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A POSTERIORI ESTIMATES FOR INVERSE PARABOLIC PROBLEMS

Abstract. The paper is concerned with a posteriori error analysis of evolutionary problems for the reaction–convection–diffusion equation, which are incorrect in the sense of Hadamard. They may have no solutions and respective approximations may be highly unstable. For these problems, the concept of a posteriori error estimation used for well-posed problems is not valid. It requires a significant reformulation. A posteriori estimates of the functional type provide the basis for addressing this issue. They generate computable quantities that estimate the distance between a function (approximate solution to the inverse problem) and exact solution of a "close" well-posed initial boundary value problem associated with the same differential equation. The solution of this close problem exactly satisfies one part of the imposed conditions and the others are satisfied approximately with the accuracy ϵ. This idea is elaborated for several evolutionary problems generated by the reaction-convection-diffusion equation: backward parabolic problem, problem with non–local time conditions, Cauchy parabolic problem, and parameter(s) identification problem. The estimates derived not only control the accuracy of approximate solutions, but also suggest suggest a method for calculating them.

§1. Introduction

1.1. A class of inverse problems. Foundations of the theory of inverse problems were laid quite a long time ago. Currently, this theory covers a large number of mathematical models that are not correct in the sense of Hadamard (e.g., see [5,6,8]). In this paper, we are concerned with one class of these problems generated by evolutionary equations of the type

$$u_t + A(t)u = 0. \tag{1.1}$$

Key words and phrases: parabolic equations, ill-posed problems, inverse problems, deviations from exact solution, error identities a posteriori estimates of the functional type.

In [6], the authors study them assuming that the operator A is generated by the bilinear form $a(t; u, v)$ defined on a Hilbert space V, which is $V-$elliptic for all t. Let H be another Hilbert such that $V \subset H$ and V is dense in H. Then for any $\xi \in H$ there exists a unique solution of (1.1) such that $u(0) = \xi$, moreover $u \in L_2(0, T; V)$ and continuously depends on ξ. This problem (which is called "forward" or "direct") is correct according to the definition of Hadamard.

Another (backward) problem is to find ξ such that $u(T, \xi) = \chi(x)$, where $\chi(x) \in H$ is a certain given function. In [6], it is reformulated as an optimal control problem: find $u \in V$ satisfying (1.1) that minimizes the cost functional

$$J(\xi) := \int\limits_{\Omega} |u(x, T; \xi) - \chi(x)|^2 dx. \tag{1.2}$$

For this class of problems,

$$\inf_{\xi \in H} J(\xi) = 0 \tag{1.3}$$

It is well known that backward problems of this type are incorrect and have a solution provided that certain additional conditions are imposed on the function χ (e.g, analyticity). These solutions are highly unstable what generates serious difficulties in quantitative analysis. Hence even if (1.3) is proven, it may be very difficult to construct a reliable approximation of the inverse problem. Known approaches to the construction of approximations to backward parabolic problems (e.g., the "quasi–reversibility method" [6] and nonlocal regularisation method [16]) use regularised problems with small parameter ϵ at the higher order differential operator or in the initial condition. Theoretically, they generate a minimising sequence for the functional (1.2) if $\epsilon \to 0$. However, exact solutions of regularised problems are generally unknown and getting sharp approximations with guaranteed small errors becomes a difficult task. Similar difficulties and analogous questions arise when studying ill-posed problems of other types. The question that rises is how to define the notion "reliable approximation" to an inverse problem.

1.2. A posteriori analysis of inverse and ill-posed problems. Direct and inverse problems are very different, so that the concept of a posteriori error control for them should be formulated differently. Indeed, a well posed problem is uniquely solvable. Let u be the solution and v be an approximation found by some numerical procedure. The question

What neighbourhood of the solution u an approximation v belongs to?

is quite logical. A posteriori estimates of the functional type give a constructive answer to this question for any v in the respective energy space (see [14], where the reader will find a systematic discussion of these estimates for a wide spectrum of problems).

An ill-posed problem may have no solution so that the above posed question is formally incorrect. It clearly needs a modification, which takes into account specific features of inverse problems. For inverse and ill-posed problems the notion of "solution" is often replaced by a "quasi-solution," which may be understood as a solution to some similar (or close) well-posed problem. Therefore, it is quite expected that it is convenient to reformulate the error control concept accordingly. Let v_ϵ be a given function in the admissible functional class viewed as an approximation for an inverse (ill-posed) problem. We wish to know

What neighbourhood of v_ϵ contains exact solution of a close well posed problem?

Below we suggest a method able to answer this question for a class of inverse parabolic problems. It defines ϵ–neighbourhood of v_ϵ that contains the solution of a direct (well–posed) problem (where ϵ is explicitly known). This solution exactly satisfies one part of the conditions while the other(s) are satisfied approximately with the accuracy ϵ. Thus, we can point out a well-posed problem, whose solution can be considered as an ϵ–solution to the problem in question.

1.3. Illustration of the main idea. The approach discussed in this paper is based on the theory of functional a posteriori estimates that has been earlier developed for direct boundary value problems (see [14, 15] and references cited therein). The main idea can be explained with the paradigm of a simple parabolic problem (1.4)–(1.6).

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = \{1, 2, 3\}$ with Lipschitz boundary Γ. The direct evolutionary problem is to find $u(x,t)$ in the space–time cylinder $Q_T := \Omega \times (0,T)$, $T > 0$, such that

$$u_t - \Delta u = f \qquad \text{in } Q_T, \tag{1.4}$$

$$u(x,t) = 0 \qquad \text{on } S_T := \Gamma \times (0,T), \tag{1.5}$$

$$u(x,0) = \xi(x). \tag{1.6}$$

It is well known that under standard assumptions on f and ξ (e.g., $\int\limits_0^T \|f\|^2_\Omega dt < \infty$ and $\xi \in L_2(\Omega)$) the problem (1.4)–(1.6) is well posed and the function $u(x,t)$ exists and is unique.

Let v and y^* be two functions considered as approximations of u and $p^* := \nabla u$, respectively. Assume that

$$v \in V_0 := \{v \in V := H^1(Q_T) \mid v(x,t) = 0 \text{ on } S_T\}$$

and

$$y^* \in Y^*_{\mathrm{div}} := \{y^* \in Y^*(Q_T) := L_2(\Omega, \mathbb{M}^{d\times d}) \mid \mathrm{div} y^* \in L_2(Q_T)\}.$$

In [13, 15], it was proven that the errors $e := v - u$ and $e^* := y^* - p^*$ meet the following estimate.

Theorem 1. *For any $v \in V_0$, $y^* \in Y^*_{\mathrm{div}}(Q_T)$, and $\beta \in (0,1]$ it holds*

$$\begin{aligned}(1-\beta)\|\nabla e\|^2_{Q_T} + \|e^*\|^2_{Q_T} + \|e(\cdot,T)\|^2_\Omega \\ \leqslant \|e(\cdot,0)\|^2_\Omega + \|\nabla v - y^*\|^2_{Q_T} + \beta^{-1}C^2_\Omega\|\mathbf{r}(v,y^*)\|^2_{Q_T},\end{aligned} \tag{1.7}$$

where

$$\mathbf{r}(y^*, v) := \mathrm{div} y^* + f - v_t$$

and C_Ω is a constant in the Friedrichs inequality for the domain Ω.

Consider now the backward evolutionary problem

$$u_t - \Delta u = f \qquad \text{in } Q_T, \tag{1.8}$$

$$u(x,t) = 0 \qquad \text{on } S_T, \tag{1.9}$$

$$u(x,T) = \chi(x) \in L_2(\Omega). \tag{1.10}$$

Assume that by some method we have found the functions $y^*_\epsilon \in Y^*_{\mathrm{div}}$ and $v_\epsilon \in V_0$ such that $v(x,T) = \chi(x)$ for all $x \in \Omega$ and

$$\Upsilon_\beta(v_\epsilon, y^*_\epsilon) \leqslant \epsilon^2, \tag{1.11}$$

where

$$\Upsilon_\beta(v, y^*) := \|\nabla v - y^*\|^2_{Q_T} + \beta^{-1}C^2_\Omega\|\mathbf{r}(v,y^*)\|^2_{Q_T}.$$

For $\xi(x) = v_\epsilon(x,0)$ the problem (1.4)–(1.6) has a solution $u_\epsilon \in V_0$. We can view v_ϵ and y^*_ϵ as approximations of u_ϵ and $p^*_\epsilon := \nabla u_\epsilon$, respectively. By (1.7) we estimate the distance between $(v_\epsilon, y^*_\epsilon)$ and $(u_\epsilon, p^*_\epsilon)$. We use (1.11) and notice that

$$e(\cdot,0) = v_\epsilon(\cdot,0) - u_\epsilon(\cdot,0) = 0.$$

Therefore, (1.7) reads as follows:

$$(1-\beta)\|\nabla(v_\epsilon - u_\epsilon)\|^2_{Q_T} + \|y^*_\epsilon - \nabla u_\epsilon\|^2_{Q_T} + \|u_\epsilon(\cdot,T)-\chi\|^2_\Omega \leqslant \epsilon^2. \quad (1.12)$$

In other words, ϵ–neighborhoods of v_ϵ and y^*_ϵ contain the exact solution $u_\epsilon(x,T)$ of a direct evolutionary problem with the initial condition $v_\epsilon(x,0)$ and the corresponding flux p^*_ϵ. Also, we can guarantee that

$$\|u_\epsilon(\cdot,T)-\chi\|_\Omega \leqslant \epsilon. \quad (1.13)$$

Hence u_ϵ is ϵ – solution of the problem (1.2)–(1.3). Moreover, direct minimization of the functional $\Upsilon_\beta(v,y^*)$ with respect to its arguments suggests a method of getting successful approximations of the ill-posed problem (this question is briefly discussed later, see Section 4.5).

1.4. Outline of the paper. The paper is organised as follows. In Section 2, we briefly discuss a class of direct initial boundary value problems associated with the reaction–convection–diffusion equation. Theorems 2 and 3 of Section 3 establish basic error relations for the direct evolutionary problems. They are used in the next section devoted to problems with non–standard conditions. We consider backward parabolic problem, problem with non–local time conditions, initial–boundary value problem with data given on a part of the boundary, and parameter identification problem. These evolutionary problems originate from various real life models in experimental sciences and engineering. Using functional relations derived in Section 3, we deduce a posteriori estimates for these generally ill-posed problems. The corresponding results are presented in Theorems 4–6. Using them we define a direct evolutionary problem that can be viewed as an ϵ-solution to the ill-posed problem under consideration.

§2. A class of initial boundary value problems

Henceforth, we assume that Γ is disjointly decomposed into a Dirichlet part Γ_1 and a Neumann part Γ_2. Consider the problem

$$u_t - \mathrm{div} p^* + \mathrm{a}\cdot\nabla u + \rho^2 u = f \text{ in } Q_T, \quad (2.1)$$

$$p^* = A\nabla u, \quad (2.2)$$

$$u(x,t) = 0 \qquad \text{on } S_{1T} := \Gamma_1 \times (0,T), \quad (2.3)$$

$$p^*\cdot n = F \qquad \text{on } S_{2T} := \Gamma_2 \times (0,T), \quad (2.4)$$

$$u(x,0) = \xi. \quad (2.5)$$

Here, u_t is the time derivative of $u(x,t)$, n is the unit outward normal to Γ, $\rho(x)$ and $\mathrm{a}(x)$ are the reaction and convection parameters, respectively, which satisfy the following conditions:

$$\mathrm{a} \in L_\infty(\Omega, \mathbb{R}^d), \ \mathrm{diva} \in L_\infty(\Omega), \tag{2.6}$$

$$\rho \in L_\infty(\Omega), \quad 0 \leqslant \rho_\ominus \leqslant \rho \leqslant \rho_\oplus, \tag{2.7}$$

$$\varkappa^2 := -\frac{1}{2}\mathrm{diva} + \rho^2 \geqslant 0. \tag{2.8}$$

It is assumed that the boundary Γ is Lipschitz and piecewise smooth, so that the unit outward normal vector n is uniquely defined at almost all points of Γ. Also, we assume that the inflow part of the boundary where $(\mathrm{a} \cdot n)(x) < 0$ is a subset of Γ_1, i.e.,

$$\delta := \mathrm{a} \cdot n \geqslant 0 \quad \text{on } \Gamma_2. \tag{2.9}$$

The problem has two important special cases: the reaction-diffusion ($\mathrm{a} = 0$) and convection-diffusion ($\rho = 0$) problems.

Coefficients of the symmetric matrix A are bounded, do not depend on t, and satisfy the condition

$$c_1^2|\xi|^2 \leqslant A\xi \cdot \xi \leqslant c_2^2|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d, \ c_1 > 0. \tag{2.10}$$

Also, we assume that [1]

$$f \ \in \ L_2(Q_T), \qquad \xi \in \overset{\circ}{W}{}^1_2(\Omega), \qquad F \in L_2(S_{2T}). \tag{2.11}$$

Throughout the paper, we use standard notation $L_p(\Omega)$ and $W^l_p(\Omega)$ (where $l, p \geqslant 1$) for the Lebesgue and Sobolev spaces, respectively. A space is marked above by $\circ$ if the respective functions vanish on S_{1T}. By $\|\cdot\|_\Omega$ we denote the L_2 norm of a function in Ω and $\|\cdot\|_{Q_T}$ denotes analogous norm in the space–time cylinder. Equivalent seminorms generated by A and A^{-1} are denoted by $\|\cdot\|_{A,\Omega}$ and $\|\cdot\|_{A^{-1},\Omega}$, respectively, i.e.,

$$\|w\|_{A,\Omega} := \left(\int\limits_\Omega A\nabla w \cdot \nabla w dx \right)^{1/2}$$

[1] The method and corresponding results presented in the paper can be extended to the problems where A, a, and ρ are bounded functions of time. Here we avoid these generalisations, which are not essential in the context of the main result presented in Section 4.

and

$$\|w\|_{A^{-1},\Omega} := \left(\int\limits_{\Omega} A^{-1}\nabla w \cdot \nabla w dx \right)^{1/2} .$$

The notation $\|\cdot\|_{A,Q_T}$ and $\|\cdot\|_{A^{-1},Q_T}$ is used for functions in Q_T.

In what follows, we use the spaces

$$W_2^{1,0}(Q_T) := L_2(0,T;W_2^1(\Omega)) \text{ and } \overset{\circ}{W}{}_2^{1,0}(Q_T) := L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega))$$

supplied with the norm

$$\|w\|_{1,Q_T} := \left(\int\limits_0^T (\|\nabla w\|_\Omega^2 + \|w\|_\Omega^2)\, dt \right)^{1/2}$$

and the space

$$\overset{\circ}{W}{}_2^{1,1}(Q_T) := \left\{ w \in W_2^1(Q_T),\ w = 0 \text{ on } S_{1T} \right\}. \tag{2.12}$$

For the latter space we also use the abridged notation $V_0(Q_T)$.

For the functions in $\overset{\circ}{W}{}_2^{1,0}(Q_T,\mathbb{R}^d)$, the norm $\|\nabla w\|_{Q_T}$ is equivalent to $\|w\|_{1,Q_T}$. Here, and later on the symbol $:=$ means "equals by definition".

Following [11], by $\overset{\circ}{W}{}_2^{\Delta,1}(Q_T)$ we denote the subspace of $\overset{\circ}{W}{}_2^{1,0}(Q_T,\mathbb{R}^d)$ that consists of the functions such that

$$\|w\|_{\Delta,1,Q_T} := \int\limits_{Q_T} (w^2 + w_t^2 + \mid \nabla w \mid^2 + (\Delta w)^2)\, dx\, dt < +\infty.$$

and the space $V_2(Q_T)$, which is the Banach space of functions from $W_2^{1,0}(Q_T)$ having finite norm

$$|\!|\!| w |\!|\!|^2 := \operatorname{vrai}\max_{t\in(0,T)} \|w(\cdot,t)\|_\Omega^2 + \|\nabla w\|_{2,Q_T}^2.$$

Spatial gradient and divergence are denoted by ∇ and div, respectively. For a function $w \in H^1(\Omega)$ vanising on Γ_1 it holds the Friedrichs inequality. A constant in this inequality is denoted by C_Ω. Then,

$$\|w\|_\Omega \ \leqslant\ C_{A,\Omega}\|\nabla w\|_{A,\Omega}, \tag{2.13}$$

where $C_{A,\Omega} \leqslant c_1^{-1}C_\Omega$.

Also, we use functional spaces associated with vector valued functions (fluxes). They are $Y^*(Q_T) := L_2(Q_T,\mathbb{R}^d)$ and the space

$$Y^*_{\rm div}(Q_T) := \{y^* \in Y^*(Q_T) \mid {\rm div} y^* \in L_2(Q_T),\ y^*\cdot n \in L_2(S_{2T}) \text{ on } S_{2T}\} \tag{2.14}$$

Generalized solution of (2.1)–(2.5) can be defined as the function

$$u \in V^{1,0}(Q_T) := C([0,T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T)$$

that satisfies the integral identity (e.g., see [10, 11])

$$\int\limits_{Q_T} \Big(A\nabla u \cdot \nabla w + (\mathrm{a} \cdot \nabla u) w + \rho^2 u w \Big)\, dx\, dt - \int\limits_{Q_T} u w_t\, dx\, dt$$
$$+ \int\limits_{\Omega} (u(x,T) w(x,T) - u(x,0) w(x,0)) dx$$
$$= \int\limits_{Q_T} f w\, dx\, dt + \int\limits_{S_{2T}} F w ds dt \quad \forall w \in V_0(Q_T) \quad (2.15)$$

for any test function $w \in V_0(Q_T)$. Under the conditions (2.6)–(2.11), the solution is more regular and belongs to $\overset{\circ}{W}{}_2^{\Delta,1}(Q_T)$.

§3. Error identities for the problem (2.1)–(2.5)

In this section, we deduce error relations that hold for the direct initial boundary value problem (2.1)–(2.5). Let $u \in V_0(Q_T)$ and two functions (considered as approximations of $u(x,t)$ and $p^*(x,t) := A\nabla u$) satisfy the conditions,

$$v(x,t) \in V_0(Q_T) \quad \text{and} \quad y^*(x,t) \in Y^*_{\mathrm{div}}(Q_T).$$

In other words, minimal regularity conditions are imposed on the approximations, and no special properties of these functions (e.g., Galerkin orthogonality) are required. The functions

$$e := v - u \quad \text{and} \quad e^* := y^* - p^*$$

are the corresponding errors. The functions

$$\mathbf{r}_f(v, y^*) := f - v_t + \mathrm{div} y^* - \mathrm{a} \cdot \nabla v - \rho^2 v, \tag{3.1}$$
$$\mathbf{r}_F(y^*) = F - y^* \cdot n, \tag{3.2}$$
$$\mathbf{r}_A(v, y^*) := A\nabla v - y^*. \tag{3.3}$$

contain known functions v and y^* and known data f, F, A, a, and ρ. Therefore, they are directly computable and represent residuals of the relations (2.1), (2.2), and (2.4).

3.1. The main error identity. The quantity $\boldsymbol{\mu}(e, e^*)$ defined by the relation

$$\boldsymbol{\mu}^2(e, e^*) := \|\nabla e\|^2_{A,Q_T} + \|e^*\|^2_{A^{-1},Q_T} + \|e(\cdot, T)\|^2_\Omega$$

is a natural measure of the distance between u and p^* and respective approximations v and y^*. First, we establish identity (3.4), where $\boldsymbol{\mu}(e, e^*)$ (plus some nonnegative terms) form the left hand side and the quantities $\mathbf{r}_f(v, y^*)$ $\mathbf{r}_F(v, y^*)$, and $\mathbf{r}_A(v, y^*)$ appear at the right one. This identity plays a key role in further analysis.

Theorem 2. *For any $v \in V_0(Q_T)$ and $y^* \in Y^*_{\mathrm{div}}(Q_T)$, it holds*

$$\begin{aligned}\boldsymbol{\mu}^2(e, e^*) + 2\|\varkappa e\|^2_{Q_T} &+ \int\limits_{S_{2T}} (\mathrm{a} \cdot n) e^2 dxdt \\ &= \|v(\cdot, 0) - \xi\|^2_\Omega + \|\mathbf{r}_A(v, y^*)\|^2_{A^{-1},Q_T} - 2 \\ &\quad \times \int\limits_{Q_T} \mathbf{r}_f(v, y^*) e \, dxdt - 2 \int\limits_{S_{2T}} \mathbf{r}_F(y^*) e dxdt. \end{aligned} \tag{3.4}$$

Proof. We have

$$\begin{aligned}\|\mathbf{r}_A(v, y^*)\|^2_{A^{-1},Q_T} &= \|A\nabla e - e^*\|^2_{A^{-1},Q_T} \\ &= \|\nabla e\|^2_{A,Q_T} + \|e^*\|^2_{A^{-1},Q_T} - 2 \int\limits_{Q_T} e^* \cdot \nabla e dxdt \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} 2\int\limits_{Q_T} \mathbf{r}_f(v, y^*) e \, dxdt &= 2 \int\limits_{Q_T} (f - v_t + \mathrm{div} y^* - \mathrm{a} \cdot \nabla v - \rho^2 v) e \, dxdt \\ &= 2 \int\limits_{Q_T} (\mathrm{div} e^* - e_t - \mathrm{a} \cdot \nabla e - \rho^2 e) \, e \, dxdt. \end{aligned} \tag{3.6}$$

By the relations

$$\int\limits_{Q_T} e_t e \, dx \, dt = \frac{1}{2} \int\limits_\Omega (e^2(x, T) - e^2(x, 0)) dx \tag{3.7}$$

and

$$\int_{\Omega}(\mathrm{a}\cdot\nabla e)e dx=\frac{1}{2}\int_{\Omega}\mathrm{a}\cdot\nabla(e^2)dx \\ =\frac{1}{2}\int_{S_{2T}}(\mathrm{a}\cdot n)e^2dx-\frac{1}{2}\int_{\Omega}\mathrm{div}\mathrm{a}\, e^2ds, \tag{3.8}$$

we rewrite (3.6) as follows:

$$2\int_{Q_T}\mathbf{r}_f(v,y^*)e\;dxdt=2\int_{Q_T}(e\mathrm{div}e^*-(\rho^2-\frac{1}{2}\mathrm{div}a)e^2)dxdt \\ -\int_{S_{2T}}(\mathrm{a}\cdot n)e^2dsdt+\int_{\Omega}\big(e^2(x,0)-e^2(x,T)\big)dx. \tag{3.9}$$

Now (3.5) and (3.9) imply

$$\|\mathbf{r}_A(v,y^*)\|^2_{A^{-1},Q_T}-2\int_{Q_T}\mathbf{r}_f(v,y^*)e\;dxdt \\ =\|\nabla e\|^2_{A,Q_T}+\|e^*\|^2_{A^{-1},Q_T}+\int_{\Omega}e^2(x,T)dx-\|v(\cdot,0)-\xi\|^2_{\Omega} \\ +\int_{S_{2T}}(\mathrm{a}\cdot n)e^2dx+2\int_{Q_T}\varkappa^2e^2dxdt-2\int_{Q_T}(e\mathrm{div}e^*+e^*\cdot\nabla e)dxdt \tag{3.10}$$

Notice that

$$\int_{\Omega}(e\mathrm{div}e^*+e^*\cdot\nabla e)dx=\int_{S_{2T}}(e^*\cdot n)eds \tag{3.11}$$

and

$$e^*\cdot n=(y^*\cdot n-F)=-\mathbf{r}_F(y^*). \tag{3.12}$$

In view of (3.11) and (3.12), (3.10) takes the form

$$\|\mathbf{r}_A(v,y^*)\|^2_{A^{-1},Q_T}-2\int_{Q_T}\mathbf{r}_f(v,y^*)\,e\,dxdt+\|v(\cdot,0)-\xi\|^2_{\Omega} \\ =\boldsymbol{\mu}^2(e,e^*)+2\|\varkappa e\|^2_{Q_T}+\int_{S_{2T}}(\mathrm{a}\cdot n)e^2dxdt+2\int_{S_{2T}}\mathbf{r}_f(y^*)\,e\,dsdt,$$

which implies (3.14). □

Remark 1. *If* $\Gamma_2 = \emptyset$, *A is the unit matrix,* $\mathrm{a} = 0$, *and* $\rho = 0$, *then* $\varkappa = 0$ *and* (3.4) *reads*

$$\|\nabla e\|^2_{Q_T} + \|e^*\|^2_{Q_T} + \|e(\cdot, T)\|^2_\Omega \\ = \|e(\cdot, 0)\|^2_\Omega + \|\nabla v - y^*\|^2_{Q_T} - 2\int\limits_\Omega (f - v_t + \mathrm{div} y^*) e dx dt. \quad (3.13)$$

This identity was earlier derived in [15]. The estimate (1.7) *follows from* (3.13).

Right hand side of (3.4) is not fully computable. It contains two directly computable norms, but two other terms (integrals over Q_T and S_{2T}) involve unknown function e. However, these integrals can be subject to the norms that form the left side of (3.4). Hence the identity generates fully computable estimates. We deduce them at the end of this section and now consider some special cases.

3.2. The case $\rho > 0$.

Theorem 3. *Let the conditions* (2.6)–(2.11) *be satisfied,* $\rho_\ominus > 0$ (*cf.* (2.7)), *and*

$$\rho_a^2 := \rho^2 - \mathrm{div}\mathrm{a} \geqslant 0.$$

Then, for any $v \in V_0(Q_T)$ *and* $y^* \in Y^*_{\mathrm{div}}(Q_T)$ *it holds*

$$\boldsymbol{\mu}^2(e, e^*) + \|\rho^{-1}\sigma(e^*, e)\|^2_{Q_T} + \int\limits_{Q_T} \rho_a^2 |e|^2 dx dt + \int\limits_{S_{2T}} (\mathrm{a}\cdot n) e^2 ds dt. \\ = \|v(\cdot, 0) - \xi\|^2_\Omega + \|\mathbf{r}_A(v, y^*)\|^2_{A^{-1}, Q_T} \quad (3.14) \\ + \|\rho^{-1}\mathbf{r}_f(v, y^*)\|^2_{Q_T} - 2\int\limits_{S_{2T}} \mathbf{r}_F(y^*)\, e ds dt,$$

where $\sigma(e^*, e) := \mathrm{div} e^* - e_t - \mathrm{a}\cdot\nabla e$.

If, in addition, $\delta > 0$ *on* S_{2T}, *then*

$$\begin{aligned}\boldsymbol{\mu}^2(e,e^*) &+ \|\rho^{-1}\sigma(e^*,e)\|^2_{Q_T} + \int\limits_{Q_T}\rho_a^2|e|^2dxdt \\ &+ \int\limits_{S_{2T}}\delta^{-1}(e^*\cdot n - \delta e)^2dsdt = \|v(\cdot,0)-\xi\|^2_\Omega \\ &+ \|\mathbf{r}_A(v,y^*)\|^2_{A^{-1},Q_T} + \|\rho^{-1}\mathbf{r}_f(v,y^*)\|^2_{Q_T} \\ &+ \int\limits_{S_{2T}}\delta^{-1}\mathbf{r}_F^2(y^*)dsdt.\end{aligned} \tag{3.15}$$

Proof. We have

$$\begin{aligned}\|\rho^{-1}\mathbf{r}_f(v,y^*)\|^2_{Q_T} &= \|\rho^{-1}(\mathrm{div}y^* + f - v_t - \mathrm{a}\cdot\nabla v - \rho^2 v)\|^2_{Q_T} \\ &= \|\rho^{-1}(\mathrm{div}e^* - e_t - \mathrm{a}\cdot\nabla e - \rho^2 e)\|^2_{Q_T} = \|\rho^{-1}\sigma(e,e^*) - \rho e\|^2_{Q_T} \\ &= \|\rho^{-1}\sigma(e^*,e)\|^2_{Q_T} + \|\rho e\|^2_{Q_T} - 2\int\limits_{Q_T}\sigma(e^*,e)\,e\,dxdt.\end{aligned} \tag{3.16}$$

By (3.7), (3.8), and the relation

$$\int\limits_{Q_T}(\mathrm{div}e^*)edxdt = -\int\limits_{Q_T}e^*\cdot\nabla edxdt + \int\limits_{S_{2T}}(e^*\cdot n)\,e\,dsdt,$$

we find that

$$\begin{aligned}2\int\limits_{Q_T}\sigma(e^*,e)edxdt &= -2\int\limits_{Q_T}e^*\cdot\nabla e\,dxdt + 2\int\limits_{S_{2T}}(e^*\cdot n)\,edsdt \\ &- \|e(x,T)\|^2_\Omega + \|e(x,0)\|^2_\Omega - \int\limits_{S_{2T}}(\mathrm{a}\cdot n)e^2dx + \int\limits_\Omega(\mathrm{div}\mathrm{a})\,e^2ds\end{aligned}$$

and modify (3.16) as follows:

$$\begin{aligned}\|\rho^{-1}\mathbf{r}_f(v,y^*)\|^2_{Q_T} &= \|\rho^{-1}\sigma(e^*,e)\|^2_{Q_T} + \|\rho e\|^2_{Q_T} + \|e(\cdot,T)\|^2_\Omega \\ &- \|e(\cdot,0)\|^2_\Omega + 2\int\limits_{Q_T}e^*\cdot\nabla edxdt - \int\limits_{Q_T}(\mathrm{div}\mathrm{a})e^2dxdt \\ &- 2\int\limits_{S_{2T}}(e^*\cdot n)\,edsdt + \int\limits_{S_{2T}}(\mathrm{a}\cdot n)\,e^2dsdt.\end{aligned} \tag{3.17}$$

Adding (3.5) and (3.17) we get the identity

$$\begin{aligned}\boldsymbol{\mu}^2(e,e^*) + \|\rho^{-1}\sigma(e^*,e)\|^2_{Q_T} + \int\limits_{S_{2T}} (\mathrm{a}\cdot n)e^2 dsdt \\ + \int\limits_{Q_T} (\rho^2 - \mathrm{diva})|e|^2 dxdt = \|v(x,0)-\xi\|^2_\Omega \\ + \|\rho^{-1}\mathbf{r}_f(v,y^*)\|^2_{Q_T} + \|\mathbf{r}_A(v,y^*)\|^2_{A^{-1}} \\ + 2\int\limits_{S_{2T}} (e^*\cdot n)\, edsdt.\end{aligned} \tag{3.18}$$

We recall (3.12) and see that (3.14) follows from (3.18).

If $\delta = \mathrm{a}\cdot n > 0$ on S_{2T}, then we can transform (3.14) using the identity

$$\int\limits_{S_{2T}} \left(\delta^{-1}\mathbf{r}_F^2(y^*) + 2\mathbf{r}_F(y^*)e + \delta\, e^2\right) dsdt = \int\limits_{S_{2T}} \delta^{-1}(e^*\cdot n - \delta e)^2 dsdt,$$

Then,

$$\int\limits_{S_{2T}} \delta\, e^2 dsdt + 2\int\limits_{S_{2T}} \mathbf{r}_F(y^*)\, e\, dsdt = \int\limits_{S_{2T}} \Big(\delta^{-1}(e^*\cdot n + \delta e)^2 - \delta^{-1}\mathbf{r}_F^2(y^*)\Big) dsdt$$

and (3.15) follows from (3.14). □

3.3. The case $\mathbf{r}_F(y^*) = 0$. Now we discuss corollaries of Theorems 2 and 3 related to the case where the term containing $\mathbf{r}_F(y^*)$ drops out. This happens if

$$\Gamma_2 = \emptyset \quad \text{or} \quad y^* \in Y_F^*(Q_T) := \{y^* \in Y^*_{\mathrm{div}}(Q_T) \mid y^*\cdot n = F \text{ on } S_{2T}\}. \tag{3.19}$$

Since y^* is at our disposal it is not difficult to correct this function so that the above condition on S_{2T} is satisfied. As a result, we obtain identities with completely computable right hand sides.

Let the conditions of Theorem 3 be satisfied and (3.19) holds. Then the last term of (3.14) vanishes and we arrive at the identity

$$\begin{aligned}\boldsymbol{\mu}^2(e,e^*) + \|\rho^{-1}\sigma(e^*,e)\|^2_{Q_T} + \int\limits_{Q_T} \rho_a^2|e|^2 dxdt + \int\limits_{S_{2T}} (\mathrm{a}\cdot n)e^2 dx \\ = \|v(\cdot,0)-\xi\|^2_\Omega + \|\mathbf{r}_A(v,y^*)\|^2_{A^{-1},Q_T} + \|\rho^{-1}\mathbf{r}_f(v,y^*)\|^2_{Q_T}\end{aligned} \tag{3.20}$$

whose right hand side contains only known functions.

Let (3.19) holds and $\varkappa > 0$. In this case, we use Theorem 2 and rewrite (3.4) in the form

$$\begin{aligned}\boldsymbol{\mu}^2(e, e^*) + \|\varkappa e\|^2_{Q_T} + \int\limits_{Q_T} (\varkappa e + \varkappa^{-1}\mathbf{r}_f(v, y^*))^2 dxdt \\ = \|v(\cdot, 0) - \xi\|^2_\Omega + \|\mathbf{r}_A(v, y^*)\|^2_{A^{-1},Q_T} + \|\varkappa^{-1}\mathbf{r}_f(v, y^*)\|^2_{Q_T}. \quad (3.21)\end{aligned}$$

Hence in this case, we obtain another error identity with computable right hand side.

3.4. Error estimates. Both parts of (3.14) and (3.20) contain integrals with weights ρ^{-1}. Therefore, for $\rho = 0$ they are not applicable. For very small ρ these theoretically correct relations may become practically inefficient. Similar difficulties arise in (3.15) and (3.21) for very small δ and $\varkappa$, respectively. In these cases, it is necessary to use other error relations (estimates) that follow from the basic identity (3.4). There are several ways to deduce computable a posteriori estimates from (3.4). Consider the simplest one.

Let (3.19) hold. Then the last integral in the right hand side of (3.4) vanishes. The remaining integral is estimated as follows:

$$\begin{aligned}2\left|\int\limits_{Q_T} \mathbf{r}_f(v, y^*)e\, dxdt\right| &\leqslant 2\int\limits_0^T \|\mathbf{r}_f(v, y^*)\|_\Omega \|e\|_\Omega\, dt \\ &\leqslant 2\frac{C_\Omega}{c_1}\int\limits_0^T \|\mathbf{r}_f(v, y^*)\|_\Omega \|\nabla e\|_{A,\Omega}\, dt \\ &\leqslant \int\limits_0^T \Big(\frac{C^2_{A,\Omega}}{\beta^2(t)}\|\mathbf{r}_f(v, y^*)\|^2_\Omega + \beta^2(t)\|\nabla e\|_{A,Q_T}\Big)dt,\end{aligned}$$

where $\beta(t)$ is a positive bounded function and $C_{A,\Omega}$ is a constant in (2.13). Then, for any $\beta(t) \in (0, 1]$ the estimate

$$\int_0^T (1-\beta^2(t))\|\nabla e\|^2_{A,\Omega}dt + \|e^*\|^2_{A^{-1},Q_T} + \|v(\cdot,T)-u(\cdot,T)\|^2_\Omega$$
$$+ \int_{S_{2T}} \delta\, e^2 dx \leqslant \|v(\cdot,0)-\xi\|^2_\Omega + \|\mathbf{r}_A(v,y^*)\|^2_{A^{-1},Q_T} \tag{3.22}$$
$$+ \int_0^T \frac{C^2_{A,\Omega}}{\beta^2(t)}\|\mathbf{r}_f(v,y^*)\|^2_{A,\Omega}dt$$

follows from (3.4). Here (unlike (3.20) and (3.21)), we arrive at the estimate instead of the identity. It is valid for small (or even zero) values of ρ and $\varkappa$ and provides an upper bound of the error measure. A lower bound is derived by the same simple procedure. It reads

$$\int_0^T (1+\beta^2(t))\|\nabla e\|^2_{A,\Omega}dt + \|e^*\|^2_{A^{-1},Q_T} + \|v(\cdot,T)-u(\cdot,T)\|^2_\Omega$$
$$+ \int_{S_{2T}} \delta\, e^2 dx \geqslant \|v(\cdot,0)-\xi\|^2_\Omega + \|\mathbf{r}_A(v,y^*)\|^2_{A^{-1},Q_T} \tag{3.23}$$
$$- \int_0^T \frac{C^2_{A,\Omega}}{\beta^2(t)}\|\mathbf{r}_f(v,y^*)\|^2_{A,\Omega}dt.$$

Certainly this bound is meaningful only if the right hand side of (3.23) is positive.

§4. Applications to analysis of inverse problems

Above derived estimates can be applied not only to the problem (2.1)–(2.5). In this section, we consider several problems based on the equation (2.1) with different conditions imposed on solution and problem data.

As in (1.12)–(1.13), by $v_\epsilon \in V_0(Q_T)$ (cf. (2.12)) and $y^*_\epsilon \in Y^*_{\rm div}(Q_T)$ (cf. (2.14)) we denote functions that satisfy all the required conditions (initial, boundary, and others). However, they may not exactly satisfy (2.1) and (2.2). These functions are considered as approximations (e.g., they may be computed by some numerical procedure).

In short, the main result is as follows. In all the cases considerd, Theorems 2 and 3 generate functionals (e.g., (4.5)) that control the distance between $(v_\epsilon, y^*_\epsilon)$ and a pair $(u_\epsilon, p^*_\epsilon)$ formed by exact solutions of a direct boundary value problem. If the functional is smaller that ϵ^2, then we guarantee that $(u_\epsilon, p^*_\epsilon)$ exactly satisfy one part of the required conditions while the others hold with the accuracy ϵ. Thus, we find a well-posed problem, the solution of which can be considered as an ϵ- approximation for the ill-posed one.

4.1. Backward problem. First, we consider the problem: find u and p^* such that

$$u_t - \mathrm{div} p^* + \mathrm{a} \cdot \nabla u + \rho^2 u = f \text{ in } Q_T, \tag{4.1}$$

$$p^* = A\nabla u, \tag{4.2}$$

$$u(x,t) = 0 \qquad \text{on } S_{1T}, \qquad p^* \cdot n = F \text{ on } S_{2T}, \tag{4.3}$$

$$u(x,T) = \chi(x) \in L_2(\Omega). \tag{4.4}$$

Problem (1.8)–(1.10) is a simple particular case of this backward parabolic problem. For $\beta \in (0,1]$ and $\gamma > 0$, we define the functional

$$\begin{aligned} \Upsilon_{\beta,\gamma}(v,y^*) := \|\mathbf{r}_A(v,y^*)\|^2_{A^{-1},Q_T} \\ + \int\limits_0^T \frac{C^2_{A,\Omega}}{\beta^2(t)} \|\mathbf{r}_f(v,y^*)\|^2_{A,\Omega} dt + \frac{1}{\gamma}\|v(\cdot,T) - \chi\|^2_\Omega. \end{aligned} \tag{4.5}$$

Theorem 4. *Let $v_\epsilon \in V_0(Q_T)$ and $y^*_\epsilon \in Y^*_{\mathrm{div}}(Q_T)$ satisfy the condition $y^*_\epsilon \cdot n = F$ on Γ_{2T}, and*

$$\Upsilon_{\beta,\gamma}(v_\epsilon, y^*_\epsilon) \leqslant \epsilon^2. \tag{4.6}$$

Then, problem (2.1)–(2.6) with the initial condition $\xi = v_\epsilon(x,0)$ has the solution u_ϵ such that

$$\|u_\epsilon(x,T) - \chi(x)\|_\Omega \leqslant \epsilon(1+\gamma)^{1/2}. \tag{4.7}$$

The solution u_ϵ and its flux p^*_ϵ belong to ϵ-neighbourhoods of v_ϵ and y^*_ϵ, respectively, i.e.,

$$\|p^*_\epsilon - y^*_\epsilon\|_{Q_T} \leqslant \epsilon, \tag{4.8}$$

and

$$\int\limits_0^T (1-\beta^2)\|\nabla(u_\epsilon - v_\epsilon)\|^2_{A,\Omega} dt + \int\limits_{S_{2T}} \delta |u_\epsilon - v_\epsilon|^2 dx \leqslant \epsilon^2. \tag{4.9}$$

Proof. We apply (3.22) to the correct initial boundary value problem (2.1)–(2.6) with the initial condition $\xi = v_\epsilon(x,0)$ and consider $v = v_\epsilon$ and $y^* = y^*_\epsilon$ as approximations of the corresponding exact functions u_ϵ and $p^*_\epsilon := A\nabla u_\epsilon$. We have

$$\int\limits_0^T (1-\beta^2(t))\|\nabla(v_\epsilon - u_\epsilon)\|^2_{A,\Omega} dt + \|y^*_\epsilon - p^*_\epsilon\|^2_{A^{-1},Q_T}$$
$$+ \|v_\epsilon(\cdot,T) - u_\epsilon(\cdot,T)\|^2_\Omega + \frac{1}{\gamma}\|v_\epsilon(\cdot,T) - \chi\|^2_\Omega$$
$$+ \int\limits_{S_{2T}} \delta\,|v_\epsilon - u_\epsilon|^2 dx \leqslant \Upsilon_{\beta,\gamma}(v_\epsilon, y^*_\epsilon) \leqslant \epsilon^2.$$

It is easy to see that

$$\|v_\epsilon(\cdot,T) - u_\epsilon(\cdot,T)\|^2_\Omega \geqslant \frac{1}{1+\gamma}\|u_\epsilon(\cdot,T) - \chi\|^2_\Omega - \frac{1}{\gamma}\|v_\epsilon(\cdot,T) - \chi\|^2_\Omega$$

for any $\gamma > 0$. Therefore,

$$\int\limits_0^T \Big((1-\beta^2(t))\|\nabla(v_\epsilon - u_\epsilon)\|^2_{A,\Omega} + \|y^*_\epsilon - p^*_\epsilon\|^2_{A^{-1},\Omega}\Big) dt$$
$$+ \int\limits_{S_{2T}} \delta|v_\epsilon - u_\epsilon|^2 dx + \frac{1}{1+\gamma}\|u_\epsilon(\cdot,T) - \chi\|^2_\Omega \leqslant \epsilon^2, \quad (4.10)$$

wherefrom the estimates (4.7), (4.8), and (4.9) follow. □

Remark 2. *Problem* (1.8)–(1.10) *is a very special case of* (4.1)–(4.4)*. It is easy to see that* (1.12) *follows from* (4.10) *if* $\mathrm{a} = 0$*,* $\rho = 0$*,* $S_{2T} = \emptyset$*, and* A *is the unit matrix. If the condition* $v_\epsilon(x,T) = \chi(x)$ *is exactly satisfied, then the last term in* (4.5) *vanishes and we can set* $\gamma = 0$*. Then* (4.5) *is reduced to* $\Upsilon_\beta(u_\epsilon, y^*_\epsilon)$ *and* (4.6) *to* (1.11)*.*

If $\rho > 0$ or $\varkappa > 0$, then modified versions of Theorem 4 follow from (3.20) and (3.21). Assume that $\rho > 0$, (3.19) holds, and

$$\Upsilon_\rho(v_\epsilon, y^*_\epsilon) := \|\mathbf{r}_A(v_\epsilon, y^*_\epsilon)\|^2_{A^{-1},Q_T} + \|\rho^{-1}\mathbf{r}_f(v_\epsilon, y^*_\epsilon)\|^2_{Q_T} \leqslant \epsilon^2.$$

for $v_\epsilon \in V_0(Q_T)$ and $y^*_\epsilon \in Y^*_{\mathrm{div}}(Q_T)$ and, in addition, $v_\epsilon(x,T) = \chi(x)$.

By (3.20) and quite analogous arguments, we prove that direct problem (2.1)–(2.6) with $\xi(x) = v_\epsilon(x,0)$ has the solution $(u_\epsilon, p^*_\epsilon)$ and

$$\boldsymbol{\mu}^2(e,e^*) + \|\rho^{-1}\sigma(e^*,e)\|^2_{Q_T} + \int\limits_{Q_T} \rho_a^2 |e|^2 dxdt + \int\limits_{S_{2T}} \delta\, e^2 dsdt \ \leqslant\ \epsilon^2, \quad (4.11)$$

where $e = v_\epsilon - u_\epsilon$ and $e^* = y^*_\epsilon - p^*_\epsilon$. Hence

$$\|\nabla(u_\epsilon - v_\epsilon)\|_{Q_T} \leqslant \epsilon,\ \|p^*_\epsilon - y^*_\epsilon\|_{Q_T} \leqslant \epsilon, \text{ and } \|u_\epsilon(x,T) - \chi(x)\|_\Omega \leqslant \epsilon. \quad (4.12)$$

We conclude that ϵ–neighbourhood of $(v_\epsilon, y^*_\epsilon)$ contains exact solution of a correct initial boundary–value problem with the initial condition $\xi = v_\epsilon(x,0)$.

Similar reasoning can be used if $\varkappa > 0$. Let

$$v_\epsilon \in V_0(Q_T),\ \ v_\epsilon(x,T) = \chi(x),\ \ \text{and}\ \ y^*_\epsilon \in Y^*_{\rm div}(Q_T)$$

be such that

$$\Upsilon_\varkappa(v_\epsilon, y^*_\epsilon) := \|\mathbf{r}_A(v_\epsilon, y^*_\epsilon)\|^2_{A^{-1},Q_T} + \|\varkappa^{-1}\mathbf{r}_f(v_\epsilon, y^*_\epsilon)\|^2_{Q_T} \ \leqslant\ \epsilon^2.$$

Then, by (3.21) we conclude that

$$\boldsymbol{\mu}^2(e,e^*) + \|\varkappa e\|^2_{Q_T} \ \leqslant \epsilon^2 \quad (4.13)$$

and, therefore, the estimates (4.12) hold.

Remark 3. *Estimate* (4.10) *shows that the sum of the four nonnegative terms is less than* ϵ^2. *The quantity* $\|\chi(x) - u_\epsilon(x,T)\|^2_\Omega$ *is only one of these terms. Therefore, we can await that the norm* $\|\chi(x) - u_\epsilon(x,T)\|_\Omega$ *is essentially smaller than* ϵ. *Analogous conclusions follow from* (4.11) *and* (4.13). *In other words, it is very likely that* u_ϵ *approximates the desired condition at* $t = T$ *much better. Since* $u_\epsilon(x,0)$ *is known, we may try to find a sharp approximation of* u_ϵ. *Stable and efficient numerical methods for solving direct initial boundary value problems are well developed* (*e.g. see [17]*). *Using them in combination with respective a posteriori error estimation methods* (*discussed in Sect.* 3) *suggests a way to get efficient and controllable approximations for the backward problem* (4.1)–(4.4) *with the much better accuracy than it follows from* (4.12).

4.2. Problem with nonlocal conditions. Consider the problem

$$u_t - \mathrm{div} p^* + \mathrm{a} \cdot \nabla u + \rho^2 u = f \text{ in } Q_T, \tag{4.14}$$
$$p^* = A\nabla u, \tag{4.15}$$
$$u(x,t) = 0 \qquad \text{on } S_T, \tag{4.16}$$
$$u(x,0) + \omega\, u(x,T) = 0, \tag{4.17}$$

where $\omega \in [-1, 0)$ is a given function. Existence and uniqueness of such type problems have been studied (e.g., see [2,9]).

Assume that we have the function $v_\epsilon \in V_0(Q_T)$ satisfing the condition

$$v_\epsilon(x,0) + \omega\, v_\epsilon(x,T) = 0 \tag{4.18}$$

and $y^*_\epsilon \in Y^*_{\mathrm{div}}(Q_T)$ such that for $\beta(t) \in (0,1]$

$$\Upsilon_\beta(v_\epsilon, y^*_\epsilon) := \|\mathbf{r}_A(v_\epsilon, y^*_\epsilon)\|^2_{A^{-1},Q_T} + \int\limits_0^T \frac{C^2_{A,\Omega}}{\beta^2(t)} \|\mathbf{r}_f(v_\epsilon, y^*_\epsilon)\|^2_{A,\Omega} dt \leqslant \epsilon^2. \tag{4.19}$$

Let $u_\epsilon(x,t)$ and y^*_ϵ solve the initial boundary value problem

$$(u_\epsilon)_t - \mathrm{div} p^*_\epsilon + \mathrm{a} \cdot \nabla u_\epsilon + \rho^2 u_\epsilon = f \text{ in } Q_T, \tag{4.20}$$
$$p^*_\epsilon = A\nabla u_\epsilon, \tag{4.21}$$
$$u_\epsilon(x,t) = 0 \qquad \text{on } S_T, \tag{4.22}$$
$$u_\epsilon(x,0) = v_\epsilon(x,0). \tag{4.23}$$

This is a well posed problem. We use (3.22) with $\xi(x) = v_\epsilon(x,0)$ (note that in our case $S_{2T} = \varnothing$). By (3.22) and (4.19), we find that

$$\int\limits_0^T (1 - \beta^2(t)) \|\nabla(v_\epsilon - u_\epsilon)\|^2_{A,\Omega} dt + \|y^*_\epsilon - p^*_\epsilon\|^2_{A^{-1},Q_T} + \omega^{-1} \|v_\epsilon(x,T) - u_\epsilon(x,T)\|^2 \leqslant \epsilon^2. \tag{4.24}$$

Since

$$\begin{aligned} \|v_\epsilon(x,T) - u_\epsilon(x,T)\|_\Omega &= \|\omega^{-1} v_\epsilon(x,0) + u_\epsilon(x,T)\|_\Omega \\ &= \omega^{-1} \|u_\epsilon(x,0) + \omega u_\epsilon(x,T)\|_\Omega, \end{aligned}$$

from (4.24) we deduce the estimate

$$\int_0^T (1-\beta^2(t))\|\nabla(v_\epsilon - u_\epsilon)\|^2_{A,\Omega} dt + \|y^*_\epsilon - p^*_\epsilon\|^2_{A^{-1},Q_T} + \omega^{-2}\|u_\epsilon(x,0) + \omega u_\epsilon(x,T)\|^2 \leqslant \epsilon^2.$$

Thus, we arrive at the following result.

Theorem 5. *Let* $v_\epsilon \in V_0(Q_T)$ *and* $y^*_\epsilon \in Y^*_{\rm div}(Q_T)$ *satisfy the conditions* (4.18) *and* (4.19). *Then, solution of* (4.20)–(4.23) *meets the estimate*

$$\|u_\epsilon(x,0) + \omega u_\epsilon(x,T)\|_\Omega \leqslant \ \omega\epsilon. \tag{4.25}$$

Moreover, u_ϵ *and* p^*_ϵ *belong to* ϵ*–neighbourhoods of* v_ϵ *and* y^*_ϵ*, respectively, i.e.,*

$$\|p^*_\epsilon - y^*_\epsilon\|_{Q_T} \leqslant \epsilon, \tag{4.26}$$

and

$$\int_0^T (1-\beta^2)\|\nabla(u_\epsilon - v_\epsilon)\|^2_{A,\Omega}\, dt \ \leqslant \ \epsilon. \tag{4.27}$$

Theorem 5 has a clear meaning. In view of (4.26) and (4.27), we guarantee that ϵ-neighbourhoods of v_ϵ and y^*_ϵ contain solutions of a direct problem with the initial condition $\xi = v_\epsilon(x,0)$ such that relations (4.14) – (4.16) are exactly satisfied and (4.25) shows that (4.17) is satisfied approximately with the accuracy $\omega\epsilon$.

4.3. Inverse Cauchy problem. Inverse Cauchy problem imposes an over-specified boundary condition on one part of the boundary while the other part remains free from any conditions. Usually, these problems are considered for elliptic equations but conditions of this type also generate a class of inverse parabolic problems (e.g., see Chapt. 5 of [6]). Modifications of these problems are studied in connection with various applied problems. At this point we address the reader to [3, 5, 7, 12] and numerous publications cited therein.

As an example, we take the problem

$$\begin{aligned} u_t - \mathrm{div} p^* + \mathrm{a} \cdot \nabla u + \rho^2 u = f & \quad \text{in } Q_T, && (4.28)\\ p^* = A \nabla u, & && (4.29)\\ u(x,t) = 0 & \quad \text{on } S_{2T}, && (4.30)\\ p^* \cdot n = F & \quad \text{on } S_{2T}, && (4.31)\\ u(x,0) = \xi(x). & && (4.32) \end{aligned}$$

Here two boundary conditions are stated on Γ_{2T} and Γ_{1T} is free of conditions.

Theorem 6. *Let $v_\epsilon \in V$ and $y^*_\epsilon \in Y^*_{\mathrm{div}}(Q_T)$ satisfy the conditions*

$$v_\epsilon(x,t) = 0 \text{ on } \Gamma_{2T}, \qquad v_\epsilon(x,0) = \xi(x), \qquad y^*_\epsilon \cdot n = F \text{ on } S_{2T},$$

and

$$\Upsilon_\beta(v_\epsilon, y^*_\epsilon) \leqslant \epsilon^2 \quad \textit{for} \quad \beta(t) \in (0,1]. \tag{4.33}$$

Then,

$$\int_0^T (1-\beta^2) \|\nabla(u_\epsilon - v_\epsilon)\|^2_{A,\Omega} \leqslant \epsilon^2, \quad \|p^*_\epsilon - y^*_\epsilon\|_{Q_T} \leqslant \epsilon, \tag{4.34}$$

$$\|u_\epsilon(x,T) - v_\epsilon(x,T)\|_\Omega \leqslant \epsilon, \tag{4.35}$$

$$\|u_\epsilon\|_{\Gamma_{2T}} \leqslant C_{\Omega,\Gamma_2} \left(1 - \beta^2_\oplus\right)^{-1/2} \epsilon, \tag{4.36}$$

where $\beta_\oplus = \sup\limits_{t\in[0,T]} \beta(t)$, C_{Ω,Γ_2} is a constant in the trace inequality associated with Γ_2, and u_ϵ and p^*_ϵ solve the initial boundary value problem

$$\begin{aligned} (u_\epsilon)_t - \mathrm{div} p^*_\epsilon + \mathrm{a} \cdot \nabla u_\epsilon + \rho^2 u_\epsilon = f & \quad \text{in } Q_T, && (4.37)\\ p^*_\epsilon = A \nabla u_\epsilon, & && (4.38)\\ u_\epsilon(x,t) = v_\epsilon(x,t) & \quad \text{on } S_{1T}, && (4.39)\\ p^*_\epsilon \cdot n = F & \quad \text{on } S_{2T}, && (4.40)\\ u_\epsilon(x,0) = \xi(x). & && (4.41) \end{aligned}$$

Proof. We apply (3.22) to the problem (4.37)–(4.41). In view of (4.33), we have

$$\int_0^T (1-\beta^2(t))\|\nabla(u_\epsilon - v_\epsilon)\|^2_{A,\Omega}dt + \|p^*_\epsilon - y^*_\epsilon\|^2_{A^{-1},Q_T} \\ + \|v_\epsilon(\cdot,T) - u_\epsilon(x,T)\|^2_\Omega \leqslant \epsilon^2,$$

wherefrom (4.34) and (4.35) follow. Since

$$\|u_\epsilon\|_{\Gamma_{2T}} \leqslant \|u_\epsilon - v_\epsilon\|_{\Gamma_{2T}} + \|v_\epsilon\|_{\Gamma_{2T}} \leqslant C_{\Omega,\Gamma_2}\|\nabla(u_\epsilon - v_\epsilon)\|_\Omega, \tag{4.42}$$

(4.36) follows from the first inequality in (4.34) and (4.42). □

Theorem 6 guarantees that ϵ-neighbourhood of known functions v_ϵ and y^*_ϵ contains exact solution of the well-posed problem (4.37)–(4.41), which satisfies the equation (4.28), relation (4.29) and conditions (4.31) and (4.32). The condition (4.30) is satisfied with ϵ-accuracy.

4.4. Identification problems. This class of problems is motivated by a situation often encountered in applications where model parameters must be determined from a set of available data (e.g., see [1, 4]). In the context of the initial boundary value problem under consideration, such unknown parameters may be f, A, a, and ρ.

For example, if f is sought and the other parameters are given, then we are dealing with a "source reconstruction" problem. The methodology discussed above is entirely applicable to this class of problems. Let us briefly consider this issue with the paradigm of the problem: find $f(x,t)$ such that the relations (4.14)–(4.16) hold and, in addition,

$$u(x,0) = \xi(x) \qquad \text{and} \qquad u(x,T) = \chi(x). \tag{4.43}$$

Here ξ and χ are certain given functions and f is a function in a set U_f of "admissible" source terms. Depending on how the set U_f is defined, this problem may or may not have a solution.

Assume that we know $f_\epsilon \in U_f$ (e.g., $U_f = L_1(0,T;L_2(\Omega))$), $v_\epsilon \in V_0(Q_T)$, and $y^*_\epsilon \in Y^*_{\rm div}(Q_T)$ such that $v_\epsilon(x,0) = \xi$, $v_\epsilon(x,T) = \chi$, and

$$\Upsilon_\beta(v_\epsilon, y^*_\epsilon) \leqslant \epsilon^2. \tag{4.44}$$

Direct problem with $f = f_\epsilon$ and initial condition $u(x,0) = \xi$ is well posed and has a solution u_ϵ. We apply (3.22) (note that $\Gamma_2 = \varnothing$), use (4.44), and obtain

$$\int_0^T \Big((1-\beta^2(t)) \|\nabla(v_\epsilon - u_\epsilon)\|^2_{A,\Omega} + \|y^*_\epsilon - p^*_\epsilon\|^2_{A^{-1},\Omega} \Big) dt + \|\chi(x) - u_\epsilon(x,T)\|^2_\Omega \leqslant \epsilon^2. \quad (4.45)$$

Hence we can define neighbourhoods of v_ϵ and y^*_ϵ (which are proportional to ϵ) that contain $u_\epsilon(x,t)$ and $p^*_\epsilon = A\nabla u_\epsilon(x,t)$, respectively. The function $u_\epsilon(x,t)$ exactly satisfies the relations (4.14) – (4.16) and the initial condition $u_\epsilon(x,0) = \xi(x)$. The condition at $t = T$ is satisfied approximately: $\|\chi(x) - u_\epsilon(x,T)\|_\Omega \leqslant \epsilon$.

This scheme is also suitable for more complex identification problems where a larger number of conditions are imposed on the solution and a larger number of parameters are to be determined based on them. For example, the conditions on A can be weakened in the sense that it is not a given matrix, but an element of some set of admissible matrices U_A (that satisfy (2.15)). Analogously, it may be known that $\mathrm{a} \in U_\mathrm{a}$ and $\rho \in U_\rho$.

Assume that we know the functions

$$(f_\epsilon, A_\epsilon, \mathrm{a}_\epsilon, \rho_\epsilon) \in U := U_f \times U_A \times U_\mathrm{a} \times U_\rho$$

that define a particular initial boundary value problem and the functions $v_\epsilon \in V_0(Q_T)$ and $y^*_\epsilon \in Y^*_{\mathrm{div}}(Q_T)$ (viewed as approximations for this problem) such that

$$\Upsilon_\beta(v_\epsilon, y^*_\epsilon; f_\epsilon, A_\epsilon, \mathrm{a}_\epsilon, \rho_\epsilon) \leqslant \epsilon^2. \quad (4.46)$$

The estimate (4.46) means that $\epsilon-$neighbourhoods of v_ϵ and y^*_ϵ contain u_ϵ and p^*_ϵ that present the solution of the direct evolutionary problem with the initial condition $\xi = v_\epsilon(x,0)$ and data f_ϵ, A_ϵ, a_ϵ, and ρ_ϵ.

4.5. Applications to numerical methods. The question of how to find functions u_ϵ and y^*_ϵ is a separate large topic which is not discussed in this article. At this point, we refer to numerous publications (e.g., see [3–8]) devoted to numerical analysis of ill-posed problems. It is worth noting that results of Section 4 also suggest a practical way to solve the problems discussed in this section.

In the above, the functionals $\Upsilon_\beta(v, y^*)$, $\Upsilon_\rho(v, y^*)$ and a more complicated functional $\Upsilon_{\beta,\gamma}$ (defined by (4.5)) are used to verify the accuracy of approximate solutions. However, these functionals can also be considered as the basis for numerical methods aimed to find an approximate solution.

For example, consider the problem (4.1)–(4.4). It is easy to see that finding suitable v_ϵ and y^*_ϵ is reduced to the quadratic minimization problem

$$\inf_{\substack{v \in V_0(Q_T),\\ y^* \in Y^*_{\rm div}(Q_T)}} \Upsilon_\beta(v, y^*). \tag{4.47}$$

In (4.47), a minimizing sequence (v_k, y^*_k), $k = 1, 2, ...$ can be computed by known numerical methods. Theorem 4 claims that the value of $\Upsilon_\beta(v_k, y^*_k)$ shows guaranteed accuracy of the approximation achieved on the step k.

For backward parabolic problems, we may await that infimum in (4.47) is equal to zero (cf. (1.3)). In this case, it is theoretically possible to find v_k, y^*_k such that $\Upsilon_\beta(v_k, y^*_k)$ is arbitrarily small (what, however, may be not an easy task). The situation is drastically different for parameter identification problems, where the distance to the set of exact solutions of "admissible" direct problems may be finite so that ϵ in (4.44) or (4.46) cannot be made smaller than some positive value. Here the notion of "solution" should be defined in the Tikhonov's sense and the value of $\Upsilon_\beta(v_k, y^*_k)$ gives an upper bound of the distance to the set formed by solutions of "close" well-posed problems that contains a "quasi-solution".

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