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T. A. Kozlovskaya

SOME DECOMPOSITIONS OF THE CACTUS GROUP ON 4 STRANDS

Abstract. We study the commutator subgroup J_4' of the cactus group J_4 on four strands and obtain generators and defining relations for it. We show that J_4' has a presentation with three generators and two defining relations. We also decompose J_4' as an HNN extension with base group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$ and infinite cyclic associated subgroups.

We prove that the pure cactus group PJ_4 is both an HNN extension of a free group and an amalgamated free product of a free group and an infinite cyclic group over a cyclic subgroup.

1. Introduction

Cactus groups were introduced by S. L. Devadoss [5] as quasibraid groups and by Davis et al. [4] as mock reflection groups. They were later shown to control coboundary categories in the same way that braid groups control braided categories [8]. The name cactus group comes from the cactus-like shape of the corresponding moduli spaces. Coboundary categories arise in the study of crystals of finite-dimensional reductive Lie algebras and, more generally, of representations of coboundary Hopf algebras.

For $n \geqslant 2$, the cactus group J_n is generated by

$$s_{p,q}, \qquad 1 \leqslant p < q \leqslant n,$$

with the following defining relations:

$$s_{p,q}^2 = 1;$$
$$s_{p,q}s_{m,r} = s_{m,r}s_{p,q}, \qquad \text{for } [p,q] \cap [m,r] = \varnothing;$$
$$s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}, \qquad \text{for } [m,r] \subset [p,q].$$

The generator $s_{p,q}$ can be represented diagrammatically by the braid on n strands in which the strands $p, p+1, \ldots, q$ intersect at one common

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point, and reverse their order after that point. Some generators and relations of J_4 are depicted in Fig. 1. The induced permutation of the strands gives an epimorphism

$$s\colon J_n \longrightarrow S_n$$

defined on generators, in one-line notation, by

$$s_{p,q} \mapsto (1, 2, \ldots, p-1, q, q-1, \ldots, p+1, p, q+1, q+2, \ldots, n).$$

The kernel $\ker(s)$ of this map is called the *pure cactus group* and is denoted by PJ_n. By [8, Theorem 9], PJ_n is isomorphic to the fundamental group of the real locus $\overline{M_{0,n+1}}\,(\mathbb{R})$ of the Deligne–Mumford compactification of the moduli space of rational curves with $n+1$ marked points [5]. In particular, $\overline{M_{0,5}}\,(\mathbb{R})$ is homeomorphic to the connected sum of five real projective planes (see [6], Example 2.5). Thus PJ_4 is isomorphic to the fundamental group of this surface:

$$\left\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \;\middle|\; \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 = 1 \right\rangle.$$

For $n = 2$, J_2 is cyclic of order 2 and PJ_2 is trivial. For $n = 3$, J_3 is the infinite dihedral group and PJ_3 is the infinite cyclic group generated by $(s_{12}s_{13})^3$. The first non-trivial pure cactus group is therefore PJ_4.

Hama and Ichihara [7] constructed an action of PJ_4 on the hyperbolic plane and a Dirichlet polygon for this action. As a corollary, they gave an alternative proof that PJ_4 is the fundamental group of the connected sum of five real projective planes.

Bellingeri, Chemini, and Lebed [2] studied several algebraic properties of J_n and PJ_n. They solved the word problem for J_n, proved that J_n has no odd torsion, and showed that elements of order 2^k occur for all k when n is sufficiently large. They also proved that PJ_n is torsion-free and that J_n and PJ_n have trivial center for $n > 2$ and $n > 3$, respectively. Their Reidemeister–Schreier computation gives the following presentation of PJ_4 [2, Appendix A]:

$$PJ_4 = \left\langle a, b, c, d, t \;\middle|\; t\,(bdad)\,t^{-1} = cb^{-1}a^{-1}c \right\rangle.$$

1.1. Overview of results. We show that PJ_4 is an HNN extension of a free group and also an amalgamated free product of a free group and an infinite cyclic group over a cyclic subgroup (Theorem 4.1).

Commutator subgroups are a classical source of information about a group. For braid groups they have been studied extensively; see e. g. [9, 1]. The abelianization $J_n^{ab} = J_n/J_n'$ is $\mathbb{Z}_2^{\oplus(n-1)}$, so J_n' is finitely presented. We compute the first non-trivial case, namely J_4'. In Corollary 3.1 we obtain the presentation

$$J_4' = \langle a, b, c \mid b^2 = 1,\ (c^{-1}ab^{-1}ca^{-1})^2 = 1 \rangle .$$

We also decompose J_4' as an HNN extension with base group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$ and infinite cyclic associated subgroups (Corollary 3.2):

$$J_4' = \langle a, b, g, e \mid b^2 = 1,\ g^2 = 1,\ aea^{-1} = beg \rangle .$$

The base group is $\langle b, g, e \mid b^2 = 1,\ g^2 = 1 \rangle$, the stable letter is a, and the associated subgroups are $\langle e \rangle$ and $\langle beg \rangle$. These decompositions yield normal forms for elements of J_4' and PJ_4, and hence for elements of J_4. They are the Britton normal form for HNN extensions and the standard normal form for amalgamated free products. They reduce concrete word computations and related structural questions for these low-dimensional cactus groups to calculations in free groups, free products, and cyclic subgroups. Appendix A contains GAP code which reproduces the computation of the presentation of J_4'.

2. Basic definitions

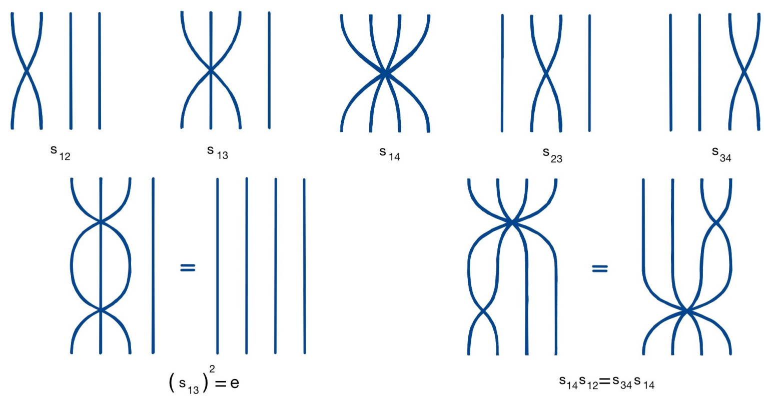

Figure 1. Some generators and relations in J_4.

Chemin and Nanda [3] and Zimireva [11] independently proved that the cactus group J_n is generated by $a_i = s_{1,i}$, for $i = 2, \dots, n$, with the following defining relations: first, $a_i^2 = 1$; second,

$$(a_i a_k a_j a_k)^2 = 1$$

for $i \leqslant j$ and $i + j \leqslant k$; third,

$$a_i a_k a_j a_k = a_{i+j-k} a_j a_{i+j-k} a_i$$

for

$$4 \leqslant j+2 \leqslant i \leqslant n, \quad j < k < i, \quad 2 \leqslant i+j-k \leqslant n, \text{ and } 2k \leqslant i+j.$$

In particular, J_4 has the presentation

$$\left\langle \begin{array}{c} s_{12}, s_{23}, s_{34}, \\ s_{13}, s_{24}, s_{14} \end{array} \middle| \begin{array}{c} s_{12}^2 = s_{23}^2 = s_{34}^2 = s_{13}^2 = s_{24}^2 = s_{14}^2 = e \\ s_{12}s_{34} = s_{34}s_{12},\ s_{12}s_{13} = s_{13}s_{23},\ s_{23}s_{24} = s_{24}s_{34}, \\ s_{12}s_{14} = s_{14}s_{34},\ s_{23}s_{14} = s_{14}s_{23},\ s_{13}s_{14} = s_{14}s_{24} \end{array} \right\rangle$$

in the standard generators, and the presentation

$$\left\langle a_2, a_3, a_4 \;\middle|\; a_2^2 = a_3^2 = a_4^2 = 1,\ (a_4 a_2)^4 = 1,\ a_4\,(a_3 a_2 a_3) = (a_3 a_2 a_3)\,a_4 \right\rangle$$

in the generators a_i.

3. Commutator subgroup J_4'

For $n = 2$, we have $J_2 = \mathbb{Z}_2$ and $J_2' = \{1\}$. For $n = 3$, the group

$$J_3 = \left\langle a_2, a_3 \mid a_2^2 = a_3^2 = 1 \right\rangle$$

is the infinite dihedral group, and J_3' is infinite cyclic, generated by $(a_2 a_3)^2$; moreover,

$$J_3^{ab} = J_3 / J_3' \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Thus J_4' is the first non-trivial case.

We compute a presentation of J_4' by the Reidemeister–Schreier method (see [10, Section 2.3]). The abelianization $J_4^{ab} = J_4/J_4'$ is generated by the images b_i of a_i, for $i = 2, 3, 4$, with the commutativity relations and

$$b_2^2 = b_3^2 = b_4^2 = 1.$$

Thus J_4^{ab} has the eight elements

$$1, b_2, b_3, b_4, b_2b_3, b_2b_4, b_3b_4, b_2b_3b_4.$$

Choose the Schreier transversal

$$\Lambda_4 = \{1, a_2, a_3, a_4, a_2a_3, a_2a_4, a_3a_4, a_2a_3a_4\}.$$

Then J_4' is generated by the Schreier generators

$$S_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},$$

where $\lambda \in \Lambda_4$, $a \in \{a_2, a_3, a_4\}$, and $\overline{\lambda a}$ denotes the representative of the coset $J_4' \lambda a$ in Λ_4.

For $\lambda = 1$,

$$S_{1,a_i} = a_i \cdot (\overline{a_i})^{-1} = a_i \cdot a_i^{-1} = 1.$$

For $\lambda = a_2$,

$$\begin{aligned} S_{a_2,a_2} &= a_2 a_2 \cdot (\overline{a_2 a_2})^{-1} = 1, \\ S_{a_2,a_3} &= a_2 a_3 \cdot (\overline{a_2 a_3})^{-1} = a_2 a_3 \cdot (a_2 a_3)^{-1} = 1, \\ S_{a_2,a_4} &= a_2 a_4 \cdot (\overline{a_2 a_4})^{-1} = a_2 a_4 \cdot (a_2 a_4)^{-1} = 1. \end{aligned}$$

For $\lambda = a_3$,

$$\begin{aligned} S_{a_3,a_2} &= a_3 a_2 \cdot (\overline{a_3 a_2})^{-1} = a_3 a_2 \cdot (a_2 a_3)^{-1} = (a_3 a_2)^2, \\ S_{a_3,a_3} &= a_3 a_3 \cdot (\overline{a_3 a_3})^{-1} = 1, \\ S_{a_3,a_4} &= a_3 a_4 \cdot (\overline{a_3 a_4})^{-1} = a_3 a_4 \cdot (a_3 a_4)^{-1} = 1. \end{aligned}$$

For $\lambda = a_4$,

$$\begin{aligned} S_{a_4,a_2} &= a_4 a_2 \cdot (\overline{a_4 a_2})^{-1} = a_4 a_2 \cdot (a_2 a_4)^{-1} = (a_4 a_2)^2, \\ S_{a_4,a_3} &= a_4 a_3 \cdot (\overline{a_4 a_3})^{-1} = a_4 a_3 \cdot (a_3 a_4)^{-1} = (a_4 a_3)^2, \\ S_{a_4,a_4} &= a_4 a_4 \cdot (\overline{a_4 a_4})^{-1} = a_4 a_4 \cdot (a_4 a_4)^{-1} = 1. \end{aligned}$$

For $\lambda = a_2 a_3$,

$$\begin{aligned} S_{a_2 a_3,a_2} &= a_2 a_3 a_2 \cdot (\overline{a_2 a_3 a_2})^{-1} = (a_2 a_3)^2, \\ S_{a_2 a_3,a_3} &= a_2 a_3 a_3 \cdot (\overline{a_2 a_3 a_3})^{-1} = 1, \\ S_{a_2 a_3,a_4} &= a_2 a_3 a_4 \cdot (\overline{a_2 a_3 a_4})^{-1} = 1. \end{aligned}$$

For $\lambda = a_2 a_4$,

$$\begin{aligned} S_{a_2 a_4,a_2} &= a_2 a_4 a_2 \cdot (\overline{a_2 a_4 a_2})^{-1} = (a_2 a_4)^2, \\ S_{a_2 a_4,a_3} &= a_2 a_4 a_3 \cdot (\overline{a_2 a_4 a_3})^{-1} = a_2 (a_4 a_3)^2 a_2, \\ S_{a_2 a_4,a_4} &= a_2 a_4 a_4 \cdot (\overline{a_2 a_4 a_4})^{-1} = 1. \end{aligned}$$

For $\lambda = a_3 a_4$,

$$\begin{aligned} S_{a_3 a_4,a_2} &= a_3 a_4 a_2 \cdot (\overline{a_3 a_4 a_2})^{-1} = a_3 a_4 a_2 a_4 a_3 a_2, \\ S_{a_3 a_4,a_3} &= a_3 a_4 a_3 \cdot (\overline{a_3 a_4 a_3})^{-1} = (a_3 a_4)^2, \\ S_{a_3 a_4,a_4} &= a_3 a_4 a_4 \cdot (\overline{a_3 a_4 a_4})^{-1} = 1. \end{aligned}$$

For $\lambda = a_2a_3a_4$,

$$\begin{aligned} S_{a_2a_3a_4,a_2} &= a_2a_3a_4a_2 \cdot (\overline{a_2a_3a_4a_2})^{-1} = a_2a_3a_4a_2a_4a_3, \\ S_{a_2a_3a_4,a_3} &= a_2a_3a_4a_3 \cdot (\overline{a_2a_3a_4a_3})^{-1} = a_2(a_3a_4)^2a_2, \\ S_{a_2a_3a_4,a_4} &= a_2a_3a_4a_4 \cdot (\overline{a_2a_3a_4a_4})^{-1} = 1. \end{aligned}$$

Hence J_4' is generated by

$$\begin{aligned} &S_{a_3,a_2},\ S_{a_4,a_2},\ S_{a_4,a_3},\ S_{a_2a_3,a_2},\ S_{a_2a_4,a_2}, \\ &S_{a_2a_4,a_3},\ S_{a_3a_4,a_2},\ S_{a_3a_4,a_3},\ S_{a_2a_3a_4,a_2},\ S_{a_2a_3a_4,a_3}. \end{aligned}$$

To obtain defining relations for J_4', we use the Reidemeister rewriting process τ. It rewrites words in the generators of J_4 representing elements of J_4' as words in the Schreier generators. For a reduced word

$$w = u_1^{\varepsilon_1} u_2^{\varepsilon_2} \dots u_\nu^{\varepsilon_\nu}, \quad \varepsilon_l = \pm 1, \quad u_l \in \{a_2, a_3, a_4\},$$

set

$$\tau(w) = S_{k_1,u_1}^{\varepsilon_1} S_{k_2,u_2}^{\varepsilon_2} \dots S_{k_\nu,u_\nu}^{\varepsilon_\nu}$$

where k_j is the representative of the $(j-1)$st initial segment of w if $\varepsilon_j = 1$ and of the jth initial segment of w if $\varepsilon_j = -1$. By [10, Theorem 2.9], the group J_4' is defined by relations

$$r_{\mu,\lambda} = \tau(\lambda\, r_\mu\, \lambda^{-1}), \quad \lambda \in \Lambda_4,$$

where r_μ is the defining relation of J_4.

For the relation $r_1 = a_2a_2$, the rewriting process gives

$$r_1 = S_{1,a_2}S_{a_2,a_2} = 1.$$

Conjugating r_1 by the coset representatives gives

$$\begin{aligned} r_{1,a_2} &= a_2r_1a_2^{-1} = 1, \\ r_{1,a_3} &= a_3r_1a_3^{-1} = S_{1,a_3}S_{a_3,a_2}S_{a_2a_3,a_2}S_{1,a_3}^{-1} = S_{a_3,a_2}S_{a_2a_3,a_2} = 1, \\ r_{1,a_4} &= a_4r_1a_4^{-1} = S_{1,a_4}S_{a_4,a_2}S_{a_2a_4,a_2}S_{1,a_4}^{-1} = S_{a_4,a_2}S_{a_2a_4,a_2} = 1, \\ r_{1,a_2a_4} &= a_2a_4r_1a_4^{-1}a_2^{-1} = S_{a_2a_4,a_2}S_{a_4,a_2} = 1, \\ r_{1,a_2a_3} &= a_2a_3r_1a_3^{-1}a_2^{-1} = S_{a_2a_3,a_2}S_{a_3,a_2} = 1, \\ r_{1,a_3a_4} &= a_3a_4r_1a_4^{-1}a_3^{-1} = S_{a_3a_4,a_2}S_{a_2a_3a_4,a_2} = 1, \\ r_{1,a_2a_3a_4} &= a_2a_3a_4r_1a_4^{-1}a_3^{-1}a_2^{-1} = S_{a_2a_3a_4,a_2}S_{a_3a_4,a_2} = 1. \end{aligned}$$

Conjugating $r_2 = a_3a_3$ gives

$$\begin{aligned}
r_{2,a_2} &= a_2r_2a_2^{-1} = a_2a_3a_3a_2^{-1} = 1,\\
r_{2,a_3} &= a_3r_2a_3^{-1} = 1,\\
r_{2,a_4} &= a_4r_2a_4^{-1} = S_{a_4,a_3}S_{a_3a_4,a_3} = 1,\\
r_{2,a_2a_4} &= a_2a_4r_2a_4^{-1}a_2^{-1} = S_{a_2a_4,a_3}S_{a_2a_3a_4,a_3} = 1,\\
r_{2,a_2a_3} &= a_2a_3r_2a_3^{-1}a_2^{-1} = 1,\\
r_{2,a_3a_4} &= a_3a_4r_2a_4^{-1}a_3^{-1} = S_{a_3a_4,a_3}S_{a_4,a_3} = 1,\\
r_{2,a_2a_3a_4} &= a_2a_3a_4r_2a_4^{-1}a_3^{-1}a_2^{-1} = S_{a_2a_3a_4,a_3}S_{a_2a_4,a_3} = 1.
\end{aligned}$$

Conjugating $r_3 = a_4a_4$ gives

$$\begin{aligned}
r_{3,a_2} &= a_2r_3a_2^{-1} = a_2a_4a_4a_2^{-1} = 1,\\
r_{3,a_3} &= a_3r_3a_3^{-1} = 1,\\
r_{3,a_4} &= a_4r_3a_4^{-1} = 1,\\
r_{3,a_2a_4} &= a_2a_4r_3a_4^{-1}a_2^{-1} = 1,\\
r_{3,a_2a_3} &= a_2a_3r_3a_3^{-1}a_2^{-1} = 1,\\
r_{3,a_3a_4} &= a_3a_4r_3a_4^{-1}a_3^{-1} = 1,\\
r_{3,a_2a_3a_4} &= a_2a_3a_4r_3a_4^{-1}a_3^{-1}a_2^{-1} = 1.
\end{aligned}$$

Conjugating $r_4 = a_4a_2a_4a_2a_4a_2a_4a_2$ gives

$$\begin{aligned}
r_{4,a_2} &= a_2r_4a_2^{-1} = S_{a_2a_4,a_2}S_{a_2a_4,a_2} = 1,\\
r_{4,a_3} &= a_3r_4a_3^{-1} = S_{a_3a_4,a_2}S_{a_2a_3,a_2}S_{a_3a_4,a_2}S_{a_2a_3,a_2} = 1,\\
r_{4,a_4} &= a_4r_4a_4^{-1} = S_{a_2a_4,a_2}S_{a_2a_4,a_2} = 1,\\
r_{4,a_2a_4} &= a_2a_4r_4a_4^{-1}a_2^{-1} = S_{a_4,a_2}S_{a_4,a_2} = 1,\\
r_{4,a_2a_3} &= a_2a_3r_4a_3^{-1}a_2^{-1} = S_{a_2a_3a_4,a_2}S_{a_3,a_2}S_{a_2a_3a_4,a_2}S_{a_3,a_2} = 1,\\
r_{4,a_3a_4} &= a_3a_4r_4a_4^{-1}a_3^{-1} = S_{a_3,a_2}S_{a_2a_3a_4,a_2}S_{a_3,a_2}S_{a_2a_3a_4,a_2} = 1,\\
r_{4,a_2a_3a_4} &= a_2a_3a_4r_4a_4^{-1}a_3^{-1}a_2^{-1} = S_{a_2a_3,a_2}S_{a_3a_4,a_2}S_{a_2a_3,a_2}S_{a_3a_4,a_2} = 1.
\end{aligned}$$

Conjugating $r_5 = a_4a_3a_2a_3a_4^{-1}a_3^{-1}a_2^{-1}a_3^{-1}$ gives

$$\begin{aligned}
r_{5,a_2} &= a_2r_5a_2^{-1} = S_{a_2a_4,a_3}S_{a_2a_3a_4,a_2}S_{a_3a_4,a_3}S^{-1}_{a_2a_3,a_2} = 1,\\
r_{5,a_3} &= a_3r_5a_3^{-1} = S_{a_3a_4,a_3}S_{a_4,a_2}S_{a_2a_4,a_3} = 1,\\
r_{5,a_4} &= a_4r_5a_4^{-1} = S_{a_3,a_2}S^{-1}_{a_2a_3a_4,a_3}S^{-1}_{a_3a_4,a_2}S^{-1}_{a_4,a_3} = 1,\\
r_{5,a_2a_4} &= a_2a_4r_5a_4^{-1}a_2^{-1} = S_{a_2a_3,a_2}S^{-1}_{a_3a_4,a_3}S^{-1}_{a_2a_3a_4,a_2}S^{-1}_{a_2a_4,a_3} = 1,\\
r_{5,a_2a_3} &= a_2a_3r_5a_3^{-1}a_2^{-1} = S_{a_2a_3a_4,a_3}S_{a_2a_4,a_2}S_{a_4,a_3} = 1,\\
r_{5,a_3a_4} &= a_3a_4r_5a_4^{-1}a_3^{-1} = S^{-1}_{a_2a_4,a_3}S^{-1}_{a_4,a_2}S^{-1}_{a_3a_4,a_3} = 1,\\
r_{5,a_2a_3a_4} &= a_2a_3a_4r_5a_4^{-1}a_3^{-1}a_2^{-1} = S^{-1}_{a_4,a_3}S^{-1}_{a_2a_4,a_2}S^{-1}_{a_2a_3a_4,a_3} = 1.
\end{aligned}$$

Lemma 3.1. *The relations above imply the following identities in J_4':*

$$\begin{aligned}
S_{a_3,a_2} &= S^{-1}_{a_2a_3,a_2}, & S_{a_4,a_2} &= S^{-1}_{a_2a_4,a_2},\\
S_{a_4,a_3} &= S^{-1}_{a_3a_4,a_3}, & S_{a_3a_4,a_2} &= S^{-1}_{a_2a_3a_4,a_2},\\
S_{a_2a_4,a_3} &= S^{-1}_{a_2a_3a_4,a_3}.
\end{aligned}$$

Also, they yield the following four relations:

$$\begin{aligned}
S^2_{a_4,a_2} &= 1,\\
S_{a_3a_4,a_2}S_{a_2a_3,a_2}S_{a_3a_4,a_2}S_{a_2a_3,a_2} &= 1,\\
S_{a_3,a_2}S^{-1}_{a_2a_3a_4,a_3}S^{-1}_{a_3a_4,a_2}S^{-1}_{a_4,a_3} &= 1,\\
S_{a_3a_4,a_3}S_{a_4,a_2}S_{a_2a_4,a_3} &= 1.
\end{aligned}$$

Set

$$a = S_{a_3,a_2},\ b = S_{a_4,a_2},\ c = S_{a_4,a_3},\ d = S_{a_3a_4,a_2},\ e = S_{a_2a_4,a_3}.$$

Theorem 3.2. *The group J_4' is generated by*

$$a,\ b,\ c,\ d,\ e,$$

with defining relations

$$b^2 = 1,\ (da^{-1})^2 = 1,\ ae = cd,\ c = be.$$

Eliminating $e = b^{-1}c$ and then $d = c^{-1}ab^{-1}c$ gives the following presentation.

Corollary 3.1. *The commutator subgroup J_4' has presentation*

$$J_4' = \left\langle a, b, c \;\middle|\; b^2 = 1,\ (c^{-1}ab^{-1}ca^{-1})^2 = 1 \right\rangle.$$

Alternatively, we can put $g = da^{-1}$, eliminate $d = ga$, and then eliminate $c = be$.

Corollary 3.2. *The group*

$$J'_4 = \left\langle a, b, g, e \;\middle|\; b^2 = 1,\ g^2 = 1,\ aea^{-1} = beg \right\rangle$$

is an HNN extension with base group

$$G = \left\langle b, g, e \;\middle|\; b^2 = 1,\ g^2 = 1 \right\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z},$$

stable letter a*, associated subgroups*

$$A = \langle e \rangle\,,\ B = \langle beg \rangle$$

and isomorphism $\varphi\colon A \to B$ *defined by* $\varphi(e) = beg$*.*

4. Pure cactus group PJ_4

We now describe two decompositions of the pure cactus group PJ_4.

Theorem 4.1.

(1) *The group* PJ_4 *is an HNN extension of a free group of rank* 4 *with cyclic associated subgroups.*
(2) *The group* PJ_4 *is an amalgamated free product of a free group of rank* 4 *and an infinite cyclic group over a cyclic subgroup.*

Proof. First, by [2], the group PJ_4 has presentation

$$PJ_4 = \left\langle a, b, c, d, t \;\middle|\; t\,(bdad)\,t^{-1} = cb^{-1}a^{-1}c \right\rangle.$$

Magnus's Freiheitssatz implies that the subgroup generated by a, b, c, d is the free group F_4 of rank 4. We may take the following basis of F_4:

$$v = cb^{-1}a^{-1}c, \quad b, \quad c, \quad d.$$

In this basis the defining relation becomes

$$t^{-1}vt = bdcv^{-1}cb^{-1}d.$$

Put

$$A = \langle v \rangle\,,\ B = \langle u \rangle\,,\ u = bdcv^{-1}cb^{-1}d.$$

The map $\varphi\colon A \to B$ given by $\varphi(v) = u$ is an isomorphism. Hence PJ_4 is an HNN extension of $F_4 = \langle v, b, c, d \rangle$ with stable letter t, associated subgroups A and B, and associated isomorphism φ.

Second, as recalled above, PJ_4 has presentation

$$\left\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \;\middle|\; \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 = e \right\rangle.$$

Let

$$X = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle, \; Y = \langle \beta \rangle, \; \beta = \alpha_5^{-1}.$$

By Magnus's Freiheitssatz, X is free of rank 4, and Y is infinite cyclic. Put

$$Z = \langle \beta^2 \rangle = \langle \alpha_1^2 \alpha_2^2 \alpha_3^2 \alpha_4^2 \rangle.$$

The defining relation is $\alpha_1^2 \alpha_2^2 \alpha_3^2 \alpha_4^2 = \beta^2$. Hence the subgroup of X generated by $\alpha_1^2 \alpha_2^2 \alpha_3^2 \alpha_4^2$ is identified with Z, and

$$PJ_4 = X \underset{Z}{*} Y. \qquad \square$$

Appendix A. GAP code for the commutator subgroup of J_4

The following GAP code reproduces the Reidemeister–Schreier computation leading to the presentation of J_4'. It was tested in GAP 4.14.0 and can be run in the command-line terminal of GAP, which is available for all major operating systems at `https://www.gap-system.org/install/`.

```
F := FreeGroup(["a2", "a3", "a4"]);;
AssignGeneratorVariables(F);

# Presentation of J_4 in the generators a2, a3, a4.
J4 := F / [
  a2^2,
  a3^2,
  a4^2,
  (a4*a2)^4,
  a4*a3*a2*a3*a4^-1*a3^-1*a2^-1*a3^-1
];

# The derived subgroup J_4' as a finitely presented group.
D := IsomorphismFpGroup(DerivedSubgroup(J4));

# Images of the generators of J_4'
# in the chosen presentation.
Display(ImagesSource(D));
```

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Regional Scientific
and Educational Mathematical Center
of Tomsk State University
pr. Lenina, 36, Tomsk, 634050, Russia
E-mail: `t.kozlovskaya@math.tsu.ru`

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