Записки научных
семинаров ПОМИ
Том 550, 2026 г.

Yu. Petrova, N. Rastegaev

KRUŽKOV-TYPE UNIQUENESS THEOREM FOR A NON-MONOTONE FLOW FUNCTION CASE WITH APPLICATION TO RIEMANN PROBLEM SOLUTIONS

Abstract. We generalize the previously obtained Kružkov-type uniqueness result for the initial-boundary value problem for the chemical flood conservation law system to the case of a nearly arbitrary flow function, without the S-shaped restriction or monotonicity with respect to chemical concentration. The result is applied to the analysis of Riemann problem solutions for an S-shaped flow function that changes monotonicity with respect to the chemical concentration exactly once. All possible Riemann problem solution structures are classified, including certain unique structures that have not been described in earlier studies.

§1. Introduction

We generalize the main result of [23], where the uniqueness of solutions of the conservation law system

$$\begin{cases} s_t + f(s,c)_x = 0, \\ (cs + a(c))_t + (cf(s,c))_x = 0, \end{cases} \tag{1}$$

was studied. This system describes the chemical flood of an oil reservoir in enhanced oil recovery methods. Here $(x,t) \in \mathbb{R}^2_+$, s is the saturation of the water phase, c is the concentration of the chemical agent dissolved in water, f denotes the fractional flow function, and a describes the adsorption of the chemical agent on the rock, usually concave like the classical Langmuir curve (see Fig. 1b). While in [23] we assumed f to be S-shaped (after

---

Key words and phrases: Initial-boundary value problem; Riemann problem; first-order hyperbolic system; conservation laws; shock waves; uniqueness theorem; vanishing viscosity; chemical flood.

The work of Y. Petrova was supported by Programa de Apoio a Novos Docentes USP, CNPq grant 406460/2023-0, 403675/2025-1. The work of N. Rastegaev was supported by the Ministry of Science and Higher Education of the Russian Federation (agreement 075-15-2025-344 dated 29/04/2025 for Saint Petersburg Leonhard Euler International Mathematical Institute at PDMI RAS) and by the Theoretical Physics and Mathematics Advancement Foundation “BASIS”.

Buckley–Leverett [4]) as well as monotone with respect to c, in this work we aim to lift those restrictions and consider a more general class of flow functions.

Similar to [23], we study the solutions of the initial-boundary value problem

$$\begin{aligned} &s(x,0)=s_0^x(x), \quad c(x,0)=c_0^x(x), \quad x\geqslant 0,\\ &s(0,t)=s_0^t(t), \quad c(0,t)=c_0^t(t), \quad t\geqslant 0, \end{aligned} \tag{2}$$

and under certain restrictions on the parameters of the problem and the class of solutions, we prove the same uniqueness theorem: that is, we prove that two different solutions from the described class with the same initial-boundary data cannot exist.

After that, we consider a simple case of an S-shaped f that changes monotonicity with respect to c exactly once, and classify the solutions to the problem describing constant injection into a homogeneously filled reservoir

$$\begin{aligned} &s_0^x(x)=s_R, \quad c_0^x(x)=c_R, \quad x\geqslant 0,\\ &s_0^t(t)=s_L, \quad c_0^t(t)=c_L, \quad t\geqslant 0, \end{aligned}$$

which, under given restrictions, is equivalent to the Riemann problem

$$(s,c)(x,0)=\begin{cases}(s_L,c_L), & \text{if } x\leqslant 0,\\ (s_R,c_R), & \text{if } x>0.\end{cases} \tag{3}$$

Riemann problems are essential in understanding hyperbolic systems of conservation laws, used in Glimm's random choice method, front tracking methods, etc. (see [9] and references therein for a more comprehensive list of possible applications). The Riemann problem for the system (1) with an S-shaped f that is monotone with respect to c was studied in [14], and solutions for it are known. The uniqueness of vanishing viscosity solutions for it was also considered in [25]. For the general case of non-monotone f and multicomponent chromatography [8] provides an algorithm for the construction of Riemann problem solutions. However, they use a different approach to distinguish physically meaningful weak solutions; therefore, our solutions may differ in some cases. Conditions (2) also cover more complicated problems, including the slug injection problem [20] or tapering [3].

The proof of the uniqueness theorem follows the scheme used in [23] almost exactly. The Lagrangian coordinate transformation described in detail in that paper is utilized, as well as the proof scheme similar to the well-known Kružkov's theorem [16]. We omit proofs that require no changes and only detail the proofs of the lemmas that need significant generalization.

We keep the admissibility criteria used in [23] based on the paradigm of the classical work by Oleinik [19] with the local variant of the vanishing viscosity condition introduced in [2] (see (W4) in Definition 1 below). The classification of admissible shocks obtained in [2] is then used when constructing the Riemann problem solutions. We study all possible Riemann solution structures and describe which values of initial parameters in (3) yield them. Compared to the monotone case [14], the layout of solution structures in the space of initial parameters is richer and includes areas with novel solution structures not observed before (see (68) and Fig. 14c).

The paper has the following structure. Section 2 lists all restrictions we place on the parameters of the problem for the generalized uniqueness theorem, i.e. on the initial-boundary conditions, on the flow function f, and on the adsorption function a. Section 3 recalls the definition of the class of admissible solutions and the travelling wave dynamical system for the dissipative system. This section also provides a lemma in which a restriction on the set of admissible shocks is derived. Section 4 describes the Lagrangian coordinate transformation. The qualities of the new flow function are derived similarly to the monotone case. Section 5 describes the mapping of the shocks in original coordinates and shocks in the Lagrangian coordinates. Here, the main change to the proof occurs, as the admissibility conditions for c-shocks need to be transferred to the new coordinates and the Kružkov-type entropy inequality for them is derived with a slightly different proof. Finally, Sec. 6 contains the application to the case of an S-shaped f that changes monotonicity with respect to c exactly once. All possible solution structures are classified in this section.

§2. RESTRICTIONS ON PROBLEM PARAMETERS

2.1. Restrictions on the initial-boundary data. The following restrictions on the functions from (2) are assumed:

(S1) $s_0^x(x) = 0$ for all $x \geqslant x^0$ for a fixed $x^0 \in [0, +\infty]$;
(S2) if $x^0 > 0$, then $s_0^x(x) \geqslant \delta^0 > 0$ for all $0 \leqslant x < x^0$;
(S3) $s_0^t(t) \geqslant \delta^0 > 0$ for all $t \geqslant 0$.

Remark 1. For the sake of including the Riemann problems with $s_L = 0$, the case when $x^0 = \infty$, $s_0^t(t) \equiv 0$ needs to be considered. In this case, the zero flow area should be investigated similarly to Sec. 4.1, and is expected to coincide with the axis $\{x = 0\}$. The solution is positive everywhere else. This creates some special behavior near the vertical axis in Lagrangian coordinates, but otherwise we expect the proofs to hold. The details of this case will be considered in future work.

2.2. Restrictions on the flow function.

See Fig. 1a for an example of a function f under the S-shaped restriction, monotone with respect to c. The following assumptions (F1), (F2), (F3$'$) for the fractional flow function f lift those restrictions and allow a much wider class of flow functions.

(F1) $f \in \mathcal{C}^2([0,1]^2)$; $f(0,c) = 0$, $f(1,c) = 1$ for $c \in [0,1]$;
(F2) $f_s(s,c) > 0$ for $0 < s < 1$, $0 \leqslant c \leqslant 1$; $f_s(0,c) = f_s(1,c) = 0$ for $c \in [0,1]$;
(F3$'$) there exists $s^f > 0$, such that $f_{ss}(s,c) > 0$ for $s \in (0, s^f)$, $c \in [0,1]$.

The assumption (F3$'$) is necessary to maintain some technical steps of the proof of the uniqueness theorem developed in [23] in lieu of the S-shaped assumption (F3) used there. We utilize it exactly once in the proof of Lemma 1, which replaces the proof of [23, Lemma 3.3].

2.3. Restrictions on the adsorption function.

The adsorption function $a = a(c)$ satisfies the following assumptions (see Fig. 1b for an example of a function a):

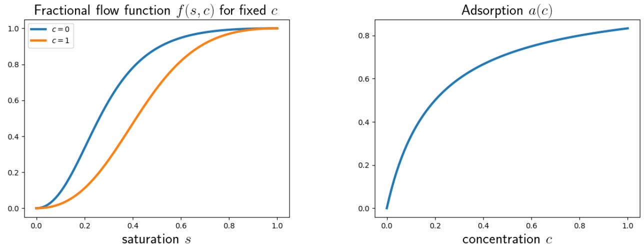
(A1) $a \in \mathcal{C}^2([0,1])$, $a(0) = 0$;
(A2) $a_c(c) > 0$ for $0 < c < 1$;
(A3) $a_{cc}(c) < 0$ for $0 < c < 1$.

§3. Admissible solutions of chemical flood system

3.1. Admissible weak solutions.

In this work, we use the local vanishing viscosity method proposed in [23]. The admissible weak solutions are required to be a classical solution almost everywhere except for a locally finite number of shocks (jump discontinuities) with a local version of the vanishing viscosity condition on the shocks.

Definition 1 (Definition 3.1, [23])**.** We call (s,c) a piecewise $\mathcal{C}^1$-smooth weak solution of (1) with vanishing viscosity admissible shocks and locally bounded "variation" of c (*W-solutions* for brevity), if:



(a) (b)

Figure 1. Examples of (a) flow function $f(s,c)$; (b) adsorption function a.

(W1) Functions s and c are continuous and piecewise continuously differentiable everywhere, except for a locally finite number of $\mathcal{C}^1$-smooth curves, where one or both of them have a jump discontinuity.

(W2) For any compact K away from the axes, the derivative $|c_x(x,t)| < C_K$ is uniformly bounded for all $(x,t) \in K$ not on jump discontinuities.

(W3) Functions s and c satisfy (1) in a classical sense inside the areas, where they are continuously differentiable.

(W4) On every discontinuity curve Γ given by $\gamma(t)$ at any point $(\gamma(t_0), t_0)$ the jump of s and c

$$s^{\pm} = s(\gamma(t_0) \pm 0, t_0), \quad c^{\pm} = c(\gamma(t_0) \pm 0, t_0)$$

with velocity $v = \gamma_t(t_0)$ could be obtained as a limit as $\varepsilon \to 0$ of travelling wave solutions

$$s(x,t) = s\Big(\frac{x-vt}{\varepsilon}\Big), \quad c(x,t) = c\Big(\frac{x-vt}{\varepsilon}\Big)$$

of the dissipative system

$$\begin{cases} s_t + f(s,c)_x = \varepsilon_c (A(s,c) s_x)_x, \\ (cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cA(s,c) s_x)_x + \varepsilon_d c_{xx}, \end{cases} \tag{4}$$

with boundary conditions

$$s(\pm\infty) = s^{\pm}, \quad c(\pm\infty) = c^{\pm}.$$

> Here $\varepsilon_c = \varepsilon$ and $\varepsilon_d = \kappa\varepsilon$ for some $\kappa \in (0, +\infty)$ are the dimensionless capillary pressure and diffusion, respectively, and $A(s, c)$ is the capillary pressure function (bounded, separated from zero, and Lipschitz continuous).

Remark 2. Note that the condition (W1) here is weaker than the condition (W1) in [23]. It allows s and c to have discontinuities in their derivatives. Careful examination of the proofs in [23] shows that this changes nothing, and the uniqueness theorem still holds in this wider class of solutions. This modification was recently introduced in [18], where obvious jumps in c derivative were observed in the slug injection problem. Similarly, we noticed possible jumps in the derivatives at the meeting points of different rarefaction waves in the Riemann problems considered here.

Observe that the system (4) differs from [14, (4.8)]. According to [2], it yields a different set of admissible shocks for different values of κ in some non-monotone cases, even though in the monotone case, the admissible shocks are the same for all κ. In this paper we consider the value of κ and function $A(s, c)$ arbitrary but fixed. The results of this paper apply to the full system [2, (3)], which accounts for capillary pressure, polymer diffusion, and dynamic adsorption, but in (4) we chose to disregard the dynamic adsorption for brevity. Note also that [8] does not use the dissipative system to distinguish admissible shocks, and uses the projection principle and the lifting algorithm instead, yielding different admissibility criteria in some cases.

Theorem 1. *Problem* (1) *with initial-boundary conditions* (2) *satisfying the restrictions* (S1)–(S3)*, with flow function satisfying* (F1), (F2), (F3′) *and adsorption satisfying* (A1)–(A3) *can only have a unique W-solution.*

3.2. Travelling wave dynamical system. The assumption (W4) for the solution is that shocks are admissible if and only if they could be obtained as a limit of travelling wave solutions for a system with additional dissipative terms as these terms tend to zero. In this section, we analyze such travelling wave solutions and derive a dynamical system that describes them.

Consider a shock between states (s^-, c^-) and (s^+, c^+) moving with velocity v. In order to check if it is admissible, we are looking for a travelling wave solution

$$s(x,t) = s\Big(\frac{x - vt}{\varepsilon}\Big), \quad c(x,t) = c\Big(\frac{x - vt}{\varepsilon}\Big)$$

for the dissipative system (4) satisfying the boundary conditions

$$s(\pm\infty) = s^{\pm}, \quad c(\pm\infty) = c^{\pm}.$$

Substituting this travelling wave ansatz into the system (4) and denoting $\xi = \frac{x-vt}{\varepsilon}$, we get the system

$$\begin{cases} -vs_{\xi} + f(s,c)_{\xi} = (A(s,c)s_{\xi})_{\xi}, \\ -v(cs + a(c))_{\xi} + (cf(s,c))_{\xi} = (cA(s,c)s_{\xi})_{\xi} + \kappa c_{\xi\xi}. \end{cases}$$

Integrating the equations over ξ, we arrive at the travelling wave dynamical system

$$\begin{cases} A(s,c)s_{\xi} = f(s,c) - v(s + d_1), \\ \kappa c_{\xi} = v(d_1 c - d_2 - a(c)). \end{cases} \tag{5}$$

The values of d_1 and d_2 are obtained from the boundary conditions:

$$\begin{aligned} vd_1 &= -vs^{\pm} + f(s^{\pm}, c^{\pm}), \\ vd_2 &= vd_1 c^{\pm} - va(c^{\pm}), \end{aligned}$$

namely, in the case when $c^+ \neq c^-$,

$$d_1 = \frac{a(c^-) - a(c^+)}{c^- - c^+}, \quad d_2 = \frac{c^+ a(c^-) - c^- a(c^+)}{c^- - c^+}. \tag{6}$$

Additionally, the same boundary conditions yield the Rankine–Hugoniot conditions

$$\begin{aligned} v[s] &= [f(s,c)], \\ v[cs + a(c)] &= [cf(s,c)], \end{aligned} \tag{7}$$

where $[q(s,c)] = q(s^+, c^+) - q(s^-, c^-)$[1]. Thus, for every set of shock parameters $(s^{\pm}, c^{\pm})$ and v satisfying (7), we can construct a phase portrait for the dynamical system (5). The points $(s^{\pm}, c^{\pm})$ are critical for this dynamical system due to (7), and we can check if there is a trajectory connecting the corresponding critical points. But even just analyzing the geometric meaning of the Rankine–Hugoniot conditions (7), we derive a lot of restrictions on admissible shock parameters.

Proposition 1. *The following restrictions on admissibility are evident from the properties* (F1), (F2), (F3'), (A1)–(A3)*, the Rankine–Hugoniot conditions* (7) *and the analysis of the sign of the right-hand side of* (5):

---

[1] Note the order of "+" and "−" terms in this definition. It could be different in different sources. We follow certain proof schemes of [24], so our order coincides with their.

- *Admissible shock velocity* v *is bounded and strictly positive*: $0 < v < \|f\|_{\mathcal{C}^1}$.
- *Shocks with* $s^- = 0$ *cannot be admissible.*
- *Shocks with* $c^+ > c^-$ *cannot be admissible.*
- *If* $s^+ = 0$ *then* $c^+ = c^-$.

Lemma 1. *There exists* $s_* \in (0,1)$ *such that when* $s^- < s_*$, *we have*

$$f_s(s^-, c^-) > v \tag{8}$$

for all admissible shock parameters.

Proof. If $c^- \neq c^+$, we rewrite (7) in the following form:

$$v = \frac{[f(s,c)]}{[s]} = \frac{f(s^\pm, c^\pm)}{s^\pm + h}, \qquad h = \frac{[a(c)]}{[c]},$$

therefore, points $(-h, 0)$, $(s^+, f(s^+, c^+))$, $(s^-, f(s^-, c^-))$ are collinear and lie on the line $l(s) = v(s+h)$. Due to (F3$'$), there exists such

$$\delta^f = \frac{\min\{f(s^f, c) : c \in [0,1]\}}{1 + \|a\|_{\mathcal{C}^1}} > 0,$$

that when $0 \leqslant v \leqslant \delta^f$, there is a unique intersection point of $l(s)$ and $f(s, c^-)$ inside the interval $(0,1)$. Indeed, by the definition of δ^f we have $\delta^f(s+h) < f(s, c^-)$ for all $s \geqslant s^f$, therefore there are no intersections on $[s^f, 1]$. And on $(0, s^f)$, there is exactly one intersection due to convexity, given by (F3$'$). We denote this intersection point $(s^c(c^-, v), f(s^c(c^-, v), c^-))$, and at this intersection point we have

$$f_s(s^c(c^-, v), c^-) > v.$$

Therefore, (8) holds for admissible shocks with $c^- \neq c^+$ for any $s^- < s^c_*$, where

$$s^c_* = \min_{c \in [0,1]} s^c(c, \delta^f).$$

If $c^- = c^+ = c$, we look at the system (5) and observe that it simplifies into one equation. A more detailed analysis of this case is given in [23, Lemma 3.3] and [23, Sect. 5.2]. We simply note that due to Oleinik admissibility, we have $f_s(s^-, c^-) \geqslant v$. To achieve equality, the graph of $f(\cdot, c)$ must be tangent to the chord connecting $(s^-, f(s^-, c))$ and $(s^+, f(s^+, c))$. Therefore, denoting by $s^s(c)$ the tangent point of the graph of $f(s,c)$ for $s \in (0, s^f)$ and the line going through the point

$$(1, \min\{f(s^f, c) : c \in [0,1]\}),$$

and by

$$s_*^s = \min_{c\in[0,1]} s^s(c),$$

we conclude that for admissible shocks with $c^- = c^+$ for any $s^- < s_*^s$, we always have a strict sign, as no tangent chord could be constructed there. Thus, (8) holds due to f being convex at s^-. By construction, $s^s(c)$ is a positive continuous function on $[0,1]$, so it admits a minimum separated from zero.

Finally, denoting $s_* = \min\{s_*^s, s_*^c\} > 0$, we complete the proof. □

3.3. Inadmissible nullcline configurations. When we get rid of the S-shaped assumption and either the monotonicity or at least the limited non-monotonicity with respect to c, it becomes untenable to write down the full classification of possible nullcline configurations, as in [2, Sec. 4.2] and [23, Sec. 3.3]. However, we can still formulate a sufficient restriction on admissible configurations that we use later when transferring admissibility to Lagrangian coordinates.

Lemma 2. *Given fixed c^+ and c^-, such that $c^+ < c^-$, consider two shocks – from (s^-, c^-) to (s^+, c^+) with speed v and from (z^-, c^-) to (z^+, c^+) with speed w. Assume*

$$s^- > z^-, \quad s^+ < z^+, \quad v < w. \tag{9}$$

Then both shocks cannot be admissible at the same time.

Proof. By (W4) and Sec. 3.2, admissible shocks must solve the travelling wave dynamical system (5). Therefore, if both shocks are admissible, there must exist two trajectories. One trajectory is the function $s(c)$ that solves

$$\frac{A(s,c)}{\kappa}\frac{ds}{dc} = \frac{f(s,c) - v(s+d_1)}{v(d_1 c - d_2 - a(c))}$$

and connects (s^+, c^+) to (s^-, c^-), the other is the function $z(c)$ that solves

$$\frac{A(z,c)}{\kappa}\frac{dz}{dc} = \frac{f(z,c) - w(z+d_1)}{w(d_1 c - d_2 - a(c))}$$

and connects (z^+, c^+) to (z^-, c^-). We observe that the values of d_1 and d_2 are defined in (6) and depend only on $c^\pm$. Due to the assumption (9), the graphs of these functions $s(c)$ and $z(c)$ must intersect, and the $s(c)$ graph must cross the $z(c)$ graph from below at one of the intersections. Denote such an intersection $(\bar{s}, \bar{c})$. At this intersection, we necessarily have

$$\frac{f(\bar{s}, \bar{c}) - v(\bar{s} + d_1)}{v(d_1\bar{c} - d_2 - a(\bar{c}))} \geqslant \frac{f(\bar{s}, \bar{c}) - w(\bar{s} + d_1)}{w(d_1\bar{c} - d_2 - a(\bar{c}))}.$$

Observe that $d_1\bar{c} - d_2 - a(\bar{c}) < 0$ due to (A3), and we arrive at $v \geqslant w$, which contradicts (9). Thus, at least one of the shocks is inadmissible. $\square$

§4. Lagrangian coordinate transformation

4.1. Zero flow area. Following [23], we utilize the Lagrangian coordinates, i.e., the coordinates tied to the flow. In order to do so, we first need to describe the area with no flow. Recall that due to (F1) and (F2), the flow function is zero only when $s = 0$. In this section we cite the statements from [23, Sec. 4.1] asserting that under the conditions (S1)–(S3) the no flow area is a connected region in $Q = \mathbb{R}^2_+$ in (x, t) space bounded by the ray $(x^0, +\infty)$ on one side and by a curve made of discontinuities of the function s on the other (see the boundaries on Fig. 2).

Lemma 3 (Lemma 4.1, [23]). *Let (x_*, t_*), $x_* > 0$, $t_* > 0$ be a zero of the solution s, i.e., $s(x_*, t_*) = 0$ inside a region of smoothness of s and c, or one of $s(x_* \pm 0, t_*) = 0$ on the shock. Then $s(x_*, t) = 0$ for all $0 \leqslant t < t_*$.*

Lemma 4 (Lemma 4.4, [23]). *For all $x > x^0$, we define*

$$t_0(x) = \sup\{t : s(x, t) = 0\}.$$

Then

- *$t_0(x) < +\infty$;*
- *$(x, t_0(x))$ is a point on a shock;*
- *$t_0(x)$ is continuous, piecewise $\mathcal{C}^1$-smooth.*

Corollary 1 (Corollary 4.5, [23]). *Define*

$$\Omega_0 = \{(x, t) : x > x^0, 0 \leqslant t < t_0(x)\}.$$

Then $s(x, t) = 0$ in Ω_0 and $s(x, t) > 0$ outside $\overline{\Omega}_0$. Moreover, $s(x, t)$ is locally separated from 0 outside $\overline{\Omega}_0$.

Proposition 2 (Proposition 4.6, [23]). *It is clear that $c_t = 0$ in Ω_0; therefore, $c(x, t_0(x)) = c_0^x(x)$.*

4.2. Lagrangian coordinates. It is hard to trace back the history of the coordinate transformation described in this subsection. Many authors describe the transformation without citing previous work. The oldest reference we found is [6] cited in [27] in the context of gas dynamics equations. The idea is also presented in the lectures by Gelfand [10] for the case of an arbitrary system of conservation laws. The splitting technique for the system (1) using the Lagrangian coordinate transformation is presented in [20]. It is later developed and applied to different systems by many authors (see [1] and references therein).

Proposition 3 (Proposition 4.3, [23]). *For any solution* (s, c) *the differential form* $f(s, c)\, dt - s\, dx$ *derived from the first equation of* (1) *is exact, i.e., on any closed curve* $\partial\Omega$ *with a finite number of shock points we have*

$$\oint\limits_{\partial\Omega} f(s, c)\, dt - s\, dx = 0. \tag{10}$$

Similarly, from the second equation of (1) *we derive the exact form* $(cs + a(c))dx - f(s, c)dt$*, therefore*

$$\oint\limits_{\partial\Omega} c(s\, dx - f(s, c)\, dt) + \oint\limits_{\partial\Omega} a(c)\, dx = 0. \tag{11}$$

We denote by φ the potential such that

$$d\varphi = f(s, c)\, dt - s\, dx. \tag{12}$$

To explain the physical meaning of φ, let us consider any trajectory ν connecting $(0, 0)$ and (x, t). When s denotes the saturation of some liquid, the potential $\varphi(x, t)$ is equal to the amount of this liquid passing through the trajectory:

$$\varphi(x, t) = \int\limits_{\nu} f(s, c)\, dt - s\, dx. \tag{13}$$

The coordinate change $(x, t) \to (\varphi, x)$ is only applicable in the area $Q_{orig} = Q \setminus \overline{\Omega}_0$, where the saturation s and the flow function $f(s, c)$ are not zero. It keeps the x coordinate, so it maps the axis $\Gamma_t = \{x = 0, t \geqslant 0\}$ onto $\Gamma_\varphi = \{\varphi \geqslant 0, x = 0\}$. The segment $[0, x^0] \times \{0\}$ maps into a curve $(\varphi_0(x), x)$, where

$$\varphi_0(x) = -\int\limits_0^x s_0^x(r)\, dr. \tag{14}$$

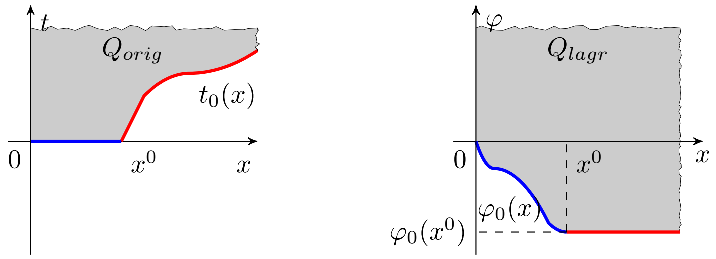


Figure 2. Areas $Q_{\rm orig}$ (grey area on the left) and $Q_{\rm lagr}$ (grey area on the right). The red curve on the left is mapped onto the red line on the right. The blue line on the left is mapped onto the blue curve on the right.

The curve $(x, t_0(x))$ for $x > x^0$ maps into a horizontal line beginning at the point $(\varphi_0(x^0), x^0)$. Therefore $Q_{\rm orig}$ maps into

$$Q_{\rm lagr} = Q \cup \{(\varphi, x) : 0 < x \leqslant x^0, 0 > \varphi > \varphi_0(x)\} \cup \big((\varphi_0(x^0), 0) \times (x^0, +\infty)\big)\,,$$

see Fig. 2.

Corollary 1 guarantees that there is a reverse transform given by

$$dt = \frac{1}{f(s,c)}\,d\varphi + \frac{s}{f(s,c)}\,dx,$$

and the denominators are locally separated from zero, therefore this coordinate change is a piecewise $\mathcal{C}^1$-diffeomorphism. Since $\mathcal{C}^1$-smooth curves preserve their smoothness properties under any diffeomorphism, all discontinuity curves map into $\mathcal{C}^1$-smooth discontinuity curves.

Substituting (12) into (11), we get that the form $-c\,d\varphi + a(c)\,dx$ is also exact in any area in Q_{lagr} where the images of (s, c) are $\mathcal{C}^1$-smooth. Therefore, it is also closed and leads to the identity

$$0 = d(c\,d\varphi - a(c)\,dx) = \left(\frac{\partial c}{\partial x} + \frac{\partial a(c)}{\partial \varphi}\right) dx \wedge d\varphi.$$

Together with the identity

$$0 = d(dt) = d\left(\frac{1}{f}\,d\varphi + \frac{s}{f}\,dx\right) = \left(\frac{\partial}{\partial x}\left(\frac{1}{f}\right) - \frac{\partial}{\partial \varphi}\left(\frac{s}{f}\right)\right) dx \wedge d\varphi$$

it yields, inside the areas of $\mathcal{C}^1$-smoothness, the classical system

$$\frac{\partial}{\partial x}\left(\frac{1}{f}\right)-\frac{\partial}{\partial \varphi}\left(\frac{s}{f}\right)=0,$$
$$\frac{\partial c}{\partial x}+\frac{\partial a(c)}{\partial \varphi}=0.$$

We use the notation

$$\mathcal{U}(\varphi,x)=\frac{1}{f(s(x,t),c(x,t))},\ \ \zeta(\varphi,x)=c(x,t),\ \ \mathcal{F}(\mathcal{U},\zeta)=-\frac{s}{f(s,c)} \tag{15}$$

to transform this system into the system of conservation laws.

$$\mathcal{U}_x+\mathcal{F}(\mathcal{U},\zeta)_\varphi=0, \tag{16}$$
$$\zeta_x+a(\zeta)_\varphi=0. \tag{17}$$

Proposition 4 (Proposition 4.7, [23]). *Suppose that* (s,c) *is a W-solution of the problem* (1) (*see Definition* 1) *with initial and boundary conditions* (2) *satisfying the conditions* (S1)–(S3). *Then the functions* $(\mathcal{U},\zeta)$ *given by the formulae* (15) *satisfy the integral equations*

$$\begin{aligned}\iint\limits_{Q_{lagr}}\mathcal{U}\widetilde{\psi}_x+\mathcal{F}(\mathcal{U},\zeta)\widetilde{\psi}_\varphi\,d\varphi\,dx+\int\limits_0^\infty\mathcal{U}_0^\varphi(\varphi)\widetilde{\psi}(\varphi,0)\,d\varphi&\\+\int\limits_{x^0}^\infty\mathcal{F}(\mathcal{U}(\varphi_0(x^0)+0,x),c_0^x(x))\widetilde{\psi}(\varphi_0(x^0),x)\,dx=0&\end{aligned} \tag{18}$$

and

$$\begin{aligned}\iint\limits_{Q_{\mathrm{lagr}}}\zeta\widetilde{\psi}_x+a(\zeta)\widetilde{\psi}_\varphi\,d\varphi\,dx+\int\limits_0^{x^0}(s_0^x(x)c_0^x(x)+a(s_0^x(x)))\widetilde{\psi}(\varphi_0(x),x)\,dx&\\+\int\limits_{x^0}^\infty a(c_0^x(x))\widetilde{\psi}(\varphi_0(x^0),x)\,dx+\int\limits_0^\infty\zeta_0^\varphi(\varphi)\widetilde{\psi}(\varphi,0)\,d\varphi=0&\end{aligned} \tag{19}$$

for all test functions $\widetilde{\psi}\in\mathcal{D}(Q_{\mathrm{lagr}})$, *where the initial values* $\mathcal{U}_0^\varphi$ *and* ζ_0^φ *are given by*

$$\mathcal{U}_0^\varphi(\varphi(0,t))=\frac{1}{f(s_0^t(t),c_0^t(t))},\qquad \zeta_0^\varphi(\varphi(0,t))=c_0^t(t). \tag{20}$$

Remark 3. The same reasoning as in [24, Lemma 2.2.1] shows that on every shock, the equations in the weak form (18) and (19) result in the Rankine–Hugoniot condition

$$\begin{aligned} v^*[\mathcal{U}] &= [\mathcal{F}(\mathcal{U},\zeta)], \\ v^*[\zeta] &= [a(\zeta)], \end{aligned} \tag{21}$$

where v^* is the velocity of the shock between states $(\mathcal{U}^-,\zeta^-)$ and $(\mathcal{U}^+,\zeta^+)$. Here, as in the original coordinates, $[q(\mathcal{U},\zeta)] = q(\mathcal{U}^+,\zeta^+) - q(\mathcal{U}^-,\zeta^-)$.

Properties of the new flow function $\mathcal{F}$ (see Fig. 3) that correspond to the properties (F1), (F2) of the function f are listed below.

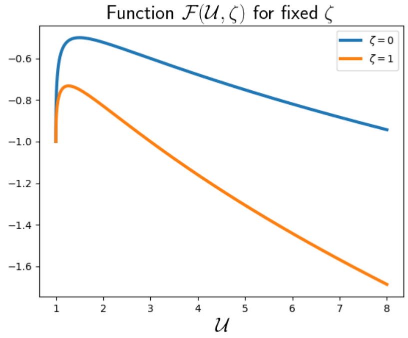


Figure 3. The function $\mathcal{F}(\mathcal{U},\zeta)$ corresponding to the flow function $f(s,c)$ plotted in Fig. 1a.

Proposition 5. *For all* $\zeta \in [0,1]$*, the following properties of the function* $\mathcal{F}$ *are fulfilled*

- $\mathcal{F} \in \mathcal{C}^2([1,+\infty)\times[0,1])$;
- $\mathcal{F}(\mathcal{U},\zeta) < 0$ *for all* $\mathcal{U} \in [1,+\infty)$;
- $\mathcal{F}(1,\zeta) = -1$;
- $\lim\limits_{\mathcal{U}\to\infty} \mathcal{F}(\mathcal{U},\zeta) = -\infty$;
- $\lim\limits_{\mathcal{U}\to 1} \mathcal{F}_{\mathcal{U}}(\mathcal{U},\zeta) = +\infty$;
- $\lim\limits_{\mathcal{U}\to\infty} \mathcal{F}_{\mathcal{U}}(\mathcal{U},\zeta) = 0$.

§5. Entropy conditions in Lagrangian coordinates

5.1. Mapping shocks to Lagrangian coordinates. Let us recall the notation from [23] pertaining to the mapping of shocks between the coordinate systems.

Consider a shock in original coordinates at the point (x_1, t_1) with one-sided limits

$$s^{\pm} = s(x_1 \pm 0, t_1), \quad c^{\pm} = c(x_1 \pm 0, t_1).$$

Denote $\theta_c(s) = \frac{1}{f(s,c)}$ and let $\vartheta_c = \theta_c^{-1}$ be its inverse function with respect to its argument s. Using these functions, we map

$$\mathcal{U}^{[+]} = \theta_{c^+}(s^+), \quad \mathcal{U}^{[-]} = \theta_{c^-}(s^-), \quad \zeta^{[+]} = c^+, \quad \zeta^{[-]} = c^-,$$

$$s^+ = \vartheta_{\zeta^{[+]}}(\mathcal{U}^{[+]}), \quad s^- = \vartheta_{\zeta^{[-]}}(\mathcal{U}^{[-]}), \quad c^+ = \zeta^{[+]}, \quad c^- = \zeta^{[-]}.$$

Note that in the original coordinates, the values $s^{\pm}$ correspond to $x \to x_1 \pm 0$, respectively. The shock velocity in original coordinates is always positive due to Proposition 1, therefore, $s^{\pm}$ correspond to $t \to t_1 \mp 0$:

$$s^{\pm} = s(x_1, t_1 \mp 0), \quad c^{\pm} = c(x_1, t_1 \mp 0).$$

Further, due to (12) and (F1), when $x = x_1$ is fixed, $t \to t_1 \mp 0$ correspond to $\varphi \to \varphi_1 \mp 0$ for the point (φ_1, x_1) on a corresponding shock in Lagrangian coordinates, and so do $\mathcal{U}^{[\pm]}$:

$$\mathcal{U}^{[\pm]} = \mathcal{U}(\varphi_1 \mp 0, x_1), \quad \zeta^{[\pm]} = \zeta(\varphi_1 \mp 0, x_1).$$

But for the equations (16), (17) in Lagrangian coordinates, the x axis plays the role of time and φ the role of space, so we would like to define $\mathcal{U}^{\pm}$ to correspond to $\varphi \to \varphi_1 \pm 0$. Thus, we denote

$$\mathcal{U}^{+} = \mathcal{U}^{[-]}, \quad \mathcal{U}^{-} = \mathcal{U}^{[+]}, \quad \zeta^{+} = \zeta^{[-]}, \quad \zeta^{-} = \zeta^{[+]},$$

$$\mathcal{U}^{\pm} = \mathcal{U}(\varphi_1 \pm 0, x_1), \quad \zeta^{\pm} = \zeta(\varphi_1 \pm 0, x_1),$$

and obtain a one-to-one mapping of shocks in original $(s^{\pm}, c^{\pm})$ and Lagrangian $(\mathcal{U}^{\pm}, \zeta^{\pm})$ coordinates:

$$\begin{aligned} (\mathcal{U}^{\pm}, \zeta^{\pm}) &\to (s^{\pm}, c^{\pm}) = (\vartheta_{\zeta^{\mp}}(\mathcal{U}^{\mp}), \zeta^{\mp}), \\ (s^{\pm}, c^{\pm}) &\to (\mathcal{U}^{\pm}, \zeta^{\pm}) = (\theta_{c^{\mp}}(s^{\mp}), c^{\mp}). \end{aligned} \tag{22}$$

We call shocks satisfying $\zeta^+ = \zeta^-$ *$\mathcal{U}$-shocks*, and refer to all other shocks as *ζ-shocks*.

5.2. Oleinik's and entropy admissibility conditions for $\mathcal{U}$-shocks.

We generalize the argument from [23, Sec. 5.2] that transfers Oleinik's E-condition to the Lagrangian coordinates and prove that it implies the entropy condition for any convex positive entropy, but specifically for entropy–entropy-flux pairs $(|\mathcal{U}-k|, \mathcal{G}(\mathcal{U},k))$, $k\in\mathbb{R}$, where

$$\mathcal{G}(\mathcal{U},k) = (\mathcal{F}(\mathcal{U},\zeta)-\mathcal{F}(k,\zeta))\operatorname{sgn}(\mathcal{U}-k).$$

Recall (see [23, Sect. 5.2]) that for

$$\Psi(s) = f(s,c) - f(s^-,c) - v(s-s^-)$$

on an s-shock we have Oleinik's entropy condition

$$\Psi(s)(s^+-s^-)\geqslant 0 \quad \text{for all } s \text{ between } s^+ \text{ and } s^-. \tag{23}$$

This follows from the simplified form of the dynamical system (5) on an s-shock. Similarly, on the corresponding $\mathcal{U}$-shock in the Lagrangian coordinates we have

$$\Psi^*(\mathcal{U}) = \mathcal{F}(\mathcal{U},\zeta)-\mathcal{F}(\mathcal{U}^-,\zeta)-v^*(\mathcal{U}-\mathcal{U}^-),$$

$$v^* = \begin{cases} \dfrac{\mathcal{F}(\mathcal{U}^-,\zeta)-\mathcal{F}(\mathcal{U}^+,\zeta)}{\mathcal{U}^- - \mathcal{U}^+}, & \mathcal{U}^- < +\infty, \\ 0, & \mathcal{U}^- = +\infty, \end{cases}$$

and the Oleinik's condition also holds. To prove this, in [23] we split Oleinik's condition into two parts (Lax condition plus $\Psi\neq 0$ on the interval) but that split was only valid because of S-shaped function f. In the general case such simplification doesn't work and we need a more general proof.

Lemma 5. *Oleinik's condition for an s-shock in original coordinates* (23) *is equivalent to Oleinik's condition for the corresponding $\mathcal{U}$-shock in Lagrangian coordinates:*

$$\Psi^*(\mathcal{U})(\mathcal{U}^+-\mathcal{U}^-)\geqslant 0 \quad \textit{for all } \mathcal{U} \textit{ between } \mathcal{U}^+ \textit{ and } \mathcal{U}^-. \tag{24}$$

When $\mathcal{U}^- = +\infty$ this should be interpreted as

$$\mathcal{F}(\mathcal{U},\zeta)\leqslant \mathcal{F}(\mathcal{U}^+,\zeta) \quad \textit{for all } \mathcal{U}\geqslant \mathcal{U}^+. \tag{25}$$

Proof. If we have (23), we rewrite it as

$$\frac{f(s,c)-f(s^-,c)}{s-s^-} \geqslant \frac{f(s^+,c)-f(s^-,c)}{s^+-s^-}.$$

Substituting s, $s^{\pm}$ and f in this relation we arrive at

$$\frac{\mathcal{U}^+ - \mathcal{U}}{\mathcal{F}(\mathcal{U}^+,\zeta)\mathcal{U} - \mathcal{F}(\mathcal{U},\zeta)\mathcal{U}^+} \geqslant \frac{\mathcal{U}^+ - \mathcal{U}^-}{\mathcal{F}(\mathcal{U}^+,\zeta)\mathcal{U}^- - \mathcal{F}(\mathcal{U}^-,\zeta)\mathcal{U}^+}.$$

This transforms into

$$\frac{\mathcal{F}(\mathcal{U},\zeta) - \mathcal{F}(\mathcal{U}^+,\zeta)}{\mathcal{U} - \mathcal{U}^+} \leqslant \frac{\mathcal{F}(\mathcal{U}^-,\zeta) - \mathcal{F}(\mathcal{U}^+,\zeta)}{\mathcal{U}^- - \mathcal{U}^+} = v^*.$$

Note that for $\mathcal{U}^- = +\infty$ we obtain (25) immediately. For $\mathcal{U}^- < +\infty$ we rewrite

$$\Big(\mathcal{F}(\mathcal{U},\zeta) - \mathcal{F}(\mathcal{U}^+,\zeta) - v^*(\mathcal{U} - \mathcal{U}^+)\Big)(\mathcal{U}^+ - \mathcal{U}^-) \geqslant 0,$$

and since $\mathcal{F}(\mathcal{U}^-,\zeta) - \mathcal{F}(\mathcal{U}^+,\zeta) - v^*(\mathcal{U}^- - \mathcal{U}^+) = 0$, we arrive at (24). $\square$

Using the Oleinik's condition in Lagrangian coordinates, we replace [23, Proposition 5.2] with the following new proposition.

Proposition 6. *For a fixed solution ζ, given two solutions $\mathcal{U}$ and $\mathcal{V}$ we have*

$$\mathcal{G}(\mathcal{U}(\varphi_0(x)+0,x), \mathcal{V}(\varphi_0(x)+0,x)) \leqslant 0, \qquad \forall x \geqslant x^0.$$

Proof. Fix $x \geqslant x^0$. Assume without loss of generality that

$$\mathcal{U}(\varphi_0(x)+0,x) \leqslant \mathcal{V}(\varphi_0(x)+0,x).$$

Consider $\mathcal{U}^+ = \mathcal{U}(\varphi_0(x)+0,x)$ and $\mathcal{U}^- = +\infty$. Then the proposition follows immediately from (25) with $\mathcal{U} = \mathcal{V}(\varphi_0(x)+0,x)$. $\square$

Recall from [23] the following lemma and the proposition that follows from it. Their proofs do not rely on the properties we eliminated.

Lemma 6 (Lemma 5.4, [23])**.** *For all $k \in \mathbb{R}$ and every positive test function $\psi \in \mathcal{D}^+(Q_{\mathrm{lagr}})$ with $\operatorname{supp}\psi$ containing only $\mathcal{U}$-shocks (and no*

ζ-shocks), the entropy condition holds:

$$
\begin{aligned}
0 \leqslant \iint\limits_{Q_{\mathrm{lagr}}} |\mathcal{U}-k|\psi_x + \mathcal{G}(\mathcal{U},k)\psi_\varphi \, d\varphi \, dx - \iint\limits_{Q_{lagr}} \mathcal{F}_\zeta(k,\zeta)\zeta_\varphi \operatorname{sgn}(\mathcal{U}-k)\psi \, d\varphi \, dx \\
+ \int\limits_0^{x^0} \Big(|\mathcal{U}(\varphi_0(x),x) - k| s_0^x(x) + \mathcal{G}(\mathcal{U}(\varphi_0(x),x),k) \Big) \psi(\varphi_0(x),x) \, dx \\
+ \int\limits_{x^0}^{\infty} \mathcal{G}(\mathcal{U}(\varphi_0(x^0)+0,x),k)\psi(\varphi_0(x^0),x) \, dx + \int\limits_0^{\infty} |\mathcal{U}(\varphi,0)-k|\psi(\varphi,0) \, d\varphi. \qquad (26)
\end{aligned}
$$

Proposition 7 (Proposition 2.7.1, [24])**.** *For a fixed solution ζ, given two solutions $\mathcal{U}$ and $\mathcal{V}$, for any positive test function $\psi \in \mathcal{D}^+(Q_{\mathrm{lagr}})$ with* $\operatorname{supp}\psi$ *containing no ζ-shocks, we have*

$$
\begin{aligned}
0 \leqslant \iint\limits_{Q_{lagr}} & |\mathcal{U}-\mathcal{V}|\psi_x + \mathcal{G}(\mathcal{U},\mathcal{V})\psi_\varphi \, d\varphi \, dx \\
& + \int\limits_0^{x^0} |\mathcal{U}(\varphi_0(x),x) - \mathcal{V}(\varphi_0(x),x)| s_0^x(x) \psi(\varphi_0(x),x) \, dx \\
& + \int\limits_0^{x^0} \mathcal{G}(\mathcal{U}(\varphi_0(x),x), \mathcal{V}(\varphi_0(x),x)) \psi(\varphi_0(x),x) \, dx \qquad (27) \\
& + \int\limits_{x^0}^{\infty} \mathcal{G}(\mathcal{U}(\varphi_0(x^0)+0,x), \mathcal{V}(\varphi_0(x^0)+0,x)) \psi(0,x) \, dx \\
& + \int\limits_0^{\infty} |\mathcal{U}(\varphi,0) - \mathcal{V}(\varphi,0)| \psi(\varphi,0) \, d\varphi.
\end{aligned}
$$

In [24], this proposition is proven for the case that corresponds to $\zeta \equiv \mathrm{const}$. For the general case, see the original proof of Kružkov's [16, Theorem 1].

5.3. Entropy admissibility condition for ζ-shocks. Similarly, the analysis of ζ-shocks in the equation (17) does not rely on any properties of the flow function, therefore the following proposition holds with no changes.

Proposition 8 (Proposition 5.8, [23])**.** *Denote*

$$\mathcal{A}(\zeta,k)=(a(\zeta)-a(k))\operatorname{sgn}(\zeta-k).$$

Then on any admissible ζ-shock given by the curve $(\Phi(x),x)$ inside Q_{lagr}, the following entropy inequality holds:

$$[\mathcal{A}(\zeta,k)]\leqslant\frac{d\Phi}{dx}[|\zeta-k|],\quad k\in\mathbb{R},$$

and therefore for every positive test function $\psi\in\mathcal{D}^+(Q_{\mathrm{lagr}})$ we have the integral entropy condition

$$\begin{aligned}0\leqslant&\iint\limits_{Q_{\mathrm{lagr}}}|\zeta-k|\psi_x+\mathcal{A}(\zeta,k)\psi_\varphi\,d\varphi\,dx\\&+\int\limits_0^{x^0}\Big(|\zeta(\varphi_0(x),x)-k|s_0^x(x)+\mathcal{A}(\zeta(\varphi_0(x),x),k)\Big)\psi(\varphi_0(x),x)\,dx\\&+\int\limits_{x^0}^{\infty}\mathcal{A}(\zeta(\varphi_0(x^0),x),k)\psi(\varphi_0(x^0),x)\,dx+\int\limits_0^{\infty}|\zeta(\varphi,0)-k|\psi(\varphi,0)\,d\varphi.\end{aligned}$$

The second change to the uniqueness proof comes when we look at the proof of [23, Lemma 9]. That proof relies on the classification of admissible shocks, and needs to be reworked based solely on Lemma 2.

Lemma 7. *Given two solutions $(\mathcal{U},\zeta)$ and $(\mathcal{V},\zeta)$ for the same ζ, we have the following entropy inequality on any ζ-shock given by the curve $(\Phi(x),x)$ inside Q_{lagr}:*

$$[\mathcal{G}(\mathcal{U},\mathcal{V})]\leqslant\frac{d\Phi}{dx}[|\mathcal{U}-\mathcal{V}|].\tag{28}$$

Proof. Let the two solutions have the ζ-shocks, which, when translated into the original coordinates via the mapping (22), connect

$(s^-=\vartheta_{\zeta^+}(\mathcal{U}^+),c^-=\zeta^+)$ to $(s^+=\vartheta_{\zeta^-}(\mathcal{U}^-),c^+=\zeta^-)$ with speed v, and

$(z^-=\vartheta_{\zeta^+}(\mathcal{V}^+),c^-=\zeta^+)$ to $(z^+=\vartheta_{\zeta^-}(\mathcal{V}^-),c^+=\zeta^-)$ with speed w.

First, we consider the case $v = w$. In this case we have equality in (28) due to the Rankine–Hugoniot condition (see (21)):

$$\frac{d\Phi}{dx} = \frac{[\mathcal{F}(\mathcal{U},\zeta)]}{[\mathcal{U}]} = \frac{[\mathcal{F}(\mathcal{V},\zeta)]}{[\mathcal{V}]}, \tag{29}$$

since all points lie on the same line in Fig. 4, and therefore, we additionally have

$$\frac{d\Phi}{dx} = \frac{\mathcal{F}(\mathcal{U}^{\pm},\zeta^{\pm}) - \mathcal{F}(\mathcal{V}^{\pm},\zeta^{\pm})}{\mathcal{U}^{\pm} - \mathcal{V}^{\pm}}.$$

Next, we can assume without loss of generality that $v < w$. Then the relations between $s^{\pm}$ and $z^{\pm}$ fall into one of four cases:

- $s^- > z^-$, $s^+ < z^+$. This satisfies the assumption (9) for our shocks; thus, due to Lemma 2, one of the considered shocks must be inadmissible. Therefore, we don't need to consider this case.
- $s^- > z^-$, $s^+ > z^+$ or $s^- < z^-$, $s^+ < z^+$. In these cases, we have

  $$\operatorname{sgn}(\mathcal{U}^+ - \mathcal{V}^+) = \operatorname{sgn}(\mathcal{U}^- - \mathcal{V}^-),$$

  therefore we have equality in (28) due to Rankine–Hugoniot condition (29).
- $s^- < z^-$, $s^+ > z^+$. In this case

  $$\begin{aligned} \operatorname{sgn}(\mathcal{U}^+ - \mathcal{V}^+) &= \operatorname{sgn}(\theta_{c^-}(s^-) - \theta_{c^-}(z^-)) = 1, \\ \operatorname{sgn}(\mathcal{U}^- - \mathcal{V}^-) &= \operatorname{sgn}(\theta_{c^+}(s^+) - \theta_{c^+}(z^+)) = -1, \end{aligned} \tag{30}$$

  so (28) transforms into

  $$\mathcal{F}(\mathcal{U}^+,\zeta^+) - \mathcal{F}(\mathcal{V}^+,\zeta^+) + \mathcal{F}(\mathcal{U}^-,\zeta^-) - \mathcal{F}(\mathcal{V}^-,\zeta^-) \leqslant \frac{d\Phi}{dx}(\mathcal{U}^+ - \mathcal{V}^+ + \mathcal{U}^- - \mathcal{V}^-).$$

  Note that due to the monotonicity of a and the Rankine–Hugoniot condition (21), the speed of the shock

  $$\frac{d\Phi}{dx} = \frac{a(\zeta^-) - a(\zeta^+)}{\zeta^- - \zeta^+} > 0. \tag{31}$$

  It can be observed geometrically in Fig. 4 that

  $$\begin{aligned} \mathcal{U}^+ &= \theta_{c^-}(s^-) > \theta_{c^-}(z^-) = \mathcal{V}^+, \\ \mathcal{U}^- &= \theta_{c^+}(s^+) < \theta_{c^+}(z^+) = \mathcal{V}^-, \end{aligned}$$

  therefore, the following inequalities hold for the inclines of lines:

  $$\frac{\mathcal{F}(\mathcal{U}^+,\zeta^+) - \mathcal{F}(\mathcal{V}^+,\zeta^+)}{\mathcal{U}^+ - \mathcal{V}^+} < \frac{d\Phi}{dx},$$

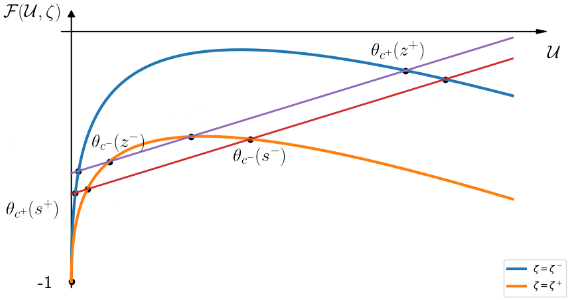


Figure 4. An illustration for the positions of $\mathcal{U}^\pm$ and $\mathcal{V}^\pm$ on the plots of $\mathcal{F}(\cdot,\zeta^\pm)$ in the last case of Lemma 7. The red line corresponding to $s^\pm$ is lower due to $v < w$.

$$\frac{\mathcal{F}(\mathcal{U}^-,\zeta^-)-\mathcal{F}(\mathcal{V}^-,\zeta^-)}{\mathcal{U}^- - \mathcal{V}^-} > \frac{d\Phi}{dx}.$$

This Fig. 4 is constructed for the S-shaped case, but the geometrical argument holds even without that assumption. Taking into account the known signs (30) of the denominators, we obtain

$$\mathcal{F}(\mathcal{U}^+,\zeta^+)-\mathcal{F}(\mathcal{V}^+,\zeta^+) < \frac{d\Phi}{dx}(\mathcal{U}^+ - \mathcal{V}^+),$$

$$\mathcal{F}(\mathcal{U}^-,\zeta^-)-\mathcal{F}(\mathcal{V}^-,\zeta^-) < \frac{d\Phi}{dx}(\mathcal{U}^- - \mathcal{V}^-).$$

Taking the sum of these inequalities concludes this case.

Thus, all four cases lead either to a contradiction or to the necessary inequality, which concludes the proof. ☐

Thus, having successfully replaced [23, Lemma 9], we can proceed with the original proof with no other changes. The following lemma follows from Proposition 7 and the new Lemma 7 we just proved as a replacement for [23, Lemma 9].

Lemma 8 (Lemma 5.11, [23])**.** *The inequality* (27) *from Proposition 7 holds for all positive test functions* $\psi \in \mathcal{D}^+(Q_{\mathrm{lagr}})$ *without restrictions on their supports (including* ζ*-shocks).*

For more details, see the proof of [23, Theorem 6.1]. It utilizes the result of Lemma 8 and the classical scheme of Kružkov's theorem to prove that the difference between $\mathcal{U}$ and $\mathcal{V}$ cannot increase, thus solutions with the same initial-boundary conditions must coincide. The proof does not fundamentally rely on any properties derived from S-shaped assumption (F3) or monotonicity assumption (F4) used in [23]. It only uses [23, Proposition 5.2], which we replaced with Proposition 6.

§6. Application to Riemann problem

In this section, we describe the solutions to the Riemann problem (1), (3) for the simple case of an S-shaped f that changes monotonicity with respect to c exactly once. For a study of the sufficient conditions for an S-shaped flow function, see [5, 22]. There are numerous cases to consider for this problem, but due to Theorem 1 we only need to provide one W-solution for each case, and the uniqueness follows. This allows us to skip checking all possible wave combinations to ascertain the uniqueness by brute force, which significantly simplifies this section.

We impose, in addition to (F1)–(F2) from Section 2, the further assumptions (F3)–(F4) on the fractional-flow function f:

(F3) f is S-shaped in s: for each $c \in [0,1]$ function $f(\cdot, c)$ has a unique point of inflection $s^I = s^I(c) \in (0,1)$, such that $f_{ss}(s,c) > 0$ for $0 < s < s^I$ and $f_{ss}(s,c) < 0$ for $s^I < s < 1$.

(F4) f has exactly one change of monotonicity in c: $\exists c^* \in (0,1)$:
- $f_c(s,c) < 0$ for $0 < s < 1$, $0 < c < c^*$;
- $f_c(s,c) > 0$ for $0 < s < 1$, $c^* < c < 1$.

We assume that $f_{cc}(s^*, c^*) > 0$, where $s^* \in [0,1]$ is a unique value that satisfies

$$f_s(s^*, c^*) = \frac{f(s^*, c^*)}{s^* + a_c(c^*)}.$$

The main result of the section states as follows.

Theorem 2. *Consider Problem* (1) *under the assumptions* (F1)–(F4) *for the fractional flow function f and* (A1)–(A3) *for adsorption function a. Then for arbitrary states $u_L = (s_L, c_L)$, $u_R = (s_R, c_R) \in [0,1]^2$, $s_L \neq 0$ there exists a unique W-solution of the Riemann problem* (3) (*see Definition* 1)*.*

The existence is proven by explicit construction, and the uniqueness is guaranteed by applying Theorem 1. For $s_L = 0$, we constructed the solution (though it satisfies the vanishing viscosity criterion only when $c_L = c_R$), but the uniqueness is not formally covered by Theorem 1 in its current form (see Remark 1). We hope to cover this case by some future generalization of the uniqueness theorem. First, in Sec. 6.1, we recall important properties of hyperbolic conservation laws in general and of the chemical flooding models in particular; then in Sec. 6.2 we provide explicit solutions of the Riemann problem.

6.1. Basic facts about conservation laws and chemical flooding models. In this section, we recall the properties of hyperbolic conservation laws – and, specifically, of the chemical flooding model (1) – that are pertinent to the analysis of the Riemann problem.

6.1.1. *Characteristic speeds.* The system (1) can be rewritten in the form

$$u_t + B(u)u_x = 0, \qquad u = (s, c),$$

where $B(u)$ is the characteristic matrix

$$B(s,c) = \begin{pmatrix} f_s & f_c \\ 0 & f/(s + a_c(c)) \end{pmatrix}. \tag{32}$$

For more details see [14, Sec. 2]. The eigenvalues of $B(u)$, i.e., the characteristic speeds for the system, are

$$\lambda^s = f_s \qquad \text{and} \qquad \lambda^c = f/(s + a_c(c)). \tag{33}$$

We choose the right eigenvectors corresponding to λ^s and λ^c to be

$$r^s := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad r^c := \begin{pmatrix} -f_c \\ \lambda^s - \lambda^c \end{pmatrix}. \tag{34}$$

Since both of the characteristic speeds are real, this model is hyperbolic when $\lambda^s \neq \lambda^c$. However, the characteristic speeds coincide, not only on the boundary line $s = 0$, but also along the curve

$$\mathcal{C} := \{\, (s,c) \in [0,1]^2 \,:\, \lambda^s(s,c) = \lambda^c(s,c) \text{ and } s \neq 0 \,\}, \tag{35}$$

thus the model is not strictly hyperbolic. Note that, the matrix $B(s,c)$ defined in (32) is not diagonalizable on $\mathcal{C}$. The coincidence locus $\mathcal{C}$ divides the domain into two regions: left Ω_L, where $\lambda^s > \lambda^c$, and right Ω_R, where $\lambda^s < \lambda^c$.

6.1.2. *Rarefaction waves.* Let λ be an eigenvalue of the characteristic matrix given by (32) with corresponding eigenvector r. The simple rarefaction waves are continuous solutions of (1) of the form

$$u(x,t) = v(\xi)\,, \quad \xi = \frac{x}{t},$$

where v corresponds to an integral curve of the vector field r. More precisely,

$$u(x,t) = \begin{cases} u_L, & \text{if } x/t < \lambda(u_L), \\ v, & \text{if } x/t = \lambda(v), \\ u_R, & \text{if } x/t > \lambda(u_R), \end{cases} \tag{36}$$

where v is an integral curve of the vector field r connecting the states u_L and u_R with the additional property that the eigenvalue λ is increasing from u_L to u_R.

Since the matrix B has two eigenvalues, λ^s and λ^c, two possible rarefaction curves pass through any given state u_L:

- If $\lambda = \lambda^s$ with eigenvector $r^s = (1,0)$, then c is constant along the integral curves. Thus, a simple rarefaction of the form (36) exists whenever $c_L = c_R$ and $\lambda^s = f_s(s, c_L)$ increases from s_L to s_R. This is precisely the Buckley–Leverett rarefaction for $f(s) = f(s, c_L)$, hereafter called an *s-rarefaction wave.*
- If $\lambda = \lambda^c$ with eigenvector r^c defined in (34), the integral curves are nontrivial in the (s,c)-plane (see Fig. 5) and correspond to solutions of the following dynamical system (we call such curves *c-rarefaction curves*):

$$\begin{pmatrix} s \\ c \end{pmatrix}_{\xi} = \alpha \begin{pmatrix} -f_c \\ \lambda^s - \lambda^c \end{pmatrix}, \tag{37}$$

where α is chosen so that $\xi = \lambda^c$:

$$\alpha = \frac{1}{\nabla\lambda^c \cdot r^c} = \frac{(s + a'(c))^2}{(\lambda^c - \lambda^s) f(s,c) a''(c)}.$$

This system is not defined on the coincidence locus $\mathcal{C}$. Changing the variable $\xi \mapsto \tilde{\xi}$, where $\frac{d\tilde{\xi}}{d\xi} = \alpha(\xi)$, we can instead study the system

$$\begin{pmatrix} s \\ c \end{pmatrix}_{\tilde{\xi}} = \begin{pmatrix} -f_c \\ \lambda^s - \lambda^c \end{pmatrix}. \tag{38}$$

The trajectories of this system are the same as (37) except for time direction in Ω_R and well-posedness on the coincidence locus (the system (37) is not defined on the coincidence locus, while the system (38) is). Note also that in the original system (37) we had $\xi = \lambda_c$, therefore all trajectories could be traversed in finite time, including the trajectories arriving at or leaving the critical point (s^*, c^*). A simple rarefaction of the form (36) exists when λ^c increases along the curve connecting u_L and u_R. We refer to such solutions as *c-rarefaction waves.*

Remark 4. We regard the constant states $s \equiv 0$ and $s \equiv 1$ as two c-rarefaction curves. Indeed, when $s_L = 0$ or $s_L = 1$, then $f_c(u_L) = 0$, and the eigenvector r^c can be taken to be $(0, 1)$. For more details, see discussion at the end of Sec. 3 in [14].

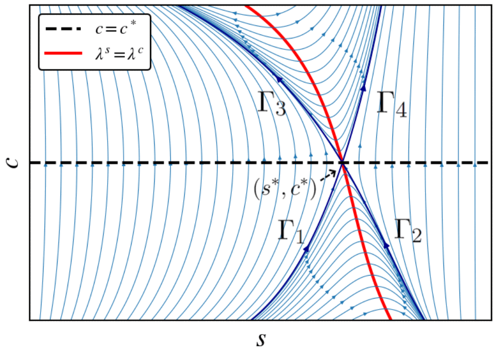


Figure 5. The blue curves represent the family of c-rarefaction curves — integral curves of the system (38) associated with c-rarefaction waves. The critical point (s^*, c^*) is of saddle-type. The arrows show the direction of increasing eigenvalue λ^c and coincide with the time direction of (37). The red curve represents the coincidence locus $\mathcal{C}$.

The following proposition collects the basic properties of c-rarefaction curves in the (s, c)-plane that we use to construct Riemann problem solutions.

Proposition 9. *For the c-rarefaction curves, the following properties hold:*

- *For any* $(s_0, c_0) \in \Omega_L \cup \Omega_R$ *the c-rarefaction curve* Γ *that passes through the point* (s_0, c_0) *can be written as a function* $s = s(c)$ *and*
$$\frac{d}{dc}\lambda^c(s(c), c) > 0.$$
- *The system* (38) *admits a unique fixed point of saddle type,* (s^*, c^*), *defined by the assumption* (F4), *with four c-rarefaction curves,* $\Gamma_i, i = 1, \dots, 4$, *intersecting at this point. We adopt the following notation for the parametrizations of* $\Gamma_i, i = 1, \dots, 4$, (*see Fig.* 5):
$$\begin{aligned} &s_1(c), c \in [0, c^*], \ \textit{such that } (s_1(c), c) \in \Omega_L \cup \mathcal{C}; \\ &s_2(c), c \in [0, c^*], \ \textit{such that } (s_2(c), c) \in \Omega_R \cup \mathcal{C}; \\ &s_3(c), c \in [c^*, 1], \ \textit{such that } (s_3(c), c) \in \Omega_L \cup \mathcal{C}; \\ &s_4(c), c \in [c^*, 1], \ \textit{such that } (s_4(c), c) \in \Omega_R \cup \mathcal{C}. \end{aligned}$$

Proof. The properties of rarefaction curves follow from these easily verifiable formulas:
$$(\lambda^c - \lambda^s) \cdot s'(c) = f_c, \qquad \frac{d}{dc}\lambda^c(s(c), c) = -\frac{a_{cc}(c) \cdot f(s(c), c)}{(s(c) + a_c(c))^2}.$$
Due to assumptions (F1)–(F4), there exists a unique point (s^*, c^*), which satisfies
$$f_c(s^*, c^*) = 0, \qquad \lambda^s(s^*, c^*) = \lambda^c(s^*, c^*),$$
thus (s^*, c^*) is a fixed point of the system (38). The linearization of the system (38) at the point $u^* = (s^*, c^*)$ states $u_{\xi} = L(u^*)u$ for the matrix L defined as:
$$L = \begin{pmatrix} -f_{sc} & -f_{cc} \\ f_{ss} - \frac{\lambda^s - \lambda^c}{s + a_c} & f_{sc} - \frac{f_c}{s + a_c} + \frac{f a_{cc}}{(s + a_c)^2} \end{pmatrix}.$$
At the fixed point u^* the expression for L simplifies:
$$L(u^*) = \begin{pmatrix} 0 & -f_{cc} \\ f_{ss} & \frac{f a_{cc}}{(s + a_c)^2} \end{pmatrix}.$$
The characteristic equation for the eigenvalues μ of $L(u^*)$ is
$$\mu^2 - \mu \cdot \frac{f a_{cc}}{(s + a_c)^2} + f_{ss} f_{cc} = 0.$$

The eigenvalues of $L(u^*)$ are

$$\mu_\pm(u^*) = \frac{1}{2}\left(\frac{fa_{cc}}{(s+a_c)^2} \pm \sqrt{\mathcal{D}}\right), \qquad \mathcal{D} = \left(\frac{fa_{cc}}{(s+a_c)^2}\right)^2 - 4f_{ss}f_{cc}.$$

Notice that at the point u^* we have $f_{ss} < 0$ and $f_{cc} > 0$ due to assumptions (F1)–(F4). Therefore, $\mu_+(u^*) > 0$ and $\mu_-(u^*) < 0$, and the point u^* is a saddle point for the dynamical system (38). See the qualitative picture of the set of the c-rarefaction curves in Fig. 5. ☐

Remark 5. Fix the state (s_0, c_0) with $c_0 < c^*$. If $s_0 \in [0, s_1(c_0)) \cup (s_2(c_0), 1]$, the c-rarefaction curve that passes through the point (s_0, c_0) is defined for all $c \in [0,1]$; meanwhile if $s_0 \in (s_1(c_0), s_2(c_0))$ this rarefaction curve reaches the coincidence locus at some point $(\tilde{s}, \tilde{c}) \in \mathcal{C}$ and cannot be defined for $c > \tilde{c}$. For more details see [14, Section 3]. In particular, this means that when $s_0 \neq s_1(c_0)$ and $s_0 \neq s_2(c_0)$, the c-rarefaction curve lies either in $\Omega_L \cup \mathcal{C}$ or in $\Omega_R \cup \mathcal{C}$. On the other hand, if $s_0 = s_2(c_0)$, the union $\Gamma_2 \cup \Gamma_3$ is also an integral curve of the system (38), and can be viewed as a unique c-rarefaction curve (similarly, $\Gamma_1 \cup \Gamma_4$). This is one of the major differences with the monotone case considered in [14].

6.1.3. *Shock waves.* Recall the standard ordering of the eigenvalues of the characteristic matrix (32) as $\lambda_1(u) < \lambda_2(u)$, referred to as the 1-family and 2-family characteristic speeds. In polymer models, $\lambda_1(u)$ equals $\lambda^c(u)$ when $u \in \Omega_L$, but equals $\lambda^s(u)$ when $u \in \Omega_R$.

A standard way to classify a discontinuity is based on the ordering of the characteristic speeds on its two sides relative to its propagation speed v, *i.e.*, $\lambda_i(u_-) - v$ and $\lambda_i(u_+) - v$ for $i = 1, 2$ (see, for example, [12, 17], and [7, Chapter 8]). Four of the possibilities, which we call the 1-family Lax, 2-family Lax, overcompressive, and crossing configurations of characteristic paths, are depicted in Fig. 6:

- 1-family Lax: $\lambda_1(u_-) > v > \lambda_1(u_+)$, $v < \lambda_2(u_-)$, and $v < \lambda_2(u_+)$;
- 2-family Lax: $\lambda_2(u_-) > v > \lambda_2(u_+)$, $v > \lambda_1(u_-)$, and $v > \lambda_1(u_+)$;
- overcompressive: $\lambda_1(u_-) > v > \lambda_1(u_+)$ and $\lambda_2(u_-) > v > \lambda_2(u_+)$;
- crossing: $\lambda_2(u_-) > v > \lambda_1(u_-)$ and $\lambda_1(u_+) < v < \lambda_2(u_+)$.

6.1.4. *Compatibility by speed.* A solution of a Riemann problem is an assembly of s-wave groups, c-waves, and constant states ordered by speed. We use the notation $u \xrightarrow{s} u'$ (respectively, $u \xrightarrow{c} u'$) to denote an s-wave group (respectively, a c-wave) connecting states u and u' in the direction

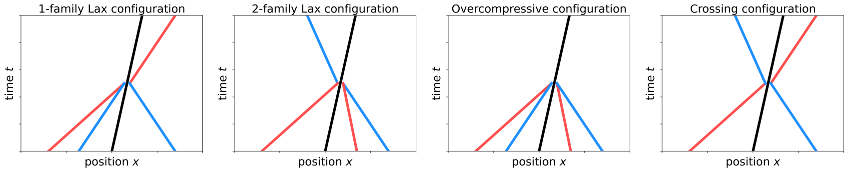


Figure 6. Four possible configurations of characteristic paths drawn in (x, t)-space: 1-family Lax, 2-family Lax, overcompressive, and crossing. The black line is the shock trajectory with speed v. Blue curves are characteristic paths for the 1-family, and red curves are for the 2-family.

of increasing speed. We say that the two waves $u \to u'$ and $u' \to u''$ are *compatible by speed* if the speed increases when we go through the state u', and therefore they can be composed to solve the Riemann problem with left state u and right state u''.

As noted in Lemma 5.1 of [14], the solution $u = (s, c)$ of a Riemann problem has the property that the function $c(x, t)$ is a monotone function of x for every $t \geqslant 0$. In their proof, the authors do not assume the monotonicity of f in c and use only the concavity of a, thus this lemma is also valid under our assumptions on f and a. For the reader's convenience, we recall this lemma below.

Proposition 10 (Lemma 5.1, [14]). *Assume that the three waves*

$$u_L \xrightarrow{c\text{-}wave} u_1 \xrightarrow{s\text{-}wave} u_2 \xrightarrow{c\text{-}wave} u_R$$

are compatible by speed. Then both c-waves are rarefaction waves.

Let $u_L = (s_L, c_L)$ and $u_R = (s_R, c_R)$ denote the left and right states of the Riemann problem, respectively. By Proposition 10, if $c_L < c_R$, any solution of the Riemann problem consists of s-waves together with c-rarefaction waves; if $c_L > c_R$, any solution consists of s-waves and a single c-shock. Moreover, if $c_L = c_R$, any solution reduces to a single s-wave. Hence, by the theory of the Buckley–Leverett equation (see [4]), the Riemann problem for the system (1) admits a unique solution when $c_L = c_R$. Therefore, it remains to prove Theorem 2 in the case $c_L \neq c_R$.

6.2. Region layout for the Riemann problem solutions. In this section, we describe the layout of the regions in the (u_L, u_R)-plane where the Riemann problem solutions have a similar structure (the same sequence of s- and c-waves). In Sec. 6.2.1 we treat the case $c_L > c_R$, while the case $c_L < c_R$ is considered in Sec. 6.2.2.

6.2.1. *Case* $c_L > c_R$. Throughout this section, we shall assume that the values of c_L and c_R are fixed with $c_L > c_R$. We aim to prove that for any $s_L, s_R \in [0,1]$ there exists a unique solution to the Riemann problem (1), (3). Recall that due to Proposition 10, the case $c_L > c_R$ corresponds to the solution of a Riemann problem with at most one c-shock, i.e., the solution has the following structure in terms of s and c-waves:

$$u_L \xrightarrow{s\text{-wave}} u^- \xrightarrow{c\text{-shock}} u^+ \xrightarrow{s\text{-wave}} u_R, \tag{39}$$

where the first, the last, or both s-waves may be absent. The solutions of the form (39) clearly satisfy the conditions (W1)–(W3) from the definition (1) of a W-solution. Moreover, as described below, the c-shock wave is obtained as a limit of travelling wave solutions of the viscous regularization (4), thus condition (W4) is also fulfilled. Therefore, Theorem 1 applies.

The case $c^* \geqslant c_L > c_R$ corresponds to the monotone dependence of f on c (decreasing in c) and the explicit solutions to the Riemann problem were constructed in [14, Sec. 7]. Note that in this case the set of admissible shock waves does not depend on the choice of $\kappa = \varepsilon_d/\varepsilon_c$ (see (4)), thus the set of admissible Riemann solutions in this paper and in [14] is the same. The case $c_L > c_R \geqslant c^*$ also corresponds to the monotone dependence f on c (increasing in c), and can be solved using the same approach. Therefore, it is enough to consider the case $c_L > c^* > c_R$.

Consider a c-shock between the states $u^- = (s^-, c_L)$ and $u^+ = (s^+, c_R)$ with the velocity v, defined by the Rankine–Hugoniot conditions (7), i.e.,

$$v = \frac{[f(s,c)]}{[s]} = \frac{f(s^+, c_R)}{s^+ + h}, \qquad h = \frac{[a(c)]}{[c]}, \tag{40}$$

where $[q(s,c)] = q(s^+, c_R) - q(s^-, c_L)$. It is convenient to have the following notation (as in [14, Sec. 7]). Condition (F3) implies that for the state $u^+ = (s^+, c_R)$ there exists at most one value $s^K = s^K(u^+) \neq s^+$ such that

$$\frac{f(s^K, c_R)}{s^K + h} = v.$$

If such s^K exists, we call it the *critical shock value*. Geometrically, s^K corresponds to the abscissa of the intersection point of the graph of $f(s, c_R)$ and the line that passes through the point $(s^+, f(u^+))$ with inclination equal to v (in particular, the line also passes through the points $(s^-, f(u^-))$ and $(-h, 0)$; see Fig. 7).

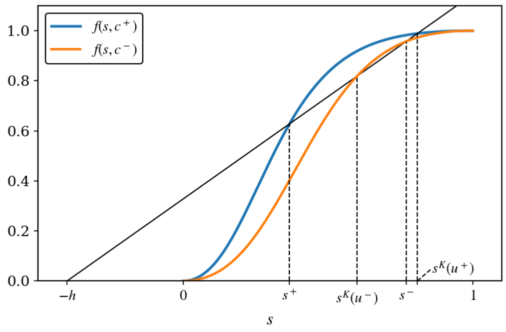


Figure 7. Geometrical interpretation of the critical values $s^K(u^+)$ and $s^K(u^-)$.

We first formulate an extension of Theorem 3.2 in [2] to a larger class of states u_L and u_R (in [2] $c_L = 1$, $c_R = 0$, but the proof is identical for any $c_L > c^* > c_R$). Lemma 9 provides a vanishing viscosity criterion for selecting admissible c-shock waves. In particular, it identifies crossing (undercompressive or transitional) shocks satisfying condition (W4).

Lemma 9. *Consider a system of conservation laws* (1) *under assumptions* (F1)–(F4) *and* (A1)–(A3). *Fix c_L and c_R such that $c_L > c^* > c_R$. Also fix the parameters $\varepsilon_c, \varepsilon_d > 0$ of the equation* (4). *Then there exist $v_{\min}$, $v_{\max}$ such that*

$$0 < v_{\min} < v_{\max} < \infty$$

and for every $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$, there exist unique

- *points $s^- := s^-(\kappa; c_L, c_R) \in [0, 1]$ and $s^+ := s^+(\kappa; c_L, c_R) \in [0, 1]$;*
- *velocity $v := v(\kappa; c_L, c_R) \in [v_{\min}, v_{\max}]$,*

such that the c-shock wave, connecting $u^- := u^-(\kappa; c_L, c_R) = (s^-, c_L)$ and $u^+ := u^+(\kappa; c_L, c_R) = (s^+, c_R)$ with velocity v, is admissible by the vanishing viscosity criterion (*see condition* (W4)) *and is of a crossing configuration.*

Moreover, the following sequence of waves is compatible by speeds

$$u_L = (s_L, c_L) \xrightarrow{s} u^- \xrightarrow{c\text{-shock}} u^+ \xrightarrow{s} u_R = (s_R, c_R)$$

for $c_L > c^* > c_R$ *and all* s_L *and* s_R *under the condition*

$$s^K(u^-) \leqslant s_L \leqslant 1 \qquad \textit{and} \qquad 0 \leqslant s_R \leqslant s^K(u^+). \tag{41}$$

Finally, let us prove Theorem 2 for the case $c_L > c^* > c_R$, c_L and c_R fixed. Consider s^-, s^+, u^-, u^+ and v from Lemma 9.

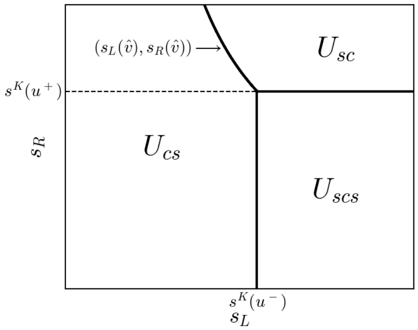


Figure 8. Subdivision into three regions in the (s_L, s_R)-plane with different structure of solutions in terms of the sequence of s-wave groups and c-waves: $\mathbf{U}_{cs}, \mathbf{U}_{sc}$ and $\mathbf{U}_{scs}$.

There exist four possible layouts for the structure of the solution to the Riemann problem as a sequence of s-wave groups and c-waves depending on $(s_L, s_R) \in [0, 1]^2$, see Fig. 8:

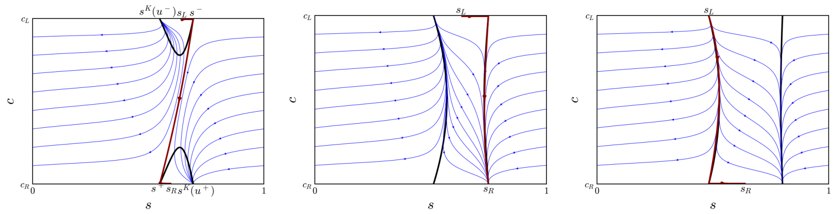
(1) **Undercompressive shock.**
Consider a region:

$$\mathbf{U}_{scs} = [s^K(u^-), 1] \times [0, s^K(u^+)].$$

If $(s_L, s_R) \in \mathbf{U}_{scs}$, then by Lemma 9 the following sequence of waves (scs) provides a solution to the Riemann problem:

$$u_L = (s_L, c_L) \xrightarrow{s} u^- \xrightarrow{c\text{-shock}} u^+ \xrightarrow{s} u_R = (s_R, c_R). \tag{42}$$

As mentioned in Lemma 9, this c-shock wave is undercompressive. If $s_L = s^-$, then the first s-wave in (42) is absent. Similarly, if $s_R = s^+$, then the last s-wave in (42) is absent.



$(s_L, s_R) \in \mathbf{U}_{scs}$ $(s_L, s_R) \in \mathbf{U}_{sc}$ $(s_L, s_R) \in \mathbf{U}_{cs}$

Figure 9. The dark red curves represent the sequence of waves (42), (45) and (46) in the (s, c)-plane for three cases: $(s_L, s_R) \in \mathbf{U}_{scs}$, $(s_L, s_R) \in \mathbf{U}_{sc}$ and $(s_L, s_R) \in \mathbf{U}_{cs}$. The light blue curves illustrate the trajectories of the corresponding dynamical system (5) and the dark blue curves illustrate the nullclines, see Sec. 3.2.

(2) **Overcompressive shock.**

Let $v_1 \in \mathbb{R}$ be defined as:

$$v_1 = \frac{1}{1+h}, \quad h = \frac{a(c_L) - a(c_R)}{c_L - c_R}.$$

For any speed $\hat{v} \in [v_1, v]$, there exist four critical points of the travelling wave dynamical system (5): $(s_L^{1,2}, c_L)$, $(s_R^{1,2}, c^R)$. Any two of these points satisfy the Rankine–Hugoniot condition (40). If we denote $s_L(\hat{v}) = s_L^1 < s_L^2$ and $s_R(\hat{v}) = s_R^2 > s_R^1$, then these saturation values have the following properties:

(a) $s_L(\hat{v}) \in [0, s^K(u^-)]$; $s_R(\hat{v}) \in [s^K(u^+), 1]$;

(b) the states $u_L = (s_L(\hat{v}), c_L)$, $u_R = (s_R(\hat{v}), c_R)$ and the speed $\hat{v}$ satisfy the Rankine-Hugoniot conditions (40);

(c) the following sequence of waves (cs) provides a solution to the Riemann problem:

$$u_L = (s_L(\hat{v}), c_L) \xrightarrow{c\text{-shock}} \hat{u} = (s^K(u_R), c_R) \xrightarrow{s\text{-shock}} u_R = (s_R(\hat{v}), c_R). \quad (43)$$

We observe that the speeds of c-shock and s-shock are both equal to $\hat{v}$.

Also there is an alternative way to represent the solution (43) as a combination of s-shock and c-shock with both speeds equal to $\hat{v}$ (the order is different, (sc)):

$$u_L=(s_L(\hat{v}),c_L)\xrightarrow{s\text{-shock}}\hat{u}=(s^K(u_L),c_L)\xrightarrow{c\text{-shock}}u_R=(s_R(\hat{v}),c_R). \tag{44}$$

Remark 6. The set of all states $(s_L(\hat{v}),s_R(\hat{v}))$ in the plane (s_L,s_R), described above forms a curve parametrized by the speed $\hat{v}\in[v_1,v]$. This curve serves as a boundary between the two regions $\mathbf{U}_{sc}$ and $\mathbf{U}_{cs}$, where the structure of the solutions is of the type (sc) and (cs), respectively (see Fig. 8). As the dependence of $s_R(\hat{v})$ is strictly monotone (by construction), we can define an inverse function, $\hat{v}(s_R)$. Hereinafter, we consider a function $\hat{s}_L(s_R)=s_L(\hat{v}(s_R))$ defined for $s_R\in[s^K(u^+),1]$.

Notice that the solution (43) can be seen formally as one c-shock wave, and following the nomenclature in Sec. 6.1, it corresponds to an overcompressive shock.

(3) **2-family Lax shock, fast.**

Take $\hat{s}_L(s_R)$ from Remark 6 and consider the region

$$\mathbf{U}_{sc}:=\{(s_L,s_R)\in[0,1]^2: s_R\in[s^K(u^+),1],\ s_L\in[\hat{s}_L(s_R),1]\}.$$

For any $(s_L,s_R)\in\mathbf{U}_{sc}$ the following sequence of waves (sc) provides a solution to the Riemann problem:

$$u_L=(s_L,c_L)\xrightarrow{s\text{-wave}}u_M=(s_M,c_L)\xrightarrow{c\text{-shock}}u_R=(s_R,c_R), \tag{45}$$

where $u_M=(s_M,c_L)$ is uniquely defined because it satisfies:

(a) Rankine–Hugoniot condition (40) for u_M, u_R and the corresponding speed v_M;

(b) $\lambda^s(u_M)<v_M$.

Notice that the c-shock from (45) is a 2-family Lax shock.

(4) **1-family Lax shock, slow.**

Consider the region

$$\mathbf{U}_{cs}:=\text{closure}([0,1]^2\setminus(\mathbf{U}_{scs}\cup\mathbf{U}_{sc}))\setminus\{s_L=0\}.$$

For any $(s_L,c_L)\in\mathbf{U}_{cs}$ the following sequence of waves (cs) provides a solution to the Riemann problem:

$$u_L=(s_L,c_L)\xrightarrow{c\text{-shock}}u_M=(s_M,c_R)\xrightarrow{s\text{-wave}}u_R=(s_R,c_R), \tag{46}$$

where $u_M=(s_M,c_L)$ is uniquely defined, as it satisfies:

(a) Rankine–Hugoniot condition (40) for u_L, u_M and corresponding speed v_M;

(b) $\lambda^s(u_M) > v_M$.
Notice that the c-shock from (46) is a 1-family Lax shock. Moreover, observe that if $s_L = 0$, then the c-shock is not (W4)-admissible. It is also physically meaningless to assign a value to the chemical concentration when water saturation is zero.

6.2.2. *Case* $c_L < c_R$. Throughout this section, we assume that the values of c_L and c_R are fixed with $c_L < c_R$. We aim to prove that for any $s_L, s_R \in [0,1]$ there exists a unique solution to a Riemann problem (1), (3). Recall that due to Proposition 10, the solution of a Riemann problem for the case $c_L < c_R$ corresponds to a combination of s-waves and c-rarefaction waves

$$u_L \xrightarrow{s\text{-wave}} u_1 \xrightarrow{c\text{-rarefaction}} u_2 \xrightarrow{s\text{-wave}} \dots \xrightarrow{c\text{-rarefaction}} u_k \xrightarrow{s\text{-wave}} u_R$$

for some $k \in \mathbb{N}$. Similarly to the previous case, if we have $c^* \geqslant c_R > c_L$ or $c^* \leqslant c_L < c_R$, it corresponds to the monotone dependence of the fractional flow function $f(s,c)$ on c (see [14, Section 6] for a full description of the solutions of the Riemann problem when $c^* \geqslant c_R > c_L$). Therefore, it is sufficient to consider the case $c_L < c^* < c_R$.

Critical value $s_{\mathcal{K}}(u)$ and its properties. Following [14, Section 6], we adopt the following notation. For the state $u = (s,c) \in [0,1]^2$, there exists at most one value $s_{\mathcal{K}}(u) \neq s$ such that

$$\lambda^c(s_{\mathcal{K}}(u), c) = \lambda^c(s,c). \tag{47}$$

If such $s_{\mathcal{K}}(u)$ exists, we call it the *critical rarefaction value*. Geometrically, $s_{\mathcal{K}}(u)$ corresponds to the abscissa of the intersection point of the graph of $f(\cdot, c)$ and the line connecting $(s, f(s,c))$ and $(-a_c(c), 0)$. The corresponding state $u_{\mathcal{K}}(u) = (s_{\mathcal{K}}(u), c)$ is called the critical rarefaction state. If $(s,c) \in \mathcal{C}$, it is convenient to consider $s_{\mathcal{K}}(s,c) = s$. If Γ is a c-rarefaction curve, then the curve $\Gamma_{\mathcal{K}}$, which consists of all critical states for the c-rarefaction curve Γ, is called the critical curve.

The following proposition explains why we call $s_{\mathcal{K}}(u)$ critical (this is a trivial generalization of [14, Lemma 6.1] for f under the conditions (F1)–(F4)).

Proposition 11. *The following is true*:

- *The two waves*

$$u_L \xrightarrow{c} u_M \xrightarrow{s} u_R$$

are compatible if and only if $u_M \in \Omega_L \cup \mathcal{C}$ *and* $s_R \in [0, s_{\mathcal{K}}(u_M)]$.

- *The two waves*

  $$u_L \xrightarrow{s} u_M \xrightarrow{c} u_R$$

  are compatible if and only if $u_M \in \Omega_R \cup \mathcal{C}$ *and* $s_L \in [s_{\mathcal{K}}(u_M), 1]$.

Remark 7. For the monotone case considered in [14], Proposition 11 implies that any solution to a Riemann problem contains at most two c-rarefaction waves. However, this is no longer true for the non-monotone case under the assumption (F4). Indeed, there exists a solution that is composed of three rarefaction waves (for more details, see Fig. 14c).

Proposition 12 follows immediately from the assumptions (F1)–(F3) for the fractional flow function f and the definition of the critical value $s_{\mathcal{K}}(u)$.

Proposition 12. *Fix the state* $u = (s, c) \in \Omega_L \cup \Omega_R$. *The following properties hold*:

- *if* $u \in \Omega_L$ *and* $s_{\mathcal{K}}(u)$ *exists, then* $u_{\mathcal{K}} = (s_{\mathcal{K}}(u), c) \in \Omega_R$. *Moreover,* $s_{\mathcal{K}}(u) > s$, *and the mapping* $s \mapsto s_{\mathcal{K}}(u)$ *is continuous and strictly decreasing.*
- *if* $u \in \Omega_R$, *then* $s_{\mathcal{K}}(u)$ *exists and* $u_{\mathcal{K}} = (s_{\mathcal{K}}(u), c) \in \Omega_L$. *Moreover,* $s_{\mathcal{K}}(u) < s$, *and the mapping* $s \mapsto s_{\mathcal{K}}(u)$ *is continuous and strictly decreasing.*
- *if* $s_{\mathcal{K}}(u)$ *exists, then* $u_{\mathcal{K}\mathcal{K}} = (s_{\mathcal{K}}(u_{\mathcal{K}}), c)$ *also exists, and* $u_{\mathcal{K}\mathcal{K}} = u$.

Proposition 13. *Fix a point* $u_0 = (s_0, c_0) \in \Omega_L \cup \Omega_R$, *such that* $s_0 < 1$ *and* $s_{\mathcal{K}}(u_0) < 1$ *exists, and consider three curves* (*here* $\delta > 0$ *is some small number such that the curves are well-defined in the* δ*-neighborhood*):

- *the c-rarefaction curve* Γ (*parametrized as in Proposition* 9) *that passes through the point* (s_0, c_0) :

  $$\Gamma = \{u(c) = (s(c), c) : c \in [c_0 - \delta, c_0 + \delta] \text{ and } s(c_0) = s_0\};$$

- *the critical curve* $\Gamma_{\mathcal{K}}$ *consisting of all critical states for the c-rarefaction curve* Γ*:*

$$\Gamma_{\mathcal{K}} = \{(s_{\mathcal{K}}(c), c) : c \in [c_0 - \delta, c_0 + \delta],\ \text{where } s_{\mathcal{K}}(c) = s_{\mathcal{K}}(u(c)), u(c) \in \Gamma\};$$

- *the c-rarefaction curve that passes through the critical state* $u_{\mathcal{K}}(u_0) \in \Gamma_{\mathcal{K}}$.

  $$\widetilde{\Gamma} = \{(\tilde{s}(c), c) : c \in [c_0 - \delta, c_0 + \delta] \text{ and } \tilde{s}(c_0) = s_{\mathcal{K}}(u_0)\}.$$

Then $s_{\mathcal{K}}(c) \in \mathcal{C}^1[c_0-\delta, c_0+\delta]$ *and we have*

$$\frac{d}{dc} s_{\mathcal{K}}(c_0) < \frac{d}{dc}\tilde{s}(c_0). \tag{48}$$

Proof. The following relations hold at $u = u_{\mathcal{K}}(u_0)$ (see [14, formula (6.3)]):

$$\frac{ds_{\mathcal{K}}}{dc} = \frac{f_c}{\lambda^c - \lambda^s} + g, \qquad \frac{d\tilde{s}}{dc} = \frac{f_c}{\lambda^c - \lambda^s}, \tag{49}$$

where

$$g = \frac{\lambda^c(u)}{\lambda^c(u) - \lambda^s(u)} \cdot a_{cc}(c_0) \cdot \frac{(s_{\mathcal{K}}(u_0) - s_0)}{s_0 + a_c(c_0)}. \tag{50}$$

For $u_0 \in \Omega_L$, we have $\lambda^c(u) > \lambda^s(u)$ and $s_{\mathcal{K}}(u_0) > s_0$, therefore $g(u_0) < 0$. Similarly, for $u_0 \in \Omega_R$ we have $\lambda^c(u) < \lambda^s(u)$ and $s_{\mathcal{K}}(u_0) < s_0$, thus $g(u_0) < 0$ as well, and the statement of this proposition follows. ☐

Notation for important points and curves. To describe the structure of solutions to a Riemann problem, we need to introduce some notation. See Fig. 10 for an illustration of this new notation.

The four rarefaction curves $s_i(c)$, $i = 1, \dots, 4$, which intersect at the fixed point (s^*, c^*) (see Proposition 9), play an important role in the analysis of solutions of Riemann problems. Denote

$$s_{1L} := s_1(c_L); \quad s_{2L} := s_2(c_L); \quad s_{3R} := s_3(c_R); \quad s_{4R} := s_4(c_R). \tag{51}$$

Consider the critical rarefaction values that correspond to the points s_{1L}, s_{2L}, s_{3R}:

$$s_{1K} := s_{\mathcal{K}}(s_{1L}, c_L); \quad s_{2K} := s_{\mathcal{K}}(s_{2L}, c_L); \quad s_{3K} := s_{\mathcal{K}}(s_{3R}, c_R). \tag{52}$$

If s_{1K} does not exist, the analysis of the solutions to the Riemann problem is easier, as there are fewer cases to consider. Thus, we assume that there exists $s_{1K} < 1$. Note also that Lemma 10 provides the existence of $s_{3K} < 1$. Consider the c-rarefaction curve Γ_0 defined by $s_0(c) \equiv 1$, $c \in [0,1]$ and its critical curve $\Gamma_{0\mathcal{K}}$, parametrized by $s_{0\mathcal{K}}(c)$, $c \in [0,1]$. Denote

$$s_{0L} = s_{0\mathcal{K}}(c_L), \qquad s_{0R} = s_{0\mathcal{K}}(c_R). \tag{53}$$

There exists a unique s_{0K} such that

$$(s_{0K}, c_L) \xrightarrow{c\text{-rare}} (s_{0R}, c_R). \tag{54}$$

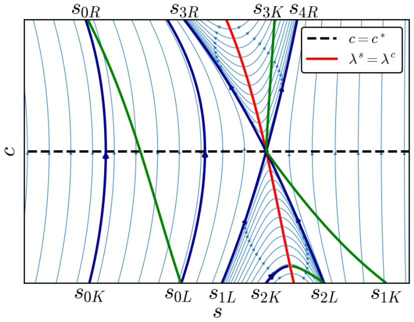


Figure 10. The illustration of the new notations. We observe that the points are arranged according to inequalities (55), (56) from Lemma 10.

Lemma 10. *The following inequalities hold*:

$$s_{0K} < s_{0L} < s_{1L} < s_{2K} < s_{2L} < s_{1K} \ (\textit{states with } c = c_L), \tag{55}$$

$$s_{0R} < s_{3R} < s_{3K} < s_{4R} < 1 \ (\textit{states with } c = c_R). \tag{56}$$

Proof. First, let us prove the inequalities (55). Due to Proposition 12, we get that the assumption $s_{1K} < 1$ implies $s_{0L} < s_{1L}$. In addition, $(s_{2L}, c_L) \in \Omega_R$ implies $(s_{2K}, c_L) \in \Omega_L$, so $s_{2K} < s_{2L}$. So, it is enough to prove:

$$s_{0K} < s_{0L}; \qquad s_{1L} < s_{2K}; \qquad s_{2L} < s_{1K}.$$

Let us prove $s_{1L} < s_{2K}$. The other two inequalities are proved similarly. The scheme of the proof is simple: we assume the opposite inequality, and get a contradiction with Proposition 13.

Assume first that $s_{1L} > s_{2K}$ (the case of equality is considered below). Consider the rarefaction curve $\Gamma = \{(s(c), c)\}$ that passes through the state (s_{2K}, c_L). By Remark 5, $\Gamma \subset \Omega_L$ and is defined for all $c \in [0, 1]$. Consider a curve $\Gamma_{\mathcal{K}} = \{(s_{\mathcal{K}}(c), c)\} \subset \Omega_R$, a critical curve to Γ. Either $\Gamma_{\mathcal{K}}$ is defined for all $c \in [0, 1]$ or there exists some $\tilde{c}$ such that $s_{\mathcal{K}}(\tilde{c}) = 1$. In both cases, we claim that there exists at least one point of intersection of the curves $\Gamma_{\mathcal{K}}$ and Γ_2 (defined in Proposition 9) where the inequality (48) is violated. Indeed, Proposition 13 implies that for c_0 sufficiently close to c_L, we have $s_{\mathcal{K}}(c_0) < s_2(c_0)$. If $\Gamma_{\mathcal{K}}$ is defined for all $c \in [0, 1]$, then for $c = c^*$, obviously, $s_{\mathcal{K}}(c^*) > s_2(c^*) = s^*$. As the curves $\Gamma_{\mathcal{K}}$ and Γ_2 are both continuous, there exists at least one intersection point where $\Gamma_{\mathcal{K}}$ crosses

Γ_2 from left to right, and therefore the inequality (48) is violated at this point. For the second case, $\Gamma_\mathcal{K}$ connects two states $(s_\mathcal{K}(c_0), c_0)$ and $(1, \tilde{c})$ on different sides of Γ_2; thus, by continuity, $\Gamma_\mathcal{K}$ and Γ_2 similarly intersect.

Assume now the case $s_{1L} = s_{2K}$. Consider $\Gamma_\mathcal{K} = \{(s_\mathcal{K}(c), c)\} \subset \Omega_L$, the critical curve to the rarefaction curve $\Gamma_2 = \{(s_2(c), c)\} \subset \Omega_R$. Note that $(s_2(c^*), c^*) = (s^*, c^*) \in \mathcal{C}$, thus $(s_\mathcal{K}(c^*), c^*) = (s^*, c^*) \in \mathcal{C}$; in particular $s_\mathcal{K}(c^*) = s^*$. Moreover, similarly to the previous case, Proposition 13 implies that there exists $c_0 > c_L$, sufficiently close to c_L, such that $s_\mathcal{K}(c_0) < s_1(c_0)$. Take any point between $(s_\mathcal{K}(c_0), c_0)$ and $(s_1(c_0), c_0)$ — for example (s_0, c_0) with $s_0 := \frac{1}{2}(s_\mathcal{K}(c_0) + s_1(c_0))$, $s_\mathcal{K}(c_0) < s_0 < s_1(c_0)$ — and consider the rarefaction curve $\Gamma_0 = \{(s(c), c)\}$ that passes through the point (s_0, c_0). Due to Remark 5, $\Gamma_0 \in \Omega_L$ and intersects $c = c^*$ at some point $(\tilde{s}, c^*)$ with $\tilde{s} < s^*$. Thus, $\Gamma_\mathcal{K}$ and Γ_0 intersect and, by construction, inequality (48) is violated at the point of intersection.

Inequalities (56) are proved in a similar manner. We omit the proof. $\square$

Riemann problem solutions structures: case of zero adsorption. Although it is not the focus of this paper, let us first discuss the structure of solutions to the Riemann problem for $a(c) \equiv 0$ (zero adsorption). This eases the transition into the more complicated case and allows us to compare the layouts qualitatively. For the monotone case with zero adsorption, the solutions of the Riemann problem were originally constructed in [11] using the generalized Lax entropy condition (see also [15] and later works [13, 26]). In [21], the vanishing adsorption limit of the Riemann problem solution was analyzed. The selection principle, introduced in [21], comes from physical considerations, justifies the admissibility criteria adopted previously for the monotone case, and selects the undercompressive contact discontinuities required to solve the general Riemann problem with non-monotone dependence (see [21, formula (4.9)], which is an analogue of formula (42) for the case of zero adsorption).

Note that for the case of zero adsorption, Proposition 13 is not valid ($g \equiv 0$ due to formula (50)). Thus, formula (49) implies that if the critical curve $\Gamma_\mathcal{K}$ intersects some c-rarefaction curve $\widetilde{\Gamma}$, then they coincide. In fact, these c-rarefaction curves are contact discontinuities. Also,

$$s_{0K} = s_{0L}; \quad s_{1L} = s_{2K}; \quad s_{1K} = s_{2L}; \quad s_{3R} = s_{4K}; \quad s_{4R} = s_{3K}.$$

This simplifies the description of all possible structures of the solution to the Riemann problem as a sequence of s-wave groups and c-waves.

Depending on $(s_L, s_R) \in [0,1]^2$, we obtain a subdivision into three regions, see Fig. 11a:

(1) If $(s_L, s_R) \in \mathbf{U}_{cs}$, then

$$u_L = (s_L, c_L) \xrightarrow{c\text{-rare}} u_M = (s_M, c_R) \xrightarrow{s} u_R = (s_R, c_R). \tag{57}$$

Here $s_L \in [0, s_{1L}]$ and $s_R \in [0, \min(1, s_{\mathcal{K}}(u_M))]$.

(2) If $(s_L, s_R) \in \mathbf{U}_{sc}$, then

$$u_L = (s_L, c_L) \xrightarrow{s} u_N = (s_N, c_L) \xrightarrow{c\text{-rare}} u_R = (s_R, c_R). \tag{58}$$

Here $s_R \in [s_{4R}, 1]$ and $s_L \in [s_{\mathcal{K}}(u_N), 1]$.

(3) If $(s_L, s_R) \in \mathbf{U}_{scs}$, then

$$u_L = (s_L, c_L) \xrightarrow{s} u_- = (s_{2L}, c_L) \xrightarrow{c\text{-rare}} u_+ = (s_{3R}, c_R) \xrightarrow{s} u_R = (s_R, c_R). \tag{59}$$

Here $s_R \in [0, s_{4R}]$ and $s_L \in [s_{1L}, 1]$.

Notice that the combinations of waves (57), (58), (59) are compatible due to Proposition 11, which is also valid for the case of zero adsorption.

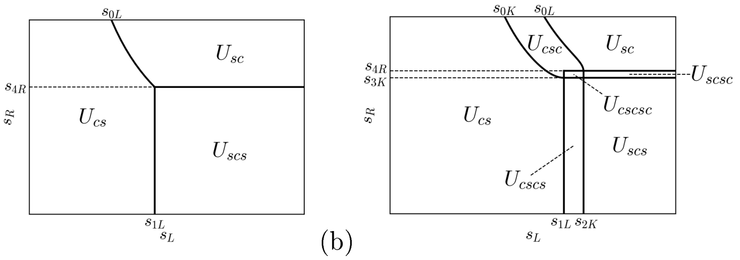
Remark 8. Notice that on the boundaries of the regions $\mathbf{U}_{cs}$, $\mathbf{U}_{sc}$, and $\mathbf{U}_{scs}$, each of the sequences (57), (58), and (59) may become degenerate. Although these sequences differ in terms of the order of s- and c-waves, the resulting solution to the Riemann problem $(s(x,t), c(x,t))$ is the same.

Riemann problem solutions structures: non-zero adsorption case. In this section, we list all possible structures of the solution to the Riemann problem for $a(c)$ satisfying (A1)–(A3) as a sequence of s-wave groups and c-waves depending on $(s_L, s_R) \in [0,1]^2$. We obtain the subdivision into 7 regions (see Fig. 11b):

Region $\mathbf{U}_{cs}$:

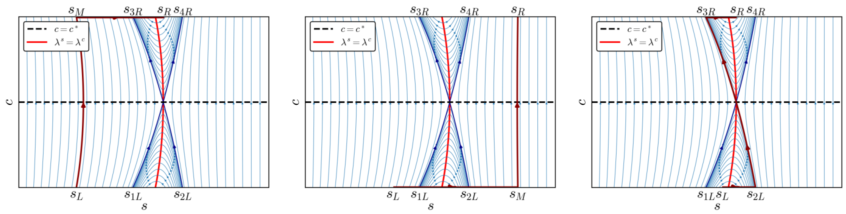
$\mathbf{U}_{cs} = \{(s_L, s_R) \in [0,1]^2 : s_L \in [0, s_{1L}]$ and $s_R \in [0, s_{\mathcal{K}}(u_M)]$
when $s_{\mathcal{K}}(u_M)$ exists, or $s_L \in [0, s_{1L}]$ and $s_R \in [0,1]$ when it does not exist;
here u_M is such that $u_L = (s_L, c_L) \xrightarrow{c\text{-rare}} u_M = (s_M, c_R)\}$.

Notice that by Remarks 4, 5 for $s_L \in [0, s_{1L})$ the rarefaction curve that passes through the state (s_L, c_L) always reaches some state on the line $\{c = c_R\}$, which we call $u_M = (s_M, c_R)$. For $s_L = s_{1L}$, we take $s_M = s_{3R}$ and the combination of two c-rarefaction curves Γ_1 and Γ_3 connects u_L



(a) (b)

Figure 11. Subdivision into regions in the (s_L, s_R)-plane with different structures of solutions in terms of the sequence of s-wave groups and c-waves:
(a) Case with zero adsorption — three regions $\mathbf{U}_{cs}, \mathbf{U}_{sc}$ and $\mathbf{U}_{scs}$.
(b) Case with non-zero adsorption — seven regions $\mathbf{U}_{cs}$, $\mathbf{U}_{sc}$, $\mathbf{U}_{scs}$, $\mathbf{U}_{csc}$, $\mathbf{U}_{scsc}$, $\mathbf{U}_{cscs}$ and $\mathbf{U}_{cscsc}$. In the limit of vanishing adsorption, $a(c) \to 0$, the regions $\mathbf{U}_{csc}$, $\mathbf{U}_{scsc}$, $\mathbf{U}_{cscs}$ and $\mathbf{U}_{cscsc}$ disappear and the diagram (b) tends to diagram (a).



(a) (b) (c)

Figure 12. (a) Representation of the sequence of waves (57) in the (s, c)-plane for $(s_L, s_R) \in \mathbf{U}_{cs}$.
(b) Representation of the sequence of waves (58) in the (s, c)-plane for $(s_L, s_R) \in \mathbf{U}_{sc}$.
(c) Representation of the sequence of waves (59) in the (s, c)-plane for $(s_L, s_R) \in \mathbf{U}_{scs}$.

and u_M. Due to Proposition 11, for any $(s_L, s_R) \in \mathbf{U}_{cs}$ the following sequence of waves (cs) provides a solution to the Riemann problem:

$$u_L = (s_L, c_L) \xrightarrow{c\text{-rare}} u_M = (s_M, c_R) \xrightarrow{s} u_R = (s_R, c_R). \tag{60}$$

Geometrically, on the plane (s_L, s_R) the region $\mathbf{U}_{cs}$ is a union of the rectangle $[0, s_{0K}] \times [0, 1]$ and a curvilinear trapezoid with three straight boundaries:

$$\{s_{0K}\} \times [0, 1]; \qquad [s_{0K}, s_{1L}] \times \{0\}; \qquad \{s_{1L}\} \times [0, s_{3K}];$$

and the fourth boundary being a continuous (strictly decreasing) curve

$$b_{cs} : s_L \in [s_{0K}, s_{1L}] \mapsto s_R \in [s_{3K}, 1], \text{ where } s_R = s_{\mathcal{K}}(u_M),$$

for u_M defined in (60). In particular, the inverse $b_{cs}^{-1} : s_R \mapsto s_L$ is well-defined.

The sequence of waves (60) degenerates into a single c-wave for $s_R = s_M$, specifically for (s_L, s_R) equal to $(0, 0)$ and (s_{1L}, s_{3R}) and a certain continuous monotone curve connecting them.

Region $\mathbf{U}_{sc}$:

$$\mathbf{U}_{sc} = \{(s_L, s_R) \in [0, 1]^2 : s_R \in [s_{4R}, 1] \text{ and } s_L \in [s_{\mathcal{K}}(u_N), 1]; \\ \text{here } u_N = (s_N, c_L) \xrightarrow{c\text{-rare}} u_R = (s_R, c_R)\}.$$

Notice that by Remark 5 for $s_R \in (s_{4R}, 1]$ the c-rarefaction curve that passes through the state (s_R, c_R) always begins at some state on the line $\{c = c_L\}$, which we call $u_N = (s_N, c_L)$. For $s_R = s_{4R}$, we take $s_N = s_{2L}$ and the combination of two c-rarefaction curves Γ_2 and Γ_4 connects u_N and u_R. Due to Proposition 11, for any $(s_L, s_R) \in \mathbf{U}_{sc}$ the following sequence of waves (sc) provides a solution to the Riemann problem:

$$u_L = (s_L, c_L) \xrightarrow{s} u_N = (s_N, c_L) \xrightarrow{c\text{-rare}} u_R = (s_R, c_R). \tag{61}$$

Geometrically, on the plane (s_L, s_R) the region $\mathbf{U}_{sc}$ is a curvilinear trapezoid with three straight boundaries:

$$[s_{2K}, 1] \times \{s_{4R}\}; \qquad \{1\} \times [s_{4R}, 1]; \qquad [s_{0L}, 1] \times \{1\};$$

and the fourth boundary being a continuous (strictly decreasing) curve

$$b_{sc} : s_R \in [s_{4R}, 1] \mapsto s_L \in [s_{0L}, s_{2K}], \text{ where } s_L = s_{\mathcal{K}}(u_N),$$

for u_N defined in (61). In particular, the inverse $b_{sc}^{-1} : s_L \mapsto s_R$ is well-defined.

The sequence of waves (61) degenerates into a single c-wave for $s_L = s_N$, specifically for (s_L, s_R) equal to $(1, 1)$ and (s_{2L}, s_{4R}) and a certain continuous monotone curve between them.

Region $\mathbf{U}_{csc}$:

$$\mathbf{U}_{csc}=\{(s_L,s_R)\in[0,1]^2 : \text{either } s_R\in[s_{4R},1] \text{ and } s_L\in[b_{cs}^{-1}(s_R),b_{sc}(s_R)]; \text{ or } s_R\in(s_{3K},s_{4R}] \text{ and } s_L\in[b_{cs}^{-1}(s_R),s_{1L}]\}.$$

We claim that there exist states $u_-, u_+ \in [0,1]^2$ such that the following sequence of waves (csc) provides a solution to the Riemann problem:

$$u_L=(s_L,c_L)\xrightarrow{c\text{-rare}}u_-\xrightarrow{s}u_+\xrightarrow{c\text{-rare}}u_R=(s_R,c_R). \tag{62}$$

Remark 9. Notice that when $(s_L,s_R)\in\mathbf{U}_{csc}\cap\mathbf{U}_{sc}$ the sequence (62) degenerates into a sequence (sc) of two waves as in (61). Similarly, when $(s_L,s_R)\in\mathbf{U}_{csc}\cap\mathbf{U}_{cs}$ the sequence (62) degenerates into a sequence (cs) of two waves as in (60). Also, for $(s_L,s_R)=(s_{1L},s_{4R})$ the solution to a Riemann problem consists of a single c-rarefaction wave that corresponds to two c-rarefaction curves $\Gamma_1\cup\Gamma_4$. In what follows, we exclude these cases from consideration.

The general idea to find the states u_- and u_+ is as follows. Consider c-rarefaction curve $\Gamma_{\mathcal{L}}$ that passes through the state (s_L,c_L), its critical curve $\Gamma_{\mathcal{K}}$ and c-rarefaction curve $\Gamma_{\mathcal{R}}$ that passes through the state (s_R,c_R). It is sufficient to prove that there exists an intersection point $u_+=(s_+,c_+)$ of $\Gamma_{\mathcal{R}}$ and $\Gamma_{\mathcal{K}}$. Then taking $u_-=(s_{\mathcal{K}}(u_+),c_+)$, the sequence of waves (62) is admissible due to Proposition 11. Let us prove that $\Gamma_{\mathcal{K}}$ and $\Gamma_{\mathcal{R}}$ intersect for the following cases separately:

1. $s_L\in(s_{0K},s_{0L})$ and $s_R\in(b_{cs}(s_L),1]$;
2. $s_L\in[s_{0L},s_{1L}]$ and $s_R\in[s_{4R},b_{sc}^{-1}(s_L))$, $(s_L,s_R)\neq(s_{1L},s_{4R})$;
3. $s_L\in[s_{0L},s_{1L}]$ and $s_R\in(b_{cs}(s_L),s_{4R})$;
4. $s_L\in(s_{1L},s_{2K})$ and $s_R\in[s_{4R},b_{sc}^{-1}(s_L))$.

Case (b) is the simplest to consider (see Fig. 13b). There exist two c-rarefaction curves $\Gamma_{\mathcal{L}}$ and $\Gamma_{\mathcal{R}}$:

$$u_L=(s_L,c_L)\xrightarrow{\Gamma_{\mathcal{L}}}u_M=(s_M,c_R); \qquad u_N=(s_N,c_L)\xrightarrow{\Gamma_{\mathcal{R}}}u_R=(s_R,c_R).$$

If $s_L=s_{1L}$, we take $\Gamma_{\mathcal{L}}=\Gamma_1\cup\Gamma_3$; if $s_R=s_{4R}$, we take $\Gamma_{\mathcal{R}}=\Gamma_2\cup\Gamma_4$. For $(s_L,s_R)\in\operatorname{int}(\mathbf{U}_{csc})$, we obtain $s_R>s_{\mathcal{K}}(u_M)$ and $s_L<s_{\mathcal{K}}(u_N)$. Due to Proposition 12, the last inequality implies $s_{\mathcal{K}}(u_L)>s_N$ as long as $s_{\mathcal{K}}(u_L)$ is well-defined (that is, for $s_L\geqslant s_{0L}$). Note that $\Gamma_{\mathcal{K}}$ is a continuous curve that connects the states

$$(s_{\mathcal{K}}(u_L),c_L)\xrightarrow{\Gamma_{\mathcal{K}}}(s_{\mathcal{K}}(u_M),c_R). \tag{63}$$

Thus, the inequalities $s_{\mathcal{K}}(u_L) > s_N$ and $s_{\mathcal{K}}(u_M) < s_R$ imply that the curves $\Gamma_{\mathcal{R}}$ and $\Gamma_{\mathcal{K}}$ have at least one intersection point (in fact, exactly one due to Proposition 13).

Case (a), see Fig. 13a. The situation here is more involved. By construction, the c-rarefaction curve $\Gamma_{\mathcal{L}}$ has exactly one intersection point with Γ_{0K} (critical curve to c-rarefaction curve $s \equiv 1$). Denote this point by $(\tilde{s}, \tilde{c})$. Thus, the critical curve $\Gamma_{\mathcal{K}}$ connects the states $(1, \tilde{c})$ and $(s_{\mathcal{K}}(u_M), c_R)$. It is clear that c-rarefaction curve $\Gamma_{\mathcal{R}}$ intersects the line $\{c = \tilde{c}\}$ at some state where $s < 1$, hence, by continuity the curves $\Gamma_{\mathcal{R}}$ and $\Gamma_{\mathcal{K}}$ intersect at some point.

Case (c), see Fig. 13c. The main difference with the case (b) is that $\Gamma_{\mathcal{R}}$ crosses the coincidence locus $\mathcal{C}$ at some point $(\tilde{s}, \tilde{c})$, and does not cross the line $\{c = c^*\}$. Notice that the critical curve $\Gamma_{\mathcal{K}} \subset \Omega_R$, thus due to (63), it intersects the line $\{c = \tilde{c}\}$ at some point in Ω_R, where $s > \tilde{s}$. The same argument as before implies that the curves $\Gamma_{\mathcal{K}}$ and $\Gamma_{\mathcal{R}}$ intersect.

Case (d), see Fig. 13d. The main difference with the case (b) is that $\Gamma_{\mathcal{L}}$ crosses the coincidence locus $\mathcal{C}$ at some point $(\tilde{s}, \tilde{c})$ and does not cross the line $\{c = c^*\}$. For $(s_L, s_R) \in \operatorname{int}(\mathbf{U}_{csc})$, we obtain $s_L < s_{\mathcal{K}}(u_N)$ or equivalently $s_{\mathcal{K}}(u_L) > s_N$. Also $\Gamma_{\mathcal{K}}$ intersects the coincidence locus at the same point $(\tilde{s}, \tilde{c})$ as the curve $\Gamma_{\mathcal{L}}$. The curve $\Gamma_{\mathcal{R}}$ intersects the segment $\{c = \tilde{c}\}$ at some point in Ω_R, where $s > \tilde{s}$. The same argument as before implies that the curves $\Gamma_{\mathcal{K}}$ and $\Gamma_{\mathcal{R}}$ intersect.

Region $\mathbf{U}_{scs}$:

$$\mathbf{U}_{scs} = \{(s_L, s_R) \in [s_{2K}, 1] \times [0, s_{3K}]\}.$$

Due to Proposition 11, the following sequence of waves (scs) provides a solution to the Riemann problem:

$$\begin{aligned} u_L = (s_L, c_L) \xrightarrow{s} u_{2L} = (s_{2L}, c_L) &\xrightarrow{c\text{-rare}} u_{3R} \\ &= (s_{3R}, c_R) \xrightarrow{s} u_R = (s_R, c_R). \end{aligned} \tag{64}$$

Here the c-rarefaction curve between u_{2L} and u_{3R} can be seen as a combination of two c-rarefaction curves Γ_2 and Γ_3:

$$u_{2L} = (s_{2L}, c_L) \xrightarrow{c\text{-rare}} (s^*, c^*) \xrightarrow{c\text{-rare}} u_{3R} = (s_{3R}, c_R). \tag{65}$$

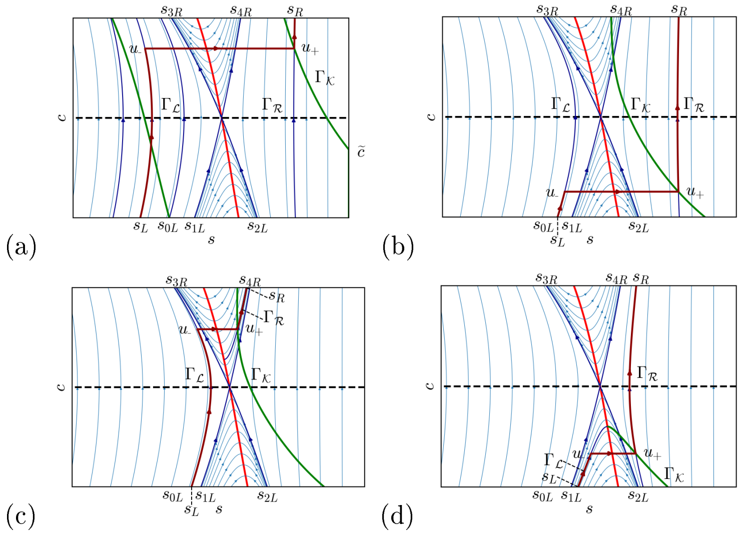


Figure 13. Schematic representation of the sequence of (csc)-waves (62) for $(s_L, s_R) \in \mathbf{U}_{csc}$, cases (a), (b), (c) and (d).

The corresponding rarefaction waves are given by formulas (36). Their concatenation gives a unique rarefaction wave that corresponds to the c-rarefaction curve between u_{2L} and u_{3R}:

$$u(x,t) = \begin{cases} u_{2L} & \text{if } \ x/t < \lambda(u_-), \\ v & \text{if } \ x/t = \lambda(v), \\ u_{3R} & \text{if } \ x/t > \lambda(u_+). \end{cases}$$

Remark 10. For $(s_L, s_R) {=} (s_{2L}, s_{3R}) {\in} \mathbf{U}_{scs}$, the sequence (64) degenerates into a single c-curve (65). Also, when $s_L {=} s_{2L}$ and $s_R {\in} [0, s_{3K}] \setminus \{s_{3R}\}$, the sequence (64) degenerates into a (cs)-sequence; when $s_R = s_{3R}$ and $s_L \in [s_{2K}, 1] \setminus \{s_{2L}\}$, the sequence (64) degenerates into a (sc)-sequence.

Region $\mathbf{U}_{cscs}$:

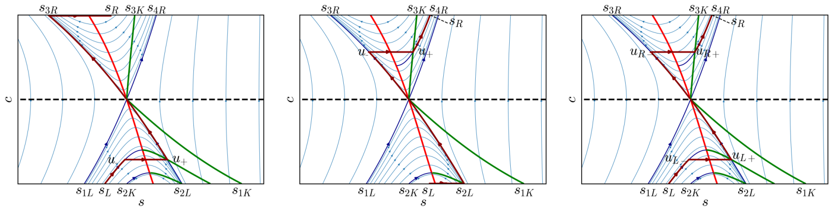
$$\mathbf{U}_{cscs} = \{(s_L, s_R) \in [s_{1L}, s_{2K}] \times [0, s_{3K}]\}.$$

There exist two states u_- and u_+ such that the following sequence of waves ($cscs$) provides a solution to the Riemann problem (see Fig. 14a):

$$u_L = (s_L, c_L) \xrightarrow{c\text{-rare}} u_- \xrightarrow{s} u_+ \xrightarrow{c\text{-rare}} (s_{3R}, c_R) \xrightarrow{s} u_R = (s_R, c_R). \quad (66)$$

Remark 11. For some pairs $(s_L, s_R) \in \mathbf{U}_{cscs}$, the sequence (66) degenerates. Specifically, when $s_L = s_{1L}$, the first (csc) sequence in the structure degenerates into a single c-wave; when $s_L = s_{2K}$, the first c-wave degenerates and disappears; finally, when $s_R = s_{3R}$, the last s-wave degenerates and disappears.

The states u_- and u_+ can be constructed as follows. Consider the c-rarefaction curve $\Gamma_{\mathcal{L}}$ that passes through the state (s_L, c_L) and its critical curve $\Gamma_{\mathcal{K}}$. As $s_L \in [s_{1L}, s_{2K}]$, $\Gamma_{\mathcal{L}}$ crosses the coincidence locus $\mathcal{C}$ at some point $(\tilde{s}, \tilde{c})$, $\tilde{c} \leqslant c^*$, and so does $\Gamma_{\mathcal{K}}$. We observe that $s_{\mathcal{K}}(u_L) \in [s_{2L}, s_{1K}]$. This implies that the curves $\Gamma_{\mathcal{K}}$ and Γ_2 intersect. We denote this point of intersection by $u_+ = (s_+, c_+)$ and take $u_- = (s_{\mathcal{K}}(u_+), c_+)$. By Proposition 11, all combinations of s and c waves are compatible by construction.



(a) (b) (c)

Figure 14. (a) Case $(s_L, s_R) \in \mathbf{U}_{cscs}$; (b) Case $(s_L, s_R) \in \mathbf{U}_{scsc}$; (c) Case $(s_L, s_R) \in \mathbf{U}_{cscsc}$.

Region $\mathbf{U}_{scsc}$:

$$\mathbf{U}_{scsc} = \{(s_L, s_R) \in [s_{2K}, 1] \times [s_{3K}, s_{4R}]\}.$$

There exist two states u_- and u_+ such that the following sequence of waves ($scsc$) provides a solution to the Riemann problem (see Fig. 14b):

$$u_L = (s_L, c_L) \xrightarrow{s} (s_{2L}, c_L) \xrightarrow{c\text{-rare}} u_- \xrightarrow{s} u_+ \xrightarrow{c\text{-rare}} u_R = (s_R, c_R). \quad (67)$$

The states u_- and u_+ can be constructed as follows. Consider c-rarefaction curve $\Gamma_{\mathcal{R}}$ that passes through the state (s_R, c_R) and the critical curve to the c-rarefaction curve Γ_3 (we denote it by $\Gamma_{3\mathcal{K}}$). As $s_R \in [s_{3K}, s_{4R}]$, $\Gamma_{\mathcal{R}}$ crosses the coincidence locus $\mathcal{C}$ at some point $(\tilde{s}, \tilde{c})$, $\tilde{c} \geqslant c^*$. This implies that the curves $\Gamma_{\mathcal{R}}$ and $\Gamma_{3\mathcal{K}}$ intersect. We denote this point of intersection by $u_+ = (s_+, c_+)$ and take $u_- = (s_{\mathcal{K}}(u_+), c_+)$. By Proposition 11, all combinations of s and c waves are compatible by construction. Similar to the previous case, some waves degenerate when $s_L = s_{2L}$, or $s_R = s_{4R}$, or $s_R = s_{3K}$.

Region $\mathbf{U}_{cscsc}$:

$$\mathbf{U}_{cscsc} = \{(s_L, s_R) \in [s_{1L}, s_{2K}] \times [s_{3K}, s_{4R}]\}.$$

There exist four states u_{L-}, u_{L+}, u_{R-} and u_{R+} such that the following sequence of waves ($cscsc$) provides a solution to the Riemann problem:

$$u_L \xrightarrow{c\text{-rare}} u_{L-} \xrightarrow{s} u_{L+} \xrightarrow{c\text{-rare}} u_{R-} \xrightarrow{s} u_{R+} \xrightarrow{c\text{-rare}} u_R. \tag{68}$$

The states u_{L-} and u_{L+} can be constructed in exactly the same way as the states u_- and u_+ for the case $\mathbf{U}_{cscs}$. The states u_{R-} and u_{R+} can be constructed exactly in the same way as the states u_- and u_+ for the case $\mathbf{U}_{scsc}$. See Fig. 14c. Similar to the previous cases, some waves degenerate at $s_L = s_{1L}$ or $s_L = s_{2K}$ and $s_R = s_{4R}$ or $s_R = s_{3K}$. We emphasize that (68) is the exact sequence of waves that contains three different c-rarefaction curves and was not described in previous works on this problem.

Acknowledgements

The authors thank Pavel Bedrikovetsky for lectures on systems of hyperbolic conservation laws, Sergey Tikhomirov for valuable advice on dynamical systems, and the reviewer for important and insightful remarks that significantly improved the paper.

References

1. F. O. Apolinário, A. P. Pires, *Oil displacement by multicomponent slug injection: An analytical solution for Langmuir adsorption isotherm.* — J. Petroleum Science and Engineering **197** (2021), 107939.
2. F. Bakharev, A. Enin, Y. Petrova,, N. Rastegaev, *Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model.* — J. Hyperbolic Differential Equations **20**, No. 02 (2023),407–432.
3. F. Bakharev, A. Enin, K. Kalinin, Y. Petrova, N. Rastegaev, S. Tikhomirov, *Optimal polymer slugs injection profiles.* — J. Comput. Appl. Math. **425** (2023),115042.
4. S. E. Buckley,, M. Leverett, *Mechanism of fluid displacement in sands.* — Transactions of the AIME **146**, No. 0 (1942), 107–116.

5. P. Castañeda, *Dogma: S-shaped.* — The Mathematical Intelligencer **38** (2016), 10–13.
6. R. Courant, *Supersonic Flow and Shock Waves: A Manual on the Mathematical Theory of Non-linear Wave Motion*, No. 62, Courant Institute of Mathematical Sciences, New York University (1944).
7. C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer (2016).
8. O. Dahl, T. Johansen, A. Tveito, R. Winther, *Multicomponent chromatography in a two phase environment.* — SIAM J. Appl. Math. **52**, No. 1 (1992), 65–104.
9. C. S. Eschenazi, W. J. Lambert, M. M. Lopez-Flores, D. Marchesin, C. F. Palmeira, B. J. Plohr, *Solving Riemann problems with a topological tool.* — J. Diff. Equations **416** (2025), 2134–2174.
10. I. M. Gelfand, *Some problems in the theory of quasilinear equations.* — Usp. Matem. Nauk **14**, No. 2 (1959), 87–158; English transl., Transactions of the American Mathematical Society **29**, No. 2 (1963), 295–381.
11. E. L. Isaacson, *Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery*, Rockefeller University, New York, NY, preprint (1981).
12. E. L. Isaacson, D. Marchesin, B. J. Plohr, *Transitional waves for conservation laws.* — SIAM J. Math. Analysis **21**, No. 4 (1990), 837–866.
13. E. L. Isaacson, J. B. Temple, *Analysis of a singular hyperbolic system of conservation laws.* — J. Diff. Equations **65**, No. 2 (1986), 250–268.
14. T. Johansen, R. Winther, *The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding.* — SIAM J. Math. Analysis **19**, No. 3 (1988), 541–566.
15. B. L. Keyfitz, H. C. Kranzer, *A system of non-strictly hyperbolic conservation laws arising in elasticity theory.* — Archive for Rational Mechanics and Analysis, **72**, No. 3 (1980), 219–241.
16. S. N. Kružkov, *First order quasilinear equations in several independent variables.* — Mathematics of the USSR-Sbornik **10**, No. 2 (1970), 217–243.
17. P. D. Lax, *Hyperbolic systems of conservation laws II.* — Communications on Pure and Applied Mathematics **10**, No. 4 (1957), 537–566.
18. S. Matveenko, N. Rastegaev, *Oil displacement by slug injection: a rigorous justification for the Jouguet principle heuristic.* arXiv preprint arXiv:2511.08533. (2025).
19. O. A. Oleinik, *Discontinuous solutions of non-linear differential equations.* — Usp. Matem. Nauk **12**, No. 3(75) (1957), 3–73; English transl., American Mathematical Society Translations **26**, No. 2 (1963), 95–172.
20. A. P. Pires, P. G. Bedrikovetsky, A. A. Shapiro, *A splitting technique for analytical modelling of two-phase multicomponent flow in porous media.* — J. Petroleum Science and Engineering **51**, No. 1-2 (2006), 54–67.
21. Y. Petrova, B. J. Plohr, D. Marchesin, *Vanishing adsorption limit of Riemann problem solutions for the polymer model.* — J. Hyperbolic Diff. Equations **21**, No. 02 (2024), 299–327.
22. N. V. Rastegaev, *On the sufficient conditions for the S-shaped Buckley–Leverett function.* — Zap. Nauchn. Semin. POMI **536** (2024), 247–260.

23. N. Rastegaev, S. Matveenko, *Kružkov-type uniqueness theorem for the chemical flood conservation law system with local vanishing viscosity admissibility.* — J. Hyperbolic Differential Equations **21**, No. 04 (2024), 1003–1043.
24. D. Serre, *Systems of Conservation Laws 1: Hyperbolicity, entropies, shock waves*, Cambridge University Press (1999).
25. W. Shen, *On the uniqueness of vanishing viscosity solutions for Riemann problems for polymer flooding.* — Nonlinear Differential Equations and Applications NoDEA **24** (2017), 1–25.
26. A. J. de Souza, D. Marchesin, *Conservation laws possessing contact characteristic fields with singularities.* — Acta Applicandae Mathematicae **51**, No. 3 (1998), 353–364.
27. D. H. Wagner, *Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions.* — J. Diff. Equations **68**, No. 1 (1987), 118–136.

Institute of Mathematics,
Statistics and Computer Science
of the University of São Paulo,
Rua do Matão, 1010, 05508-090,
São Paulo, Brazil
E-mail: `yu.pe.petrova@gmail.com`

St. Petersburg Department
of Steklov Mathematical Institute
of Russian Academy of Sciences,
27 Fontanka, 191023,
St. Petersburg, Russia
E-mail: `rastmusician@gmail.com`

Поступило 8 февраля 2026 г.