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DYNAMIC REPRESENTATION OF THE WEYL SOLUTION FOR THE SCHRÖDINGER EQUATION ON THE HALF-LINE

Abstract. We derive a representation formula for the Weyl solution to the Schrödinger equation on the half-line for certain classes of potentials. Our approach is based on relations with the initial-boundary value problem for the wave equation with the same potential on the half-line.

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§1. Introduction

We consider the Schrödinger operator

$$H=-\frac{d^2}{dx^2}+q(x), \quad x>0 \tag{1.1}$$

on $L_2(\mathbb{R}_+)$ with a real-valued locally integrable potential q.

We assume that (1.1) is the limit point case at ∞, that is, for each $z\in\mathbb{C}_+:=\{z\in\mathbb{C}:\operatorname{Im} z>0\}$ the equation

$$-u''+q\left(x\right)u=zu \tag{1.2}$$

has a unique up to multiplication by constant solution u_+ which is in L_2:

$$\int\limits_{\mathbb{R}_+}\left|u_+\left(x,z\right)\right|^2dx<\infty, \quad z\in\mathbb{C}_+.$$

Such a solution u_+ is called a *Weyl solution* and its existence is the central point of the Titchmarsh-Weyl theory.

The *Titchmarsh-Weyl m-function*, $m(z)$, is defined for $z\in\mathbb{C}_+$ as

$$m\left(z\right):=\frac{u_+'\left(0,z\right)}{u_+\left(0,z\right)}. \tag{1.3}$$

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The function $m(z)$ is analytic in $\mathbb{C}_+$ and satisfies the Herglotz property, i.e. $m : \mathbb{C}_+ \mapsto \mathbb{C}_+$.

Simon and Gesztesy in [8, 4] proposed the following representation for the Weyl function:

$$m(-k^2) = -k - \int_0^\infty A(\alpha) e^{-2\alpha k}\, d\alpha, \tag{1.4}$$

where the absolute convergence of integral was proven for $q \in L^1(\mathbb{R}_+)$ and $q \in L^\infty(\mathbb{R}_+)$. They called the function A in (1.4) the A-*amplitude*.

In [2] the authors proposed the natural physical interpretation of the A-amplitude. It is as follows: for the same potential q, the initial-value problem for the wave equation is considered:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0, \quad x > 0,\ t > 0, \\ u(x,0) = u_t(x,0) = 0,\ u(0,t) = f(t). \end{cases} \tag{1.5}$$

where f is an arbitrary $L^2_{loc}(\mathbb{R}_+)$ function referred to as a *boundary control*. The solution to (1.5) is denoted by u^f. The *response operator* (the dynamical Dirichlet-to-Neumann map) R^T for the system (1.5) is defined in $\mathcal{F}^T := L^2(0,T)$ by

$$(R^T f)(t) = u^f_x(0,t), \quad t \in (0,T),$$

with the domain $\{f \in C^2([0,T]) : \ f(0) = f'(0) = 0\}$. The response operator has the representation (see [1, 2]):

$$(R^T f)(t) = -f'(t) + \int_0^t r(s) f(t-s)\, ds,$$

where r is called the *response function*. In [2] the authors have shown that

$$m(-k^2) = -k + \int_0^\infty r(\alpha) e^{-k\alpha}, \tag{1.6}$$

$$A(\alpha) = -2r(2\alpha). \tag{1.7}$$

and the integrals in (1.4) and (1.6) converge for wider class of potentials, namely for $q \in l^\infty(L^1(\mathbb{R}_+)) := \left\{ q \mid \int_n^{n+1} |q(x)|\, dx \in l^\infty \right\}$. Formula (1.7) shows that $A-$amplitude, initially introduced in some artificial way, is in fact (taking into account the sign and scaling of coordinates) a response function, i.e., the kernel of the response operator, a classical object in the theory of dynamic inverse problems [3].

Although the authors in [2] pointed out the connection between (1.2) and (1.5) using the Laplace transform, they dealt directly with the integral equation for the $A-$amplitude and did not show that the solution to (1.5) after applying the Laplace transform is in fact the Weyl solution. This drawback can be overcome by appealing to the finite wave propagation speed in the system (1.5), as was done for the system associated with Jacobi matrices in [7, 5] (note that, nevertheless in [6] the authors proved that the corresponding solution is from l_2), but here we propose a different approach and show how to obtain the Weyl solution for (1.2) from the solution u^f of (1.5). Thus, the goal of this paper is to obtain a new representation for the Weyl solution which is interesting in itself and eliminates the drawback from [2]. Note that our results do not follow from [2], but rather the opposite.

The paper is organized as follows: in the second section we follow [1]: formulate "standard" statements on the solvability of (1.5) depending on the potential, show a representation of u^f in terms of the solution of a certain Goursat problem. We rewrite this Goursat problem as an integral equation and formulate some statements about its solvability for a potential from $L^1_{\mathrm{loc}(\mathbb{R}_+)}$. In the third section, we show that for $L^1(\mathbb{R}_+)$ potentials, a representation for the Weyl solution follows from the standard estimates from [1]. We then analyze the integral equation for the solution of the Goursat problem in the case of $q \in l^\infty\left(L^1(\mathbb{R}_+)\right)$, which also provides a representation for the Weyl solution in this case.

§2. Dynamical system.

In [1] the authors shown that the solution to (1.5) admits the following representation

$$u^f(x,t) = \begin{cases} f(t-x) + \int\limits_x^t w(x,s) f(t-s)\, ds, & x \leqslant t, \\ 0, \quad x > t. \end{cases} . \tag{2.1}$$

Where the kernel $w(w,t)$ is the unique solution to the Goursat problem:

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) + q(x) w(x,t) = 0, \quad 0 < x < t, \\ w(0,t) = 0, \; w(x,x) = -1/2 \int\limits_0^x q(s)\, ds. \end{cases} \tag{2.2}$$

For u^f they proved the following

Proposition 1. a) *If $q\in C^1(\mathbb{R}_+)$, $f\in C^2(\mathbb{R}_+)$ and $f(0)=f'(0)=0$, then the solution to* (1.5) *given by formula* (2.1) *is a classical solution to (1.5).*

b) *If $q\in L^1_{\text{loc}}(\mathbb{R}_+)$ and $f\in L^2(0,T)$, then formula* (2.1) *represents a unique generalized solution to the initial-boundary value problem* (1.5) *$u^f\in C([0,T];\mathcal{H}^T)$, where $\mathcal{H}^T=L^2(0,T)$.*

Proposition 2. a) *If $q\in L^1_{\text{loc}}(\mathbb{R}_+)$, then the generalized solution $w(x,s)$ to the Goursat problem* (2.2) *is a continuous function and*

$$|w(x,s)|\leqslant\Big(\frac{1}{2}\int_0^{\frac{s+x}{2}}|q(\alpha)|\,d\alpha\Big)\exp\Big\{\frac{s-x}{4}\int_0^{\frac{s+x}{2}}|q(\alpha)|\,d\alpha\Big\},\tag{2.3}$$

$$w_x(\cdot,s),w_s(\cdot,s),w_x(x,\cdot),w_s(x,\cdot)\in L_{1,\,loc}(\mathbb{R}_+).\tag{2.4}$$

Partial derivatives in (2.4) *continuously in $L^1_{\text{loc}}(\mathbb{R}_+)$ depend on parameters x, s. The equation in* (2.2) *holds almost everywhere and the boundary conditions are satisfied in the classical sense.*

b) *If $q\in C_{\text{loc}}(\mathbb{R}_+)$, then the generalized solution $w(x,s)$ to the Goursat problem* (2.2) *is C^1-smooth, equation and boundary conditions are satisfied in the classical sense.*

c) *If $q\in C^1_{\text{loc}}(\mathbb{R}_+)$, then the solution to the Goursat problem* (2.2) *is classical, all its derivatives up to the second order are continuous.*

We outline the scheme of the proof of the Proposition 2, which is given in [1]

By setting $\xi=s-x$, $\eta=s+x$, and

$$v(\xi,\eta)=w\Big(\frac{\eta-\xi}{2},\frac{\eta+\xi}{2}\Big),$$

equation (2.2) is reduced to

$$\begin{cases} v_{\xi\eta}-\frac{1}{4}q(\frac{\eta-\xi}{2})v=0, \quad 0<\xi<\eta,\\ v(\eta,\eta)=0,\ v(0,\eta)=-\frac{1}{2}\int\limits_0^{\eta/2}q(\alpha)\,d\alpha.\end{cases}\tag{2.5}$$

The boundary value problem (2.5) is equivalent to the integral equation

$$v(\xi,\eta)=-\frac{1}{2}\int_{\xi/2}^{\eta/2}q(\alpha)\,d\alpha-\frac{1}{4}\int_0^{\xi}d\xi_1\int_{\xi}^{\eta}d\eta_1 q\Big(\frac{\eta_1-\xi_1}{2}\Big)v(\xi_1,\eta_1).\tag{2.6}$$

We introduce a new function

$$Q(\xi,\eta)=-\frac{1}{2}\int\limits_{\xi/2}^{\eta/2} q(\alpha)\,d\alpha \tag{2.7}$$

and the operator $K: C(\mathbb{R}^2)\mapsto C(\mathbb{R}^2)$ by the rule

$$\Big(Kv\Big)(\xi,\eta)=\frac{1}{4}\int\limits_{0}^{\xi} d\xi_1\int\limits_{\xi}^{\eta} d\eta_1 q\Big(\frac{\eta_1-\xi_1}{2}\Big)v(\xi_1,\eta_1).$$

Rewriting (2.6) as

$$v=Q-Kv$$

and formally solving it by iterations, we get

$$v(\xi,\eta)=Q(\xi,\eta)+\sum_{n=1}^{\infty}(-1)^n(K^nQ)(\xi,\eta). \tag{2.8}$$

To prove the convergence of (2.8) we need suitable estimates for $|K^nQ|(\xi,\eta)$. In [1] the authors obtain such estimates for the case of $q\in L_{1\,\mathrm{loc}(\mathbb{R}_+)}$, in the third Section we refine these estimates for $q\in l^\infty(L_1(\mathbb{R}_+))$.

§3. Weyl solution representation.

For the special control $f(t)=\delta(t)$, we denote by $u^\delta(x,t)$ the fundamental solution of (1.5). From (2.1) it is clear that any solution of (1.5) has the form $u^f=u^\delta * f$.

Let $f\in C_0^\infty(0,\infty)$ and

$$\widehat{f}(k):=\int\limits_0^\infty f(t)\,e^{ikt}\,dt$$

be its Fourier transform. Function $\widehat{f}(k)$ is well defined for $k\in\mathbb{C}$. Going formally in (1.5) over to the Fourier transforms (we note that the authors in [8, 4, 2] used the Laplace transform), one has

$$\begin{cases}-\widehat{u^f_{xx}}(x,k)+q(x)\widehat{u^f}(x,k)=k^2\widehat{u^f}(x,k),\\ \widehat{u^f}(0,k)=\widehat{f}(k),\end{cases}$$

and for the response operator

$$\widehat{(Rf)}(k)=\widehat{u^f_x}(0,k)\,,$$

respectively. We note that in view of (2.1) $\widehat{u^f}(x,k) = \widehat{u^\delta}(x,k)\widehat{f}(k)$, so we see that the values of the function $\widehat{u}(0,k)$ and its first derivative at the origin, $\widehat{u}_x(0,k)$, are related through the Titchmarsh–Weyl m-function (cf. (1.3)):

$$\widehat{u^f}_x(0,k) = m(k^2)\widehat{f}(k)\,.$$

Thus the formula (1.6) is valid provided we can justify two things: taking the Fourier transform of (1.5) and that $\widehat{u^\delta}(x,k)$ is a Weyl solution, where

$$\widehat{u^\delta}(x,k) = e^{ikx} + \int\limits_0^\infty w(x,t)e^{ikt}\,dt. \tag{3.1}$$

3.1. Case of $L^1(\mathbb{R}_+)$ potentials. We note that the first term in (3.1) is in $L_2(\mathbb{R}_+)$ as soon as $\operatorname{Im} k > 0$. Therefore, we need to estimate the second term in (3.1). Using (2.3) we obtain:

$$|w(x,t)| \leqslant \frac{1}{2}\|q\|_{L^1} e^{\frac{\|q\|_{L_1}}{4}(t-x)}. \tag{3.2}$$

The last estimate implies that $\int\limits_0^\infty w(x,t)e^{ikt}\,dt$ converges as $\operatorname{Im} k > \frac{\|q\|_{L^1}}{4}$. Then from the same estimate (3.2) it follows that the second term in (3.1) is from $L^2(\mathbb{R}_+)$. Summarizing all of the above we obtain

Proposition 3. *If $q \in L^1(\mathbb{R}_+)$ then formula (3.1) yields the Weyl solution in the domain $\operatorname{Im} k > \frac{\|q\|_{L^1}}{4}$.*

3.2. Case of $l_\infty(L^1(\mathbb{R}_+))$ potentials. The following Lemma was proved in [2]:

Lemma 1. *Let $f(x)$ be a non-negative function and*

$$||f|| := \sup_{x\geqslant 0} \int\limits_x^{x+1} f(s)\,ds < \infty.$$

Then for any $a, b \geqslant 0$ and natural n

$$\int\limits_0^a (x+b)^n f(x)\,dx \leqslant \frac{(a+b+1)^{n+1}}{n+1}||f||.$$

We need to show the convergence of (2.8), to this aim we rewrite

$$Q(\xi,\eta) = -\frac{1}{2}\int\limits_{\xi/2}^{\eta/2} q(\alpha)\,d\alpha = \Big[\alpha = \frac{\gamma}{2}\Big] = -\frac{1}{4}\int\limits_{\xi}^{\eta} q\left(\frac{\gamma}{2}\right)\,d\gamma.$$

Then we introduce the notation

$$\widetilde{q}(\gamma) := q\left(\frac{\gamma}{2}\right) \tag{3.3}$$

and use Lemma 1 to estimate:

$$|Q(\xi,\eta)| \leqslant \frac{1}{4}\int\limits_{\xi}^{\eta} \left|q\left(\frac{\gamma}{2}\right)\right|\,d\gamma \leqslant \frac{1}{4}(\eta+1)\|\widetilde{q}\|.$$

The second term in (2.8) is estimated by:

$$|KQ(\xi,\eta)| \leqslant \frac{1}{4}\int\limits_{0}^{\xi} d\xi_1 \int\limits_{\xi}^{\eta} d\eta_1 \left|q\left(\frac{\eta_1-\xi_1}{2}\right)\right| \frac{1}{4}(\eta_1+1)\|\widetilde{q}\|$$

$$\leqslant [\alpha = \eta_1 - \xi_1] \leqslant \frac{\|\widetilde{q}\|}{4^2}\int\limits_{0}^{\xi} d\xi_1 \int\limits_{\xi-\xi_1}^{\eta-\xi_1} d\alpha \left|q\left(\frac{\alpha}{2}\right)\right| (\alpha+\xi_1+1)$$

$$\leqslant \frac{\|\widetilde{q}\|}{4^2}\int\limits_{0}^{\xi} d\xi_1 \int\limits_{0}^{\eta-\xi_1} d\alpha \left|q\left(\frac{\alpha}{2}\right)\right| (\alpha+\xi_1+1) \leqslant \frac{\|\widetilde{q}\|^2}{4^2}\xi\frac{(\eta+2)^2}{2},$$

where we use Lemma 1 to get the last inequality. Then the next term is estimated by

$$|K^2Q(\xi,\eta)| \leqslant \frac{1}{4}\int\limits_{0}^{\xi} d\xi_1 \int\limits_{\xi}^{\eta} d\eta_1 \left|q\left(\frac{\eta_1-\xi_1}{2}\right)\right| \frac{\|\widetilde{q}\|^2}{4^2}\xi_1\frac{(\eta_1+2)^2}{2}$$

$$\leqslant \frac{\|\widetilde{q}\|^2}{4^3}\int\limits_{0}^{\xi} d\xi_1 \int\limits_{0}^{\eta} d\eta_1 \left|q\left(\frac{\eta_1}{2}\right)\right| \xi_1\frac{(\eta_1+2)^2}{2} \leqslant \frac{\|\widetilde{q}\|^3}{4^3}\frac{\xi^2}{2}\frac{(\eta+3)^3}{3!}.$$

Using induction and Lemma 1 it is not hard to show that

$$|K^nQ(\xi,\eta)| \leqslant \frac{\|\widetilde{q}\|^{n+1}}{4^{n+1}}\frac{\xi^n}{n!}\frac{(\eta+n+1)^{n+1}}{(n+1)!}.$$

Then for the general term of (2.8) one can write an estimate using the inequality $(a+b)^n \leqslant 2^{n-1}\left(a^n+b^n\right)$:

$$\begin{aligned}|K^nQ(\xi,\eta)| &\leqslant \frac{\|\widetilde{q}\|^{n+1}}{4^{n+1}}\frac{\xi^n}{n!}\frac{(\eta+n+1)^{n+1}}{(n+1)!}\\ &\leqslant \frac{\|\widetilde{q}\|^{n+1}}{4^{n+1}}\frac{\xi^n}{n!}\frac{2^n\left(\eta^{n+1}+(n+1)^{n+1}\right)}{(n+1)!}\\ &\leqslant \frac{\|\widetilde{q}\|^{n+1}}{2^{n+2}}\left(\frac{\xi^n}{n!}\frac{\eta^{n+1}}{(n+1)!}+\frac{\xi^n}{n!}\frac{(n+1)^{n+1}}{(n+1)!}\right).\end{aligned}$$

Using the above estimate and the Stirling inequality

$$n! \geqslant \sqrt{2\pi}\left(\frac{n}{e}\right)^n,$$

and taking some $\varkappa>0$ we can proceed to the estimate of (2.8):

$$|v(\xi,\eta)| \leqslant \sum_{n=0}^{\infty}\frac{\|\widetilde{q}\|^{n+1}}{2^{n+2}}\frac{\xi^n}{n!}\frac{\eta^{n+1}}{(n+1)!}+\sum_{n=0}^{\infty}\frac{\|\widetilde{q}\|^{n+1}}{2^{n+2}}\frac{\xi^n}{n!}\frac{(n+1)^{n+1}}{(n+1)!} \tag{3.4}$$

$$\leqslant \sum_{n=0}^{\infty}\frac{\|\widetilde{q}\|\eta}{4(n+1)}\frac{\left(\frac{\xi\|\widetilde{q}\|\varkappa}{2}\right)^n}{n!}\frac{\left(\frac{\eta}{\varkappa}\right)^n}{n!}+\sum_{n=0}^{\infty}\frac{\|\widetilde{q}\|e}{4\sqrt{2\pi}}\frac{\left(\frac{\xi e\|\widetilde{q}\|}{2}\right)^n}{n!}$$

$$\leqslant \sum_{n=0}^{\infty}\frac{\|\widetilde{q}\|\eta}{4}\frac{\left(\frac{\xi\|\widetilde{q}\|\varkappa}{2}\right)^n}{n!}\frac{\left(\frac{\eta}{\varkappa}\right)^n}{n!}+\sum_{n=0}^{\infty}\frac{\|\widetilde{q}\|e}{4\sqrt{2\pi}}\frac{\left(\frac{\xi e\|\widetilde{q}\|}{2}\right)^n}{n!} \tag{3.5}$$

$$\leqslant \frac{\|\widetilde{q}\|\eta}{4}e^{\frac{\xi\|\widetilde{q}\|\varkappa}{2}}e^{\frac{\eta}{\varkappa}}+\frac{\|\widetilde{q}\|e}{4\sqrt{2\pi}}e^{\frac{\xi e\|\widetilde{q}\|}{2}}.$$

The inequality (3.5) is valid due to the fact that $\eta,\xi \geqslant 0$. Rewriting (3.4) for $w(x,t)$ we get that

$$|w(x,t)| \leqslant \frac{\|\widetilde{q}\|(x+t)}{4}e^{\frac{(t-x)\|\widetilde{q}\|\varkappa}{2}+\frac{t+x}{\varkappa}}+\frac{\|\widetilde{q}\|e}{4\sqrt{2\pi}}e^{\frac{(t-x)e\|\widetilde{q}\|}{2}}. \tag{3.6}$$

Then (3.6) implies that the Fourier transform (3.1) of u^δ exists as soon as $\operatorname{Im} k > \max\left\{\frac{e\|\widetilde{q}\|}{2}, \frac{\varkappa\|\widetilde{q}\|}{2}+\frac{1}{\varkappa}\right\}$, and that (3.1) is from $L_2(\mathbb{R}_+)$ if $\frac{\varkappa\|\widetilde{q}\|}{2} > \frac{1}{\varkappa}$. Finally we arrive at the following

Proposition 4. *If $q \in l_\infty(L^1(\mathbb{R}_+))$ then formula* (3.1) *gives the representation for the Weyl solution of the Schrödinger operator H in the region $\operatorname{Im} k > \max\left\{\frac{e\|\widetilde{q}\|}{2}, \sqrt{\frac{\|\widetilde{q}\|}{2}}\right\}$, where $\widetilde{q}$ is defined in* (3.3).

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