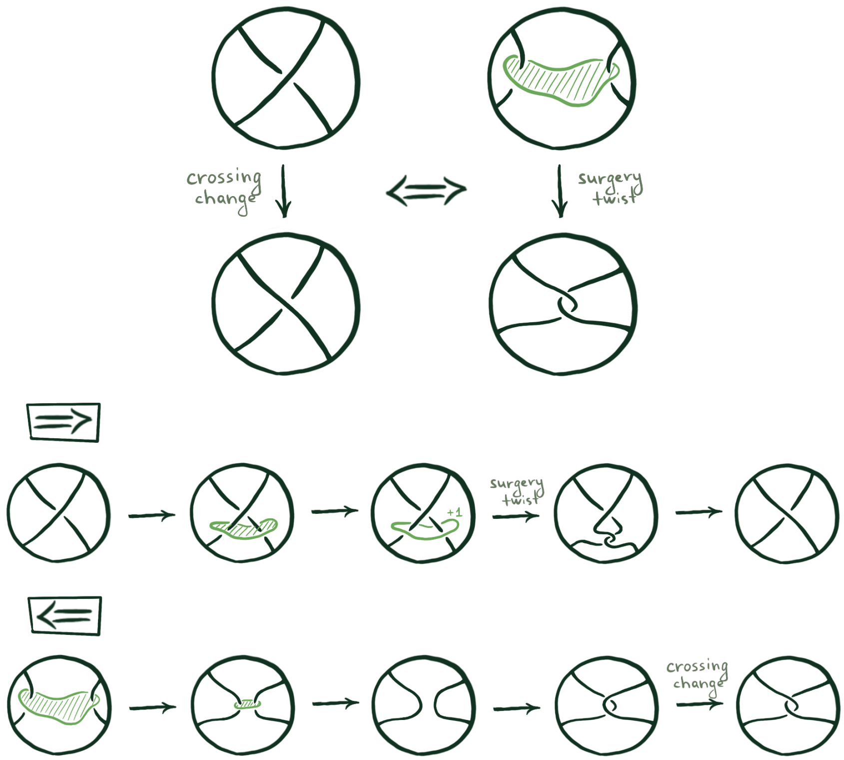


Figure 4. The crossing change is equivalent to the surgery twist.

Let $k_0 \subset S^3$ be a simple closed curve representing the ambient isotopy class of the knot $K_0$. Since $K_1$ differs from $K_0$ by a single crossing change, there exists an embedded disk $D_1 \subset S^3$ intersecting $k_0$ transversely in exactly two points (both lying in the interior of $D_1$) such that performing $+1$-framed surgery along $\partial D_1$ yields a curve representing $K_1$. Applying an ambient isotopy, we can localize $D_1$ within a small 3-ball $B$. A surgery twist along $D_1$ changes the crossing and yields a curve $k_1$ that represents $K_1$, as shown in Fig. 5.

Since $K_2$ differs from $K_1$ by a single crossing change, there exists an embedded disk $D_2 \subset S^3$ intersecting $k_1$ transversely in exactly two points (both lying in the interior of $D_2$), such that performing a surgery twist along $D_2$ yields a curve representing $K_2$, see Fig. 6 (a). We assume without loss of generality that the intersection points of $D_2$ and $k_1$ lie outside $B$. If $D_2$ meets $B$, we use an ambient isotopy that keeps $k_1$ fixed to push $D_2$ out of $B$, see Fig. 6 (b) (note that $B \setminus \operatorname{Int}\big(N(k_1)\big)$ is a handlebody). Next, we perform an ambient isotopy of $D_2$ that keeps $B$ fixed and localizes $D_2$ within a small 3-ball $B_1$ centered at one of the two points of $D_2 \cap k_1$, see Fig. 6 (c). This isotopy takes $k_1$ to a curve $k_1'$.

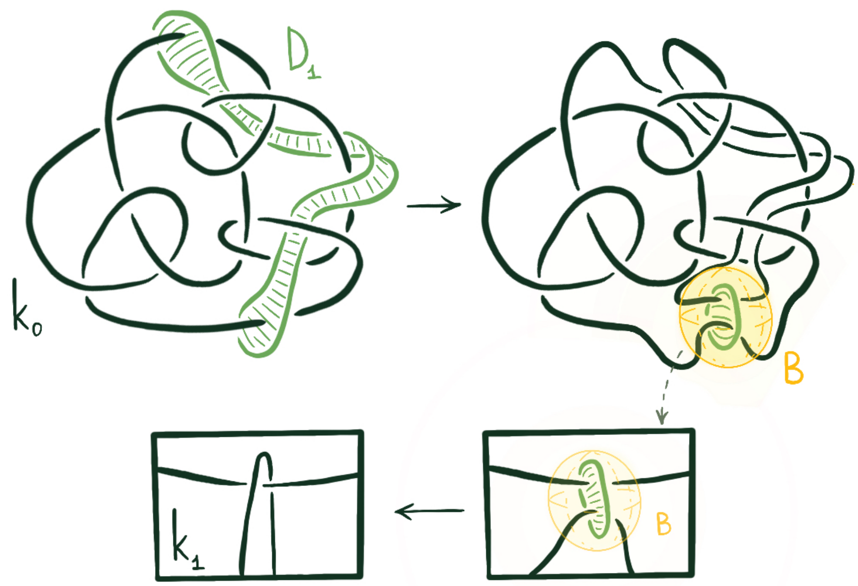


Figure 5. The isotopy and the surgery twist.

Subsequently, we slide $B_1$ along the curve $k_1'$ by an ambient isotopy until it enters the ball $B$, see Fig. 6 (d). This isotopy takes $k_1'$ to a curve $k_1''$. Whenever $k_1''$ intersects $B$ in a collection of arcs that are disjoint from $D_2$, we may remove each such arc from $B$ by applying the same handlebody-argument. Finally, a surgery twist along $D_2$ produces a curve $k_2$ representing $K_2$, see Fig. 6 (e) and Fig. 6 (f).

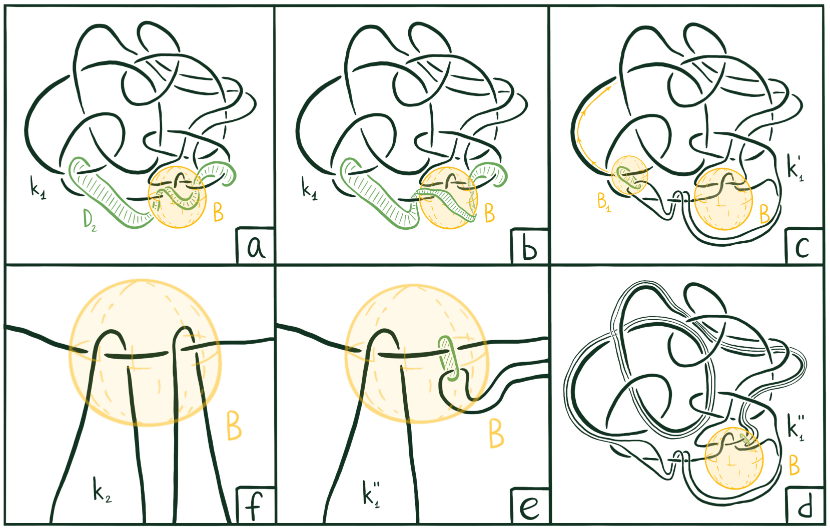


Figure 6. The Next Step.

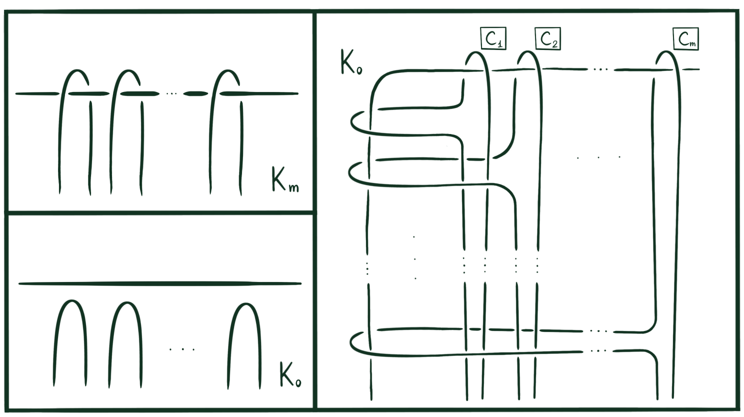

Figure 7. The final isotopy after the crossing changes.

Proceeding similarly, we obtain a curve $k_m$ representing $K_m$. We change the appropriate crossings in $B$ and perform elementary isotopies to obtain a curve representing $K_0$, thereby producing the required special diagram of $K_0$, see Fig. 7 (by reversing the order of the isotopies described above, one can verify that the intermediate diagrams in Fig. 3 represent the knots $K_{m-1}, K_{m-2}, \ldots, K_1$). □

## §3. Proof of the main theorem

**Proof of Theorem 1.** Let $K_0, K_1, \ldots, K_m$ be knots that form a path in $G$ connecting the vertices corresponding to $K_0$ and $K_m$. By Lemma 3, the knots $K_0, K_1, \ldots, K_m$ admit a sequence of consistent diagrams, as illustrated in Fig. 3. The structure of these diagrams allows us to construct the family of knots $\{K_i^j\}_{i=1,\, j=1}^{m-1,\, \infty}$ diagrammatically.

First, we describe the knots $K_1^1, K_2^1, \ldots, K_{m-1}^1$ diagrammatically. For this purpose, we construct an auxiliary 2-tangle $P_1^\circ$ in the following manner. Schubert's Theorem [6, Satz 6] implies that for any knot $Q$, the set $C(Q)$ of its companions is finite. Take $P_1$ to be an arbitrary prime knot outside the finite union $\bigcup_{i=0}^m C(K_i)$. Cut $P_1$ at a point to obtain a $P_1$-knotted arc, then place this arc in a 3-ball so that only its endpoints lie on the boundary; this yields a 1-tangle. Doubling the strand of this 1-tangle by a parallel shift along a continuous normal vector field yields the required 2-tangle $P_1^\circ$ (see Fig. 8).


Figure 8. The auxiliary tangle $P_1^\circ$ obtained from the knot $P_1$.

For $i = 1, 2, \ldots, m-1$, the relevant part of the diagram of $K_i^1$ we are now constructing is shown in Fig. 9. The diagram contains the diagram of the 2-tangle $P_1^\circ$ constructed earlier, along with crossings labeled $\boxed{c_1}, \boxed{c_2}, \ldots, \boxed{c_m}$ and a single crossing labeled $\boxed{1}$. Outside the indicated region, the diagram of $K_i^1$ agrees with the initial diagram of $K_i$.

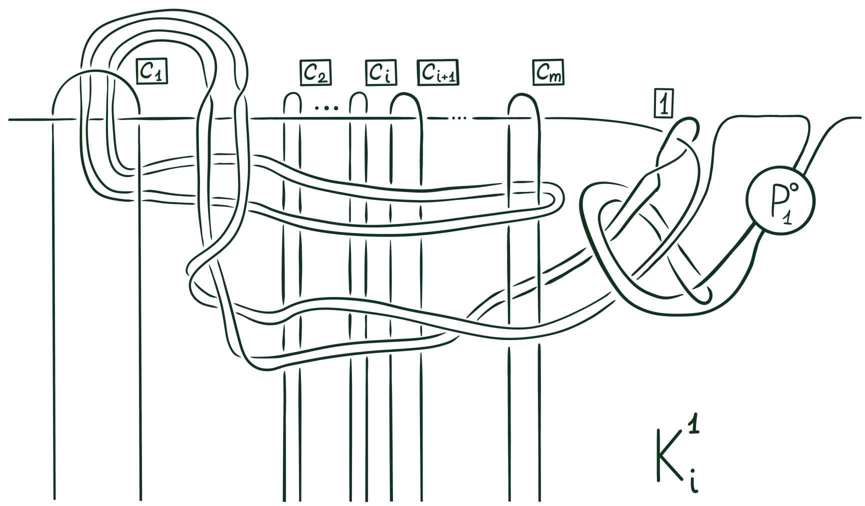


Figure 9. A part of the diagram of $K_i^1$.

Having reached this point, we are in a position to diagrammatically describe the knots $K_1^j, K_2^j, \ldots, K_{m-1}^j$ for $j \geqslant 2$. Assume that all knots in the family $\{K_s^t\}_{s=1,\, t=1}^{m-1,\, j-1}$ have been constructed in the previous steps. We then fix an arbitrary prime knot $P_j$ outside the finite union

$$\bigcup_{s=1,\, t=1}^{m-1,\, j-1} C(K_s^t) \,\cup\, \bigcup_{i=0}^{m} C(K_i).$$

As $P_1^\circ$ was formed from $P_1$, we now form a 2-tangle $P_j^\circ$ from $P_j$.

For $i = 1, 2, \ldots, m-1$, the relevant part of the diagram of $K_i^j$ we are now constructing is shown in Fig. 10. The diagram contains diagrams of 2-tangles $P_1^\circ, P_2^\circ, \ldots, P_j^\circ$ constructed earlier, along with crossings labeled $\boxed{c_1}, \boxed{c_2}, \ldots, \boxed{c_m}$ and crossings labeled $\boxed{1}, \boxed{2}, \ldots, \boxed{j}$. Outside the indicated region, the diagram of $K_i^j$ agrees with the initial diagram of $K_i$.

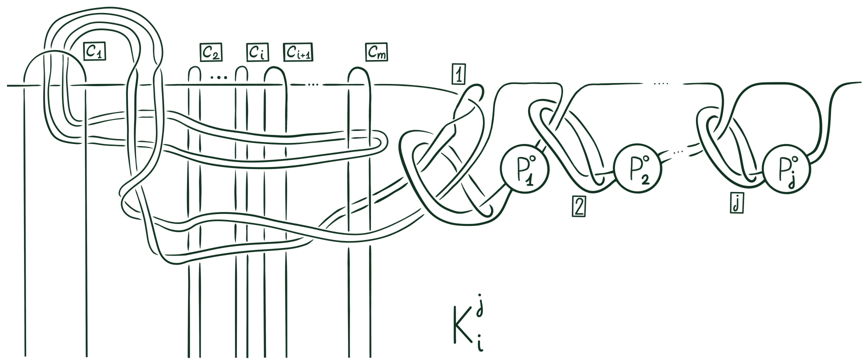


Figure 10. A part of the diagram of $K_i^j$.

For each $i \in \{1, 2, \ldots, m-2\}$ and each $j \geqslant 1$, the diagram of $K_i^j$ is obtained from the diagram of $K_{i+1}^j$ by a crossing change at the crossing labeled $\boxed{c_{i+1}}$. Additionally, a diagram of $K_0$ is obtained from the diagram of $K_1^j$ by a crossing change at the crossing labeled $\boxed{c_1}$ and a diagram of $K_m$ is obtained from the diagram of $K_{m-1}^j$ by a crossing change at the crossing labeled $\boxed{c_m}$. Moreover, for each $i \in \{1, 2, \ldots, m-1\}$ and each $a > b \geqslant 1$, a diagram of $K_i^b$ is obtained from the diagram of $K_i^a$ by a crossing change at the crossing labeled $\boxed{b+1}$. Finally, for each $i \in \{1, 2, \ldots, m-1\}$ and each $j \geqslant 1$, a diagram of $K_i$ is obtained from the diagram of $K_i^j$ by a crossing change at the crossing labeled $\boxed{1}$.

We are now in a position to show that all knots in the family

$$\{K_s^t\}_{s=1,\, t=1}^{m-1,\, \infty} \cup \{K_i\}_{i=0}^{m}$$

are pairwise distinct. Indeed, for any fixed $j \geqslant 1$, the knots

$$K_0, K_1^j, K_2^j, \ldots, K_{m-1}^j, K_m$$

are all distinct; otherwise, we have $d(K_0, K_m) < m$ and reach a contradiction with the assumption of the theorem. It remains to show that for any quadruple $(j, i, a, b)$ such that

$$j \geqslant 1, \quad i, a \in \{1, 2, \ldots, m-1\} \quad \text{and} \quad b \in \{1, 2, \ldots, j-1\}$$

the knot $K_i^j$ differs from the knot $K_a^b$ and from the knots $K_1, K_2, \ldots, K_{m-1}$.

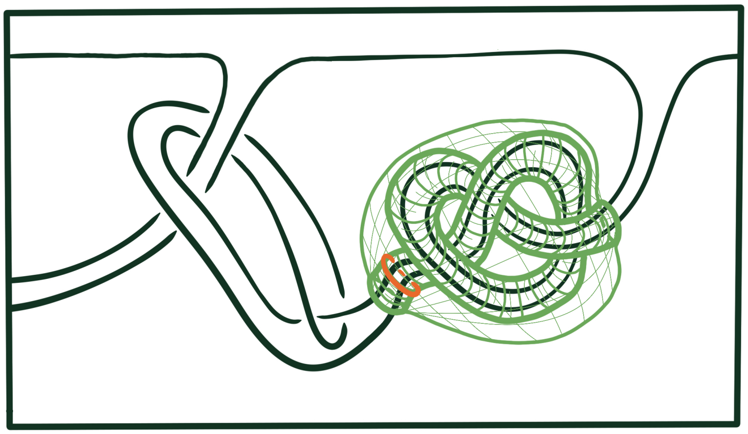

Figure 11. The green knotted torus is incompressible in $S^3 \setminus \mathrm{Int}\,\big(N(K_i^j)\big)$ if and only if the orange meridian curve is linked to $K_i^j$.

The construction yields a solid torus $W \subset S^3$ that is knotted into the nontrivial knot $P_j$ and contains $K_i^j$, see Fig. 11. Note that the torus $\partial W$ is a non-boundary-parallel torus in $S^3 \setminus \mathrm{Int}\,\big(N(K_i^j)\big)$, where $N(K_i^j)$ denotes the closed tubular neighborhood of $K_i^j$. Indeed, each meridian curve of $\partial W$ has zero linking number with $K_i^j$. If $\partial W$ is incompressible in $S^3 \setminus \mathrm{Int}\,\big(N(K_i^j)\big)$, then by Remark 1 the knot $P_j$ is a companion of $K_i^j$. Since $P_j$ is not a companion of any knot from the family

$$\{K_s^t\}_{s=1,\,t=1}^{m-1,\,j-1} \cup \{K_i\}_{i=0}^{m}$$

it suffices to show that $\partial W$ is incompressible in $S^3 \setminus \mathrm{Int}\,\big(N(K_i^j)\big)$.

The incompressibility of $\partial W$ in $S^3 \setminus \mathrm{Int}\,\big(N(K_i^j)\big)$ follows from the incompressibility of $\partial W$ in $S^3 \setminus K_i^j$. By Lemma 1, $\partial W$ is incompressible in $S^3 \setminus K_i^j$ if and only if a meridian curve of $\partial W$ is linked to $K_i^j$, see Fig. 11. To see that a meridian curve is linked to $K_i^j$, we apply Lemma 2 step by step. Our first step is to locate an unknotted one-holed torus $H$ in $S^3 \setminus K_i^j$ such that $\partial H$ is a meridian of $\partial W$, see Fig. 12 (top). At this point, Lemma 2 reduces the original problem to checking that each element of a meridian-longitude

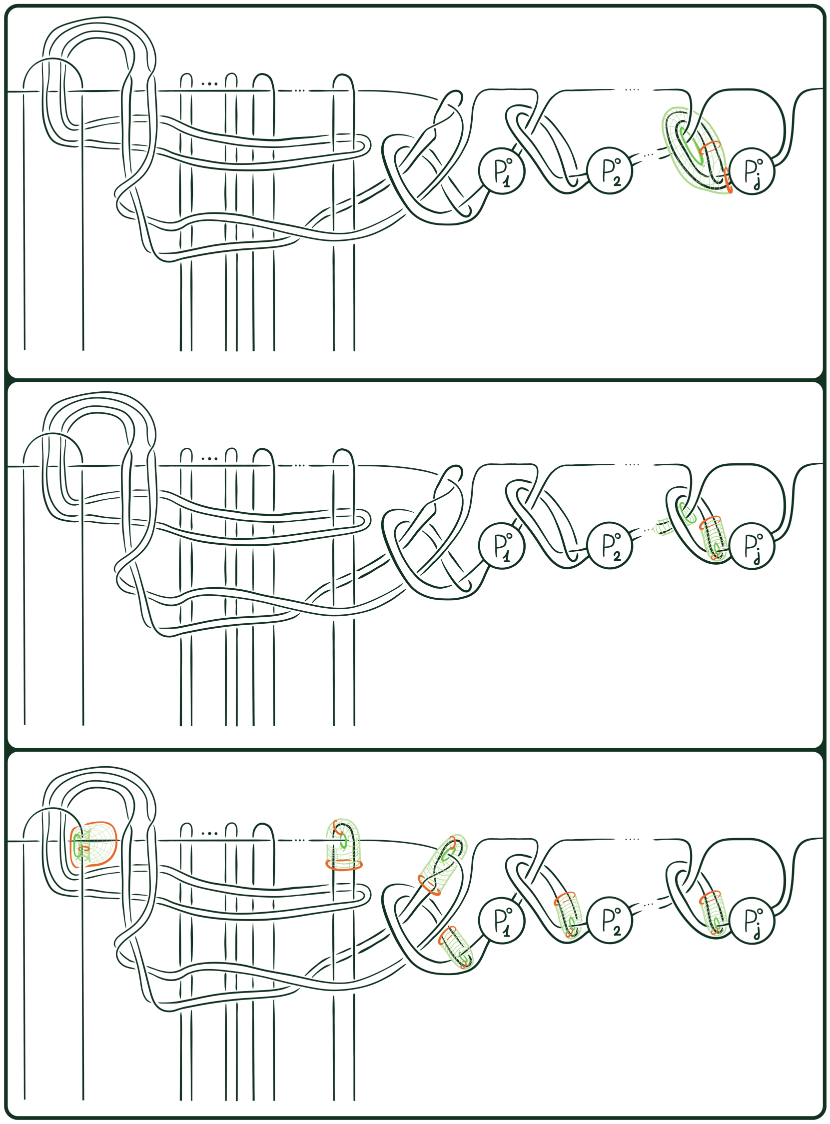


Figure 12. The Process.

pair of $H$ is linked to $K_i^j$. We perform two further steps, constructing for each of these curves a suitable one-holed torus in $S^3 \setminus K_i^j$, and proceed in the same manner. As illustrated in Fig. 12, this process always terminates. After finitely many steps, the required curves have a nonzero linking number with $K_i^j$, see Fig. 12 (bottom). Thus, we have shown that the knot $K_i^j$ differs from the knot $K_a^b$ and from the knots $K_1, K_2, \ldots, K_{m-1}$.

Note that for each $j \geqslant 1$, the knots $K_0, K_1^j, \ldots, K_{m-1}^j, K_m$ form a path in $G$ connecting the vertices corresponding to $K_0$ and $K_m$. We denote this path by $\alpha_j$. For distinct $i, j \geqslant 1$, the paths $\alpha_i$ and $\alpha_j$ are internally vertex-disjoint. Furthermore, for each $k \in \{1, 2, \ldots, m-1\}$, the collection of $k$-th vertices on these paths forms a clique. This completes the proof. □

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