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SPECTRAL PROPERTIES OF THE FRÉCHET DERIVATIVES OF STRATIFIED STEADY STOKES WAVES

Abstract. We consider stratified steady water waves in a two-dimensional channel. Our main subject is the spectral properties of the Fréchet derivatives of steady water Stokes waves. One of the main results is the absence of subharmonic water waves in a neighborhood of a Stokes wave. The main assumption is formulated in terms of the eigenvalues of the Fréchet derivative evaluated at this wave and considered in the class of periodic solutions of the same period. The first eigenvalue is always negative. We show that if the second eigenvalue is positive then there are no waves with multiple periods in a neighborhood of the Stokes wave, except of this Stokes wave.

§1. Introduction

We consider stratified steady water waves in a two-dimensional channel. We use the classical formulation of the problem based on the Euler equations. The surface tension is neglected and the water motion can be rotational. The aim of this paper is to present some results on the uniqueness of water waves.

The problem of uniqueness for water waves is connected with the Benjamin–Lighthill conjecture, which suggests estimates for the main parameters of the problem, such that the volume flux, the Bernoulli constant and the flow force. In the case of an incompressible and irrotational fluid such estimates were proved in [1, 2] and [3]. For an incompressible and rotational fluid such estimates were obtained [16], where the main references in this field can be found. We also mention the papers [8, 9] and [14], where uniqueness results are proved for near critical Bernoulli constant for incompressible fluid.

Key words and phrases: water waves, vorticity, Fréchet derivative, subharmonic bifurcation, uniqueness of water waves.

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Here we study the following local uniqueness property. Assume that we have a certain periodic, even solution to the water waves problem, where the flux and the Bernoull constant are fixed. Is this solution unique in a small L_∞ neighborhood? We give certain conditions on the first eigenvalues of the Fréchet derivative, which guarantee such uniqueness for periodic waves in the class of solutions with multiple periods.

1.1. Statement of the main result. Our object of study is the class of two-dimensional stratified steady water waves traveling with constant speed c under the influence of gravity. To eliminate the dependence on time, we use a moving reference frame, where the fluid occupies a domain

$$D = D_\xi = \{(x, y) \,:\, -d < y < \xi(x),\ x \in \mathbb{R}\}$$

in the channel with the flat bottom B given by $y = -d$ and with the free surface of the flow $S = S_\xi$ given by $y = \xi(x)$. The density of the fluid ρ, defined in $\overline{D}$, is assumed to be positive and is not necessarily constant. To describe a water wave we use the velocity of the flow (u, v), the pressure P and the density ρ. Corresponding relations can be found in the paper by Walsh [18]. We recall these relations for the reader's convenience:

$$u_x + v_y = 0 \text{ in } D \text{ (incompressibility)} \tag{1.1}$$

$$(u - c)\rho_x + v\rho_y = 0 \text{ in } D \text{ (the conservation of mass)} \tag{1.2}$$

and the conservation of momentum

$$\begin{aligned} (u - c)u_x + vu_y + \frac{P_x}{\rho} &= 0 \text{ in } D, \\ (u - c)v_x + vv_y + \frac{P_y}{\rho} &= -g \text{ in } D, \end{aligned} \tag{1.3}$$

where g is the gravitational constant. The boundary conditions are

$$v = (u - c)\xi_x \text{ and } P = P_{atm} \text{ on } S, \tag{1.4}$$

where P_{atm} is the atmospheric pressure, and

$$v = 0 \text{ on } B. \tag{1.5}$$

The pseudostream function $\psi = \psi(x, y)$ is defined by

$$\psi_x(x, y) = -\sqrt{\rho}\,v(x, y),\ \ \psi_y(x, y) = \sqrt{\rho}\,(u(x, y) - c).$$

Then equations (1.1) and (1.2) are satisfied if and only if ρ is constant along stream lines of ψ. The relative pseudomass is

$$p_0 = \int_{-d}^{\xi(x)} \sqrt{\rho}(u(x,y) - c)dY$$

and it does not depend on x. From (1.2) it follows that the function ρ is constant along the stream lines. If

$$\psi_y \leqslant 0 \ \text{ in } \overline{D}, \tag{1.6}$$

then ρ can be written as

$$\rho = \rho(-\psi). \tag{1.7}$$

In what follows we will assume that $p_0 < 0$ and that the function ρ is a positive function of $-\psi$ even in the case, when ψ is not monotone with respect to y.

One can verify that the quantity

$$E := \frac{1}{2}(\psi_x^2 + \psi_y^2) + P + g\rho y \tag{1.8}$$

is constant along the stream lines of ψ. This allows to define the Bernoulli function

$$\beta(\psi) = \frac{dE}{d\psi}. \tag{1.9}$$

It can be verified that

$$\beta(\psi) = \Delta\psi - gy\rho'(-\psi) = 0 \ \text{ in } D. \tag{1.10}$$

Boundary conditions for ψ are

$$\psi(x, \xi(x)) = 0 \ \text{ and } \ \psi(x, -d) = -p_0 \tag{1.11}$$

together with the Bernoulli boundary condition

$$\frac{1}{2}|\nabla\psi|^2 + g\rho(0)(\xi(x) + d) = R, \tag{1.12}$$

which is obtained from (1.8) by setting $y = \xi(x)$ and using that $\psi(x, \xi(x)) = 0$. The constant R is called the Bernoulli constant.

We set

$$\omega(y, \psi) = -gy\rho'(-\psi) - \beta(\psi). \tag{1.13}$$

Then equation (1.10) takes the form

$$\Delta\psi + \omega(y, \psi) = 0 \ \text{ in } D. \tag{1.14}$$

In order to formulate smoothness assumptions, let us introduce some functional spaces. For $k = 0, 1, \ldots$, $\alpha \in (0,1)$ and $-\infty < a < b < \infty$ we denote by $C^{k,\alpha}(\overline{D}_{a,b})$ and $C^{k,\alpha}([a,b])$ the Hölder spaces of functions in $\overline{D}_{a,b} = \{(x,y) \in \overline{D} : a \leqslant x \leqslant b\}$ and $[a,b]$ respectively.

The space $C^{k,\alpha}_{\rm b}(\overline{D})$ consists of functions u defined on $\overline{D}$ such that

$$\sup_{a\in\mathbb{R}} ||w||_{C^{k,\alpha}(\overline{D}_{a,a+1})} < \infty.$$

The subspaces $C^{k,\alpha}_{\rm per}(\overline{D})$, $C^{k,\alpha}_{0,e,{\rm per}}(\overline{D})$ consists of Λ-periodic (Λ–periodic, even) functions in $C^{k,\alpha}_{\rm b}(\overline{D})$ vanishing for $y = -d$.

Similarly, one can define the spaces $C^{k,\alpha}_{\rm b}(\mathbb{R})$, $C^{k,\alpha}_{\rm per}(\mathbb{R})$ and $C^{k,\alpha}_{e,{\rm per}}(\mathbb{R})$.

We will assume that the density function ρ is of class $C^{2,\alpha}$ for a certain $\alpha \in (0,1)$ and the Bernoulli function β is of class $C^{1,\alpha}$. In what follows the solution $\psi \in C^{2,\alpha}_{0,e,{\rm per}}(\overline{D})$ is assumed to be fixed. We always assume that $\psi_y \neq 0$ on the free surface S_ξ, which together with (1.11) and (1.12) implies $\xi \in C^{2,\alpha}_{e,{\rm per}}(\mathbb{R})$.

Remark 1.1. The functions ρ and β are more or less arbitrary and are chosen to model some important properties of the flow under consideration. For example to model a unidirectional flow we assume that

$$u - c < 0 \ \text{ in } \overline{D}, \tag{1.15}$$

which implies

$$\psi_y < 0 \ \text{ inside } \overline{D}.$$

In this case p_0 is negative and the function ρ can be chosen to guarantee the monotonicity of the function $\rho(-\psi)$.

The next example involves a linear function ρ with respect to ψ, and a constant β, see [6]. This choice allows to include in consideration flows with counter-currents.

1.2. The Fréchet derivative. Let us present a formal derivation of the Fréchet derivative for the nonlinear operator in the problem (1.10)–(1.12) at a solution $(\widehat{\psi}, \widehat{\xi})$. We consider a small perturbation of this solution

$$\psi = \widehat{\psi} + tu, \ \ \xi = \widehat{\xi} + t\widetilde{\xi}.$$

Inserting these functions in (1.10)–(1.12) and collecting terms of order t, we get the following expressions

$$\Delta u + gy\rho''(-\widehat{\psi})u - \beta'(\widehat{\psi})u \ \text{ in } D, \tag{1.16}$$

$$\nabla\widehat{\psi}\cdot\nabla u+\partial_y|\nabla\widehat{\psi}|^2\widetilde{\xi}+g\rho(0)\widetilde{\xi}(x), \tag{1.17}$$

defined for functions subject to

$$u(x,\widehat{\xi}(x))+\widehat{\psi}_y(x,\widehat{\xi}(x)\widetilde{\xi}=0 \text{ and } u(x,-d)=0. \tag{1.18}$$

From the first relation in (1.18) we find

$$\widetilde{\xi}=-\frac{u}{\widehat{\psi}_y} \text{ for } y=\widehat{\xi}(x)$$

and rewrite the expressions (1.16) and (1.17) as

$$\begin{aligned} A(x,y,\partial_x,\partial_y)u=Au&:=\Delta u+gy\rho^{''}(-\widehat{\psi})u-\beta'(\widehat{\psi})u \text{ in } D,\\ B(x,y,\partial_x,\partial_y)u=Bu&:=\nabla\widehat{\psi}\cdot\nabla u-\sigma(x)u \text{ for } y=\widehat{\xi}(x) \end{aligned} \tag{1.19}$$

defined for functions subject to

$$u(x,-d)=0. \tag{1.20}$$

Here

$$\sigma(x)=\frac{\nabla\widehat{\psi}\cdot\partial_y\nabla\widehat{\psi}+g\rho(0)}{\widehat{\psi}_y}, \text{ where } y=\widehat{\xi}(x). \tag{1.21}$$

In what follows, we use the notation

$$\omega_*=\omega_*(x,y)=gy\rho^{''}(-\widehat{\psi}(x,y))-\beta'(\widehat{\psi}(x,y)). \tag{1.22}$$

Our smoothness assumptions lead to

$$\omega_*\in C^{0,\alpha}_{0,e,\mathrm{per}}(\overline{\Omega}) \text{ and } \sigma\in C^{0,\alpha}_{0,e,\mathrm{per}}(\mathbb{R}).$$

There are several spectral problems connected with the Fréchet derivative. The first one, which is the main subject of the paper, is the following

$$\begin{aligned} &((\partial_x+z)^2+\partial_y^2)u+\omega_*(x,y)u=0 \text{ in } D,\\ &\nabla\widehat{\psi}\cdot(\partial_x+z,\partial_y)u-\sigma(x)u=0 \text{ for } y=\widehat{\xi}(x), \end{aligned} \tag{1.23}$$

where $z\in\mathbb{C}$ is a spectral parameter. Here the function u is Λ-periodic and subject to (1.20). We do not assume that the function u is even in this spectral problem. If (z,u) solves the problem then the pair $(z+ik\tau_*, e^{-k\tau_* x}u)$, where $\tau_*=2\pi/\Lambda$ and k is an integer, is also a solution to this problem. The spectral parameter z is usually called quasimomentum. This problem is important in study of asymptotic behaviour of solutions of boundary value problem with periodic coefficients (see [17]) and corresponding non-linear problems (see [7])). The problem (1.23) appears when we are looking for the Floquet solutions

$$U(x,y)=e^{zx}\sum_{k=0}^{N}a_k(x,y)x^k$$

of the problem

$$\begin{aligned}&(\partial_x^2+\partial_y^2)U+\omega_*(x,y)U=0 \ \text{ in } D,\\ &\nabla\widehat{\psi}\cdot\nabla U-\sigma(x)U=0 \ \text{ for } y=\widehat{\xi}(x),\\ &U(x,-d)=0.\end{aligned}\tag{1.24}$$

Here a_k are Λ–periodic coefficients.

The second spectral problem is

$$\begin{aligned}&\Delta u+\omega_*(x,y)u+\mu u=0 \ \text{ in } D,\\ &\nabla\widehat{\psi}\cdot\nabla u-\sigma(x)u=0 \ \text{ for } y=\widehat{\xi}(x),\\ &u(x,-d)=0,\end{aligned}\tag{1.25}$$

This problem is equivalent to the spectral problem for the Fréchet derivative after application of partial hodograph transformation, which can be used for unidirectional flows, i.e. $\psi_y\neq 0$ inside $\overline{D}$, see Sect. 2.1 and Sect. 2.2.

Finally, the third spectral problem has the form

$$\begin{aligned}&\Delta u+\omega_*(x,y)u=0 \ \text{ in } D,\\ &\nabla\widehat{\psi}\cdot\nabla u-\sigma(x)u=\theta u \ \text{ for } y=\widehat{\xi}(x),\\ &u(x,-d)=0.\end{aligned}\tag{1.26}$$

This problem appears in the case $\psi_y\neq 0$ on S, see Sec. 2.3. Both problems (1.25) and (1.26) are considered for Λ-periodic, even functions. Usually the Fréchet derivatives are defined for operators acting in fixed spaces. In order to define the Fréchet derivatives for operators connected to a free boundary value problem it is useful to reduce it to a problem defined in a fixed domain. It can be done by using a flattening change of variables. We consider two cases. The first one serves for unidirectional flows, i.e. $\widehat{\psi}_y<0$ in $\overline{D}$. In this case we can use the partial hodograph transformation. The second case covers the problems when $\widehat{\psi}_y(x,\widehat{\xi}(x))<0$.

1.3. Results. In Sect. 2.4 we prove that the spectral problems (1.25) and (1.26) have the same number of negative eigenvalues. Namely, let

$$\mathbf{a}(u,v)=\int_\Omega (\nabla u\cdot\nabla\overline{v}-\omega_* u\overline{v})dxdy-\int_{-\Lambda/2}^{\Lambda/2}\sigma(x)u(x,\widehat{\xi}(x))\overline{v}(x,\widehat{\xi}(x))\frac{dx}{\widehat{\psi}_y},\quad (1.27)$$

where

$$\Omega=\{(x,y)\,:\,x\in(-\Lambda/2,\Lambda/2),\ y\in(-d,\widehat{\xi}(x))\}.\qquad (1.28)$$

We assume that

$$a(u,u)>0\ \text{ for all nonzero } u\in\widehat{H}^1(\Omega),\qquad (1.29)$$

where $\widehat{H}^1(\Omega)$ consists of functions u in $H^1(\Omega)$ such that $u(-\Lambda/2,y)=u(\Lambda/2,y)$ and $u(x,-d)=u(x,\widehat{\xi}(x))=0$.

Inequality (1.29) is true in the case of a unidirectional flow or a flow with constant vorticity and linear density.

Proposition 1.2. *Assume that* (1.29) *is valid. Let a and b be continuous, even, positive and Λ-periodic functions defined in $\overline{D}$ and in $\mathbb{R}$ respectively. Consider the spectral problems*

$$\begin{aligned}&Au+\mu au=0\ \textit{ in } \Omega,\\&Bu=0\ \textit{ for } y=\xi(x),\\&u=0\ \textit{ for } p=-d\end{aligned}\qquad (1.30)$$

and

$$\begin{aligned}&Au=0\ \textit{ in } \Omega,\\&Bu=\theta bu\ \textit{ for } y=\xi(x),\\&u=0\ \textit{ for } y=-d,\end{aligned}\qquad (1.31)$$

where μ and θ are spectral parameters and u is an even, Λ–periodic function. Then these problems have the same number of negative eigenvalues (*accounting for their multiplicities*). *Moreover, if we consider these spectral problems for $M\Lambda$-periodic, even functions, then they have the same number of negative eigenvalues also.*

The problem (1.25), considered for even, Λ–periodic functions, is a spectral problem for self-adjoint operator and its spectrum consists of isolated eigenvalues of finite multiplicity, which are uniformly bounded from below. We numerate the eigenvalues in non-decreasing order

$$\mu_1\leqslant\mu_2\leqslant\ldots,\qquad (1.32)$$

taking into account the multiplicities of the eigenvalues.

It was proved in Lemma 2.5 that the first eigenvalue μ_1 is always negative. The main our assumption is

$$\text{the second eigenvalue } \mu_2 \text{ is positive.} \tag{1.33}$$

By Proposition 1.2 both spectral problems (1.25) and (1.26) has one negative eigenvalue and the second eigenvalue is positive. Therefore we choose the spectral problem (1.25) as the basic auxiliary problem for investigating spectrum of (1.23). We put

$$\tau_* = \frac{2\pi}{\Lambda}. \tag{1.34}$$

Our main concern is the spectral problem (1.23). The main result about the spectral problem (1.23) is the following

Theorem 1.3. *Let (1.33) be valid and $\widehat{\psi}_x \neq 0$ in*

$$\Omega_+ = \{(x, y) \,:\, x \in (0, \Lambda/2),\ y \in (-d, \widehat{\xi}(x))\}. \tag{1.35}$$

Then the set $z = i\tau$, $\tau \in [0, \tau_*)$, contains the only eigenvalue $z = 0$ of the problem (1.23) *which is simple with the odd eigenfunction $\widehat{\psi}_x$.*

The proof is given in Section 2.6. The next important corollary is proved in Sect. 2.7.

Corollary 1.4. *Consider the problem* (1.10)–(1.12) *in the space of $m\Lambda$-periodic functions, where m is an odd integer. Then there are no $m\Lambda$ periodic, even solutions in a small neighborhood of $(\widehat{\psi}, \widehat{\xi})$, except $(\widehat{\psi}, \widehat{\xi})$.*

§2. Proofs

In Sections 2.1, 2.2 and 2.3 we prove an equivalence of the formal Fréchet derivative, constructed in Section 1.2, with the real Fréchet derivatives, which are calculated after changes of variables. We consider two changes of variables: the partial hodograph transformation, which can be applied when $\psi_y < 0$ in $\overline{D}$, and a flattening the domain applying in the case when $\psi_y \neq 0$ on S. It appears that corresponding spectral problems has a spectral parameter in different places, the first in the equation of the problem and the second in the boundary condition. In Section 2.4 we show that these problems have the same number of negative eigenvalues. The proof of Theorem 1.3 is based on the estimates of eigenvalues of spectral problems for the Fréchet derivatives obtained in Section 2.5. Proofs of Theorem 1.3 and Corollary 1.4 are presented in Sections 2.6 and 2.7, respectively. Finally, in Section 2.8 we discuss generalized eigenfunctions corresponding to the eigenvalue $z = 0$ of the spectral problem (1.23).

2.1. Partial hodograph transformation. Here we assume that

$$\widehat{\psi}_y < 0 \quad \text{in } \overline{D} \tag{2.1}$$

and we will use the notation ψ instead of $\widehat{\psi}$ Let

$$q = x, \ \ p = -\psi(x,y), \ \ (x,y) \in \overline{D}.$$

Then

$$(q,p) \in \overline{Q}, \ \ Q = \{q \in \mathbb{R}, \ p \in (p_0, 0)\}.$$

We put

$$h(q,p) = d + y.$$

Then

$$\psi_y = -\frac{1}{h_p}, \ \ \psi_x = \frac{h_q}{h_p}.$$

Applying this change of variables, we arrive at

$$\begin{aligned}
&F(h) := \Big(\frac{1+h_q^2}{2h_p^2}\Big)_p - \Big(\frac{h_q}{h_p}\Big)_q + g(h-d)\rho_p - \beta(-p) = 0 \ \text{ in } Q,\\
&G(h) := \frac{1+h_q^2}{2h_p^2} + gh\rho - R = 0 \ \text{ for } p = 0,\\
&h(q,p_0) = 0 \ \ q \in \mathbb{R}.
\end{aligned} \tag{2.2}$$

From our smoothness assumptions, formulated at the end of Section 1.1, it follows that h is an even, Λ–periodic function from the space $C_{\mathrm{b}}^{2,\alpha}(\overline{Q})$ with

$$\delta \leqslant h_p(q,p) \ \text{ on } \overline{Q} \text{ and } \rho \in C^{1,\alpha}([p_0,0]),\ \beta \in C^{0,\alpha}([p_0,0]), \tag{2.3}$$

where δ is a positive constant. The nonlinear operators in (2.5) is continuous in the following spaces

$$(F,G) \, : \, C^{2,\alpha}_{0,e,\mathrm{per}}(\overline{Q}) \to C^{0,\alpha}_{0,e,\mathrm{per}}(\overline{Q}) \times C^{1,\alpha}_{e,\mathrm{per}}(\mathbb{R}).$$

Here we consider only positive functions h_p.

Now the spaces for the operators F and G are fixed and the Fréchet derivatives of the operators in (2.5) are defined in a usual way;

$$\begin{aligned}
&\mathcal{F}w := \Big(\frac{h_q w_q}{h_p^2} - \frac{(1+h_q^2)w_p}{h_p^3}\Big)_p - \Big(\frac{w_q}{h_p} - \frac{h_q w_p}{h_p^2}\Big)_q + gw\rho_p \text{ in } Q,\\
&\mathcal{G}w := \frac{h_q w_q}{h_p^2} - \frac{(1+h_q^2)w_p}{h_p^3} + g\rho w \text{ for } p=0,\\
&w(q,p_0) = 0 \;\; q\in\mathbb{R}, \qquad (2.4)
\end{aligned}$$

where $w \in C^{2,\alpha}_{0,e,\mathrm{per}}(\overline{Q})$.

2.2. The Fréchet derivative (2.4) in (x,y) variables. We start from a function u depending on $(x,y)\in\overline{D}$ and put

$$w(q,p) = u(q,h(q,p))h_p(q,p). \qquad (2.5)$$

To evaluate $\mathcal{F}w$ and $\mathcal{G}w$, we note first that

$$w_q = u_x h_p + u_y h_q h_p + u h_{qp}, \;\; w_p = u_y h_p^2 + u h_{pp}.$$

Therefore

$$\begin{aligned}
I_1 &:= \frac{h_q w_q}{h_p^2} - \frac{(1+h_q^2)w_p}{h_p^3}\\
&= \frac{h_q(u_x h_p + u_y h_q h_p + u h_{qp})}{h_p^2} - \frac{(1+h_q^2)(u_y h_p^2 + u h_{pp})}{h_p^3}\\
&= u_x\frac{h_q}{h_p} - u_y\frac{1}{h_p} + \Big(\frac{1+h_q^2}{2h_p^2}\Big)_p u\\
&= \psi_x u_x + \psi_y u_y - \frac{1}{2\psi_y}(\psi_x^2+\psi_y^2)_y u
\end{aligned} \qquad (2.6)$$

and

$$\begin{aligned}
I_2 := \frac{(u_x h_p + u_y h_q h_p + u h_{qp})}{h_p} - \frac{h_q(u_y h_p^2 + u h_{pp})}{h_p^2} &= u_x + u\Big(\frac{h_q}{h_p}\Big)_p\\
&= u_x + h_p\psi_{xy}u.
\end{aligned}$$

This implies

$$\begin{aligned}
I_{1p} - I_{2q} = &-\frac{1}{\psi_y}\Big(\psi_x u_x + \psi_y u_y - \frac{u}{\psi_y}(\psi_x\psi_{xy} + \psi_y\psi_{yy})\Big)_y\\
&- (\partial_x - \frac{\psi_x}{\psi_y}\partial_y)(u_x - \frac{\psi_{xy}}{\psi_y}u).
\end{aligned}$$

Taking all terms containing u, we get

$$\frac{u}{\psi_y}\Big(\frac{\psi_x\psi_{xy}+\psi_y\psi_{yy}}{\psi_y}\Big)_y+u(\partial_x-\frac{\psi_x}{\psi_y}\partial_y)\Big(\frac{\psi_{xy}}{\psi_y}\Big)=\frac{u}{\psi_y}(\psi_{xx}+\psi_{yy})_y.$$

Taking terms containing the first derivatives of u, we obtain

$$-\frac{1}{\psi_y}\Big(\psi_{xy}u_x+\psi_{yy}u_y-\frac{u_y}{\psi_y}(\psi_x\psi_{xy}+\psi_y\psi_{yy})\Big)+\frac{\psi_{xy}}{\psi_y}(\partial_x-\frac{\psi_x}{\psi_y}\partial_y)u=0.$$

Finally, collecting all terms containing second derivatives of u, we get

$$-\frac{1}{\psi_y}\Big(\psi_x u_{xy}+\psi_y u_{yy}\Big)-(\partial_x-\frac{\psi_x}{\psi_y}\partial_y)u_x=-u_{xx}-u_{yy}.$$

Using these calculations, we obtain

$$I_{1p}-I_{2q}=\frac{u}{\psi_y}(\psi_{xx}+\psi_{yy})_y-u_{xx}-u_{yy}. \tag{2.7}$$

Now we can give a connection between the Fréchet derivatives (1.19) and (2.4).

Lemma 2.1. *Let ψ and ξ satisfy equations* (1.10) *and* (1.12). *Let also F and g continuous, even and Λ-periodic functions. If $u=u(x,y)$ satisfies the problem*

$$\begin{aligned} Au&=f,\\ Bu&=g, \end{aligned} \tag{2.8}$$

where the operators A and B are defined by (1.19), *then the function w given by* (2.5) *satisfies the equations*

$$\begin{aligned} &\mathcal{F}w=-f(q,h),\\ &\mathcal{G}w=g \ \ \textit{for } p=0,\\ &w=0 \ \ \textit{for } p=p_0. \end{aligned} \tag{2.9}$$

Proof. Using (2.7), we get

$$\mathcal{F}w=\frac{u}{\psi_y}(\psi_{xx}+\psi_{yy})_y-u_{xx}-u_{yy}-gw\rho_p \tag{2.10}$$

Differentiating (1.14) with respect to y, we obtain

$$\Delta\psi_y-g\rho'(-\psi)+g\rho^{''}(-\psi)\psi_y-\beta'(\psi)\psi_y=0,$$

which allows to rewrite (2.10) as

$$\mathcal{F}w=\frac{u}{\psi_y}(g\rho'(-\psi)-gy\rho^{''}(-\psi)\psi_y+\beta'(\psi)\psi_y)-u_{xx}-u_{yy}+gw\rho_p.$$

Thus
$$\mathcal{F}w = u(-gy\rho''(-\psi) + \beta'(\psi)) - u_{xx} - u_{yy}.$$
This proves the first relation in (2.9).

Furthermore,
$$\mathcal{G}w = I_1 + g\rho w = \nabla\psi \cdot \nabla u - \sigma u = g.$$
This proves the second relation in (2.9). □

Remark 2.2. This lemma implies: if
$$f = -\mu u(q,h) \text{ and } g = 0 \text{ then}$$
in (2.8), then
$$\mathcal{F}w = \mu \frac{w}{h_p},$$
$$\mathcal{G}w = 0 \text{ for } p = 0.$$

Thus we obtain a connection between spectral problems for the Fréchet derivatives in (x,y) and (q,p) variables.

2.3. Flattening the boundary in the second case. In the case $\psi_y \neq 0$ only on S we cannot apply the partial hodograph change of variables. In this case we use the following flattening change of variables
$$\widehat{x} = x, \quad \widehat{y} = \frac{d(y+d)}{\xi(x)+d},$$
to reduce the problem to a fix strip
$$Q = \{(\widehat{x},\widehat{y}) : \widehat{x} \in \mathbb{R}, \ 0 < \widehat{y} < d\}.$$
Since
$$\partial_x = \partial_{\widehat{x}} - \frac{\widehat{y}\xi'}{\xi+d}\partial_{\widehat{y}}, \quad \partial_y = \frac{d}{\xi+d}\partial_{\widehat{y}},$$
where $'$ means $\partial_{\widehat{x}}$, the equations (1.14) and (1.12) takes the form
$$\begin{aligned}
F(\widehat{\psi},\xi) &:= \Big(\Big(\partial_{\widehat{x}} - \frac{\widehat{y}\xi'}{\xi+d}\partial_{\widehat{y}}\Big)^2 + \Big(\frac{d}{\xi+d}\partial_{\widehat{y}}\Big)^2\Big)\widehat{\psi} + \widehat{\omega}(\widehat{y},\widehat{\psi}) = 0 \text{ in } Q,\\
G(\widehat{\psi},\xi) &:= \frac{1}{2}\Big(\Big|\Big(\partial_{\widehat{x}} - \frac{\widehat{y}\xi'}{\xi+d}\partial_{\widehat{y}}\Big)\widehat{\psi}\Big|^2 + \Big|\frac{d}{\xi+d}\partial_{\widehat{y}}\widehat{\psi}\Big|^2\Big) \\
&\quad + g\rho(0)(\xi(x)+d) - R = 0 \text{ for } \widehat{y} = d,\\
\widehat{\psi} &= 0 \text{ for } \widehat{y} = d,
\end{aligned} \tag{2.11}$$

$$\widehat{\psi} = -p_0 \quad \text{for } \widehat{y} = 0, \tag{2.12}$$

where

$$\widehat{\psi}(\widehat{x}, \widehat{y}) = \psi\Big(\widehat{x}, \frac{\widehat{y}(\xi(\widehat{x}) + d)}{d} - d\Big) \quad \text{or} \quad \psi(x, y) = \widehat{\psi}(\widehat{x}, \widehat{y})$$

and

$$\widehat{\omega}(\widehat{y}, \widehat{\psi}) = \omega(y, \psi(x, y)) = \omega\Big(\frac{\widehat{y}(\xi(\widehat{x}) + d)}{d} - d, \widehat{\psi}(\widehat{x}, \widehat{y})\Big).$$

Then the problem (2.11) is equivalent to

$$(F(\widehat{\psi}, \xi), G(\widehat{\psi}, \xi)) = 0,$$

which is defined on Λ–periodic, even functions from $C^{2,\alpha}(Q) \times C^{2,\alpha}(\mathbb{R})$ satisfying $\widehat{\psi}(\widehat{x}, 0) = -p_0$, $\widehat{\psi}(\widehat{x}, d) = 0$ and $\xi + d > 0$.

We calculate the Fréchet derivative at $(\widehat{\psi}, \xi)$:

$$\begin{aligned}\mathcal{F}(u, \zeta) &:= \partial_t F(\widehat{\psi} + tu, \xi + t\zeta)|_{t=0} \\ &= \Big(\Big(\partial_{\widehat{x}} - \frac{\widehat{y}\xi'}{\xi + d}\partial_{\widehat{y}}\Big)^2 + \Big(\frac{d}{\xi + d}\partial_{\widehat{y}}\Big)^2\Big)u + \widetilde{\omega} \\ &\quad - \Big(\frac{\zeta}{\xi + d}\Big)'\widehat{y}\partial_{\widehat{y}}\Big(\partial_{\widehat{x}} - \frac{\xi'}{\xi + d}\widehat{y}\partial_{\widehat{y}}\Big)\widehat{\psi} - \Big(\partial_{\widehat{x}} - \frac{\widehat{y}\xi'}{\xi + d}\partial_{\widehat{y}}\Big)\Big(\frac{\zeta}{\xi + d}\Big)'\widehat{y}\partial_{\widehat{y}}\widehat{\psi} \\ &\quad - 2\frac{d^2\zeta}{(\xi + d)^3}\partial_{\widehat{y}}^2\widehat{\psi},\end{aligned} \tag{2.13}$$

where

$$\widetilde{\omega} = \omega_\psi(y, \psi)u + (\omega_y + \omega_\psi\psi_y)\widehat{y}\frac{\zeta}{d},$$

and

$$\begin{aligned}\mathcal{G}(u, \zeta) &:= \partial_t G(\widehat{\psi} + tu, \xi + t\zeta)|_{t=0} \\ &= \Big(\partial_{\widehat{x}}\widehat{\psi} - \frac{\widehat{y}\xi'}{\xi + d}\partial_{\widehat{y}}\widehat{\psi}\Big)\Big(\partial_{\widehat{x}}u - \frac{\widehat{y}\xi'}{\xi + d}\partial_{\widehat{y}}u\Big) \\ &\quad + \frac{d^2}{(\xi + d)^2}\partial_{\widehat{y}}\widehat{\psi}\partial_{\widehat{y}}u + g\rho(0)\zeta - \Big(\partial_{\widehat{x}}\widehat{\psi} - \frac{\widehat{y}\xi'}{\xi + d}\partial_{\widehat{y}}\widehat{\psi}\Big)\Big(\frac{\zeta}{\xi + d}\Big)'\widehat{y}\partial_{\widehat{y}}\widehat{\psi} - \frac{d^2\zeta}{(\xi + d)^3}\partial_{\widehat{y}}\widehat{\psi} \\ &= \psi_x u_x + \psi_y u_y + g\rho(0)\zeta - \psi_x\psi_y\Big(\frac{\zeta}{\xi + d}\Big)_x(\xi + d) - \psi_y^2\frac{\zeta}{\xi + d}.\end{aligned} \tag{2.14}$$

Here $u = 0$ for $\widehat{y} = 0$ and $\widehat{y} = d$.

Let us introduce the transformation

$$v(x,y) = u(\widehat{x},\widehat{y}) - \psi_y(x,y)\frac{(y+d)\zeta}{\xi+d}. \tag{2.15}$$

Lemma 2.3. (i) *Assume that the functions ψ and ξ satisfy* (1.14) *in the domain D_ξ. If the function v is given by* (2.15) *then*

$$(\partial_x^2+\partial_y^2)v + \omega_* v = \mathcal{F}(u,\zeta),$$

where ω_* is defined by (1.22).

(ii) *Furthermore,*

$$\psi_x v_x + \psi_y v_y + \widehat{\sigma}\zeta = \mathcal{G}(u,\zeta) \quad \text{on } \mathcal{S}_\xi,$$

where

$$\widehat{\sigma} = \psi_x\psi_{xy} + \psi_y\psi_{yy} + g\rho(0).$$

Proof. (i) Using relations (2.13) and (1.14), we get

$$\begin{aligned}
&(\partial_x^2+\partial_y^2)v + \omega_* v = \Big(\Big(\partial_{\widehat{x}} - \frac{\widehat{y}\xi'}{\xi+d}\partial_{\widehat{y}}\Big)^2 + \Big(\frac{d}{\xi+d}\partial_{\widehat{y}}\Big)^2)\Big)u(\widehat{x},\widehat{y}) + \widetilde{\omega}\\
&-\Big(\partial_x^2+\partial_y^2+\omega_*\Big)\Big(\psi_y(x,y)\frac{(y+d)\zeta}{\xi+d}\Big)\\
&= \Big(\Big(\partial_{\widehat{x}} - \frac{\widehat{y}\xi'}{\xi+d}\partial_{\widehat{y}}\Big)^2 + \Big(\frac{d}{\xi+d}\partial_{\widehat{y}}\Big)^2)\Big)u(\widehat{x},\widehat{y}) + \widetilde{\omega}\\
&-\frac{(y+d)\zeta}{\xi+d}\Big(\partial_x^2+\partial_y^2+\omega_*\Big)\psi_y(x,y) - I,
\end{aligned}$$

where

$$I = \Big(2(y+d)\psi_{xy}\Big(\frac{\zeta}{\xi+d}\Big)' + (y+d)\psi_y\Big(\frac{\zeta}{\xi+d}\Big)''\Big) + 2\psi_{yy}\frac{\zeta}{\xi+d}.$$

Comparing this with the second line in (2.13), we arrive at the assertion (i).

(ii) We have

$$\begin{aligned}
&\psi_x v_x + \psi_y v_y + \widehat{\sigma}\zeta = \psi_x u_x + \psi_y u_y + \widehat{\sigma}\zeta\\
&-(\psi_x\psi_{xy}+\psi_y\psi_{yy})\zeta - \psi_x\psi_y(y+d)\Big(\frac{\zeta}{\xi+d}\Big)' - \psi_y^2\frac{\zeta}{\xi+d}\\
&= \psi_x u_x + \psi_y u_y + g\rho(0)\zeta - \psi_x\psi_y(y+d)\Big(\frac{\zeta}{\xi+d}\Big)' - \psi_y^2\frac{\zeta}{\xi+d}.
\end{aligned}$$

This together with (2.14) leads to the required proof of (ii). $\square$

Corollary 2.4. *Let the functions ψ and ξ satisfy the first equation in* (2.11) *in the domain Q. Assume that u and ζ satisfy*

$$\begin{aligned}
&\mathcal{F}(u,\zeta) = 0 \ \ \textit{in } Q, \\
&\mathcal{G}(u,\zeta) = \mu b\zeta \ \ \textit{for } y = d, \\
&u = 0 \ \ \textit{for } y = d.
\end{aligned} \tag{2.16}$$

Then the functions v and ζ satisfy

$$\begin{aligned}
&Av = 0 \ \ \textit{in } D_\xi, \\
&Bv = \mu bv \ \ \textit{on } S_\xi, \\
&v = 0 \ \ \textit{for } y = 0
\end{aligned} \tag{2.17}$$

and

$$\zeta = -v/\psi_y \ \ \textit{on } S_\xi.$$

2.4. Proofs of Proposition 1.2 and the negativity of μ_1. Let us prove Proposition 1.2.

Proof. The problems (1.31) and (1.30) present eigenvalue problems for self-adjoint operators and their spectrum consist of eigenvalues of finite multiplicity with the only accumulation point at ∞. Introduce spaces

$$\mathcal{X} = \{u \in H^1(\Omega) \,:\, u(-\Lambda/2, y) = u(\Lambda/2, y) \ \text{ and } \ u(x, -d) = 0\}$$

and

$$\mathcal{Y} = \{u \in H^1(\Omega) \,:\, u(-\Lambda/2, y) = u(\Lambda/2, y), \ Au = 0 \ \text{ and } \ u(x, -d) = 0\}.$$

Denote by X the finite dimensional subspace in $\mathcal{X}$ generating by the eigenfunctions corresponding to negative eigenvalues of the problem (1.30). Then $\dim X$ coincides with the number of negative eigenvalues counting together with their multiplicities. This number also coincides with the dimension of a largest subspace where the form $\mathbf{a}$ is negative. The same property is true for the subspace Y consisting of eigenfunctions corresponding to the negative eigenvalues of the problem (1.31). This implies that

$$\dim X \geqslant \dim Y.$$

To prove the opposite inequality, we proceed as follows. First, we note that

$$\mathbf{a}(u, v) = 0 \ \text{ for } u \in \mathcal{Y} \text{ and } v \in \widehat{H}^1(\Omega).$$

For each $w \in \mathcal{X}$ we can find unique $\widetilde{w} \in \mathcal{Y}$ such that $w = \widetilde{w}$ for $y = \xi(x)$. Therefore $\widehat{w} := w - \widetilde{w} \in \widehat{H}^1(\Omega)$. Since

$$\mathbf{a}(\widetilde{w} + \widehat{w}, \widetilde{w} + \widehat{w}) = \mathbf{a}(\widetilde{w}, \widetilde{w}) + \mathbf{a}(\widehat{w}, \widehat{w}) < 0 \ \text{ if } w \neq 0$$

and $\mathbf{a}(\widehat{w}, \widehat{w}) \geqslant 0$ by (1.29), we have that $\mathbf{a}(\widetilde{w}, \widetilde{w}) < 0$, which prove the opposite inequality. □

Let $\mu_1 \leqslant \mu_2 \leqslant \dots$ be the eigenvalues of the spectral problem (1.25) considered for even, Λ–periodic functions, introduced by (1.32).

Lemma 2.5. *Let $\psi_x \neq 0$ in Ω_+. Then the eigenvalue μ_1 is negative and simple and the corresponding eigenfunction is positive in Ω.*

Proof. The simplicity of the lowest eigenvalue and positivity (up to the sign) of the corresponding eigenfunction w_1 are quite standard facts.

Let $v = \psi_x$. By differentiating (1.25) with respect to x one can verify that the function v satisfies the problem (1.10), (1.12) with $\mu = 0$. The function v is odd and hence $v(-\Lambda/2, y) = v(\Lambda/2, y) = v(x, -d) = 0$ for $y \in (-d, \xi(\Lambda/2))$. This implies that $\mu_1 \leqslant 0$. In the case $\mu_1 = 0$ we get that $v = cw_1$, which implies that $w_1 = 0$. Hence $\mu_1 < 0$.

□

Remark 2.6. Let us discuss the assumption $\psi_x \neq 0$ in Ω_+. Usually large amplitude solutions are constructed as elements of a continuous (analytic) branch of solutions (ψ, ξ) of (1.10)–(1.12) depending on a parameter t and starting from a laminar flow. If the laminar flow satisfies $\psi_x \neq 0$ inside Ω_+ then the same is true for all elements on the branch. This fact is well-known, see for example [4, 5, 13] and [19].

2.5. Estimates of eigenvalues. Here we study eigenvalues of the spectral problem (1.25) in the case when the function w is not necessary Λ-periodic and even. These problems will be used in the study our main spectral problem (1.23).

Let the domain Ω be defined by (1.28) and Ω_+ by (1.35). Introduce two spectral boundary value problems. The first one is described by

$$\begin{aligned}
&\Delta u + \omega_* u + \mu u = 0 \ \text{ in } \Omega,\\
&\nabla\widehat{\psi} \cdot \nabla u - \sigma(x) u = 0 \ \text{ for } y = \widehat{\xi}(x),\\
&u(x, -d) = 0,
\end{aligned} \tag{2.18}$$

together with the Neumann boundary condition on the remaining part of the boundary:

$$\partial_x w(-\Lambda/2, y) = \partial_x w(\Lambda/2, y) = 0 \quad \text{for } y \in (-d, \xi(\Lambda/2)). \tag{2.19}$$

For a weak formulation of this problem we introduce some spaces

$$H_0^1(\Omega) = \{w \in H^1(\Omega) \,:\, w = 0 \,\text{for}\, y = -d\},$$
$$H_{00}^1(\Omega) = \{w \in H_0^1(\Omega) \,:\, w = 0 \,\text{for}\, x = \pm\Lambda/2\}.$$

Then the Neumann spectral problem is formulated as finding of $w \in H_0^1(\Omega)$ and $\mu \in \mathbb{R}$ satisfying

$$\mathbf{a}(w, v) = \mu(w, v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega).$$

In this formulation we do not assume that the function w is even. Certainly even, Λ–periodic solutions to the spectral problem (1.25) solve just introduced spectral problem. The second spectral problem is described by the equations (2.18) together with the Dirichlet boundary conditions:

$$w(-\Lambda/2, y) = w(\Lambda/2, y) = 0 \quad \text{for } y \in (-d, \xi(\Lambda/2)). \tag{2.20}$$

The weak formulation of the Dirichlet spectral problem is the following. Find $w \in H_{00}^1(\Omega)$ and $\mu \in \mathbb{R}$ solving the equation

$$\mathbf{a}(w, v) = \mu(w, v)_{L^2(\Omega)} \quad \text{for all } v \in H_{00}^1(\Omega).$$

Here we also do not assume that the function is even or odd, but we note that odd, Λ periodic solutions to the spectral problem (1.25) deliver solutions to this problem. We denote by $\{\mu_{jN}\}$ and $\{\mu_{jD}\}$, $j = 1, \dots,$ the eigenvalues for the Neumann and Dirichlet spectral problems respectively. As usual the numeration takes into account the multiplicity and the increasing order:

$$\mu_{1N} \leqslant \mu_{2N} \leqslant \cdots, \quad \mu_{1D} \leqslant \mu_{2D} \leqslant \cdots.$$

Clearly,

$$\mu_{jN} \leqslant \mu_j \ \text{ and } \ \mu_{jN} \leqslant \mu_{jD}, \quad j = 1, 2, \dots$$

Since the coefficients in (2.18) belongs to $C^{0,\alpha}$, we obtain that all eigenfunctions belong to $C^{2,\alpha}(\overline{\Omega})$, see [15], Chapter 4.

Lemma 2.7. *Let $\psi_x \neq 0$ in Ω_+. Then the eigenvalues μ_{1D} and μ_{1N} are simple and*

$$0 > \mu_{1D} > \mu_{1N}.$$

Furthermore,

$$\mu_{1N} = \mu_1, \quad 0 = \mu_{2D} > \mu_{2N}$$

and the corresponding eigenfunction to μ_{2D} is odd and coincides with ψ_x. Moreover, $\mu_{3N} = \mu_2$ and $\mu_{3D} > 0$.

Proof. The proof of simplicity of μ_{1N} and μ_{1D} together with the inequality $\mu_{1D} > \mu_{1N}$ is quite standard and we omit it.

Let us show that $0 > \mu_{1D}$. Consider the function $v = \psi_x$. It satisfies the problem (1.25) with $\mu = 0$. Since the corresponding function is odd it changes sign in Ω. This implies $0 > \mu_{1D}$. Moreover using that $v \neq 0$ in Ω_+, we obtain that v is the second eigenfunction and $\mu_{2D} = 0$ and the corresponding eigenfunction is v. Since the eigenfunctions corresponding to the eigenvalues μ_1 and μ_{1N} do not change sign in Ω we conclude $\mu_1 = \mu_{1N}$.

The important property for the eigenfunctions of the Dirichlet and Neumann problems for (2.18) is the following. One can verify that, if $w(x,y)$ is an eigenfunction then $w(-x,y)$ is also an eigenfunction corresponding to the same eigenvalue and hence the functions

$$v_\pm = w(x,y) \pm w(-x,y)$$

are also eigenfunctions or one of them can be zero. To proceed further it is convenient to introduce some more eigenvalue problems. Considering the equations (2.18) only on the domain Ω_+ with Dirichlet or Neumann boundary conditions for $x = 0$ and $x = \Lambda/2$ we get four spectral problems with DD, DN, ND and NN boundary conditions at $x = 0$ and $x = \Lambda/2$. We denote corresponding eigenvalues by

$$\begin{aligned} &\mu_{1DD} < \mu_{2DD} \leqslant \cdots, \quad \mu_{1DN} < \mu_{2DN} \leqslant \cdots, \\ &\mu_{1ND} < \mu_{2ND} \leqslant \cdots, \quad \mu_{1NN} < \mu_{2NN} \leqslant \cdots \end{aligned}$$

Clearly,

$$\mu_{jDD} > \mu_{jDN} > \mu_{jNN} \text{ and } \mu_{jDD} > \mu_{jND} > \mu_{jNN} \text{ for } j = 1, 2, \ldots.$$

Eigenfunctions corresponding to μ_{jDD} and μ_{jDN} can be extended for all x as odd, and eigenfunctions corresponding to μ_{jND} and μ_{jNN} can be extended as even. Furthermore, all eigenvalues with the first index 1 are simple and corresponding eigenfunctions do not change sign in Ω_+ and

$$\mu_{1D} = \mu_{1ND}, \quad \mu_{1N} = \mu_{1NN} = \mu_1 < 0, \quad \mu_{2D} = \mu_{1DD} = 0, \quad \mu_{2NN} = \mu_2 > 0.$$

The eigenvalue μ_{2N} coincides with μ_{2NN} or μ_{2DN}, which is greater than μ_{2NN}, and hence greater than 0. The eigenvalue μ_{3D} coincides with μ_{2ND} or μ_{2NN} and hence also greater than 0. This completes the proof. $\square$

2.6. Proof of Theorem 1.3.

Proof. We introduce an auxiliary spectral problem

$$
\begin{aligned}
&((\partial_x + z)^2 + \partial_y^2)u + \omega_* u + \mu u = 0 \ \text{ in } D,\\
&\nabla\widehat{\psi}\cdot\nabla_z u - \sigma(x)u = 0 \ \text{ for } y = \widehat{\xi}(x),\\
&u(x,-d) = 0,
\end{aligned}
\tag{2.21}
$$

where $z = i\tau$, τ is real and w is Λ–periodic. One can check that this is a spectral problem for self-adjoint operator for every real τ. We denote by

$$\widehat{\mu}_1(\tau) \leqslant \widehat{\mu}_2(\tau) \leqslant \cdots$$

its eigenvalues numerated in non-decreasing order and taking into account the multiplicities of eigenvalues.

Let

$$
\begin{aligned}
&H^1_0(\Omega,\tau) = \{w = e^{i\tau x}v \,:\, v \in H^1_{0,\mathrm{per}}(\Omega)\},\\
&H^1_{0,\mathrm{per}}(\Omega) = \{v \in H^1_0(\Omega) \,:\, v(-\Lambda/2, y) = v(\Lambda/2, y), y \in (-d, \xi(x))\}.
\end{aligned}
$$

These spaces coincide for $\tau = k\tau_*$, $k \in \mathbb{Z}$. The above spectral problem can be reformulated as follows

$$a(w,u) = \mu(w,u)_{L^2(\Omega)} \ \text{ for all } v \in H^1_0(\Omega,\tau), \tag{2.22}$$

where $w \in H^1_0(\Omega,\tau)$.

Let us prove that

$$\mu_{jN} < \widehat{\mu}(\tau)_j < \mu_{jD}, \ \ j = 1,2,\ldots, \tag{2.23}$$

for $\tau \in (0,\tau_*)$. Indeed, since $\tau \in (0,\tau_*)$,

$$H^1_{0,\mathrm{per}}(\Omega) \supset H^1_0(\Omega,\tau) \supset H^1_{0,0}(\Omega)$$

and both inclusions are strick. Using the mini-max definition of the eigenvalues one can show that

$$\mu_{jN} \leqslant \widehat{\mu}(\tau)_j \leqslant \mu_{jD}, \ \ j = 1,2,\ldots. \tag{2.24}$$

To prove that the inequalities are strong assume that $\mu_{jN} = \widehat{\mu}(\tau)_j$ for a certain j. Then one can show that there is an eigenfunction v, which simultaneously satisfies the Neumann and Dirichlet spectral problems with the same eigenvalue. This implies that the eigenfunction together with its normal derivative vanish for $q = \pm\Lambda/2$. Hence the eigenfunction is identically zero.

The inequality (2.24) together with Lemma 2.7 imply

$$\widehat{\mu}_2(\tau) < 0 \text{ and } \widehat{\mu}_3(\tau) > 0,$$

which in turn leads to the proof of our theorem.

□

2.7. Proof of Corollary 1.4. Let M be a positive integer and v be a $(2M+1)\Lambda$–periodic function in D. Introduce the following transformation

$$V(v)(\tau_m, x, y) = V(\tau_m, x, y) = \sum_{k=-M}^{M} e^{-i\tau_m(x+k\Lambda)} v(x + k\Lambda, y) \qquad (2.25)$$

and

$$(\mathcal{M}v)(x, y) = (V(\tau_{-M}, x, y), V(\tau_{-M+1}, x, y), \ldots, V(\tau_M, x, y)), \qquad (2.26)$$

where

$$\tau_m = m\tau_1, \quad \tau_1 = \frac{2\pi}{(2M+1)\Lambda}.$$

One can verify that the function $V(\tau_m, x, y)$ is periodic with respect of x with the period Λ for every m and $\mathcal{M}v$ is a vector function consisting of $2M+1$, Λ–periodic functions. Moreover, if v is an even function, then

$$V(\tau_m, -x, y) = V(-\tau_m, x, y) \text{ for } m = 0, \pm\tau_1, \ldots, \pm\tau_M, \qquad (2.27)$$

and hence $V(\tau_0, x, y)$ is even with respect to x.

We introduce one more operator by

$$(\mathcal{N}V)(x, y) = \sum_{m=-M}^{M} e^{i\tau_m x} V(\tau_m, x, y),$$

where V is a Λ–periodic function with respect to x for each τ_m. Applying the operator $\mathcal{N}$ to V given by (2.25), we have

$$(\mathcal{N}V)(x, y) = \sum_{m=-M}^{M} e^{i\tau_m x} \sum_{k=-M}^{M} e^{-i\tau_m(x+k\Lambda)} v(x + k\Lambda, y)$$

$$= \sum_{k=-M}^{M} v(x + k\Lambda, y) \sum_{m=-M}^{M} \left(e^{-ik2\pi/(2M+1)}\right)^m = (2M+1)v(x, y).$$

Thus the operator $(2M+1)^{-1}\mathcal{N}$ is inverse to $\mathcal{M}$.

To estimate norms of just introduced operators we note that

$$\sum_{m=-M}^{M} ||(\mathcal{M}v)(\tau_m,x,y)||^2_{L^2(\Omega)} = (2M+1)||v||^2_{L^2(\Omega_M)},$$

where

$$\Omega_M = \{(x,y)\,:\,(-M-1/2)\Lambda < x < (M+1/2)\Lambda,\ -d<y<\xi(x)\}.$$

Similar to [17] one can show

$$\sum_{m=-M}^{M} ||(V)(\tau_m,x,y)||^2_{H^2(\Omega)} \leqslant C_1||v||^2_{H^2(\Omega_M)}$$

and

$$||\mathcal{N}(V)||^2_{H^2(\Omega_M)} \leqslant C_2 \sum_{m=-M}^{M} ||V(\tau_m,x,y)||^2_{H^2(\Omega)}.$$

The important property of the transformation (2.25) is the following

$$A(x,y,\partial_x,\partial_y)(\mathcal{N}V)(x,y) = \sum_{m=-M}^{M} e^{i\tau_m x} A(x,y,\partial_x+i\tau_m,\partial_y)V(\tau_m,x,y)$$

and

$$B(x,y,\partial_x,\partial_y)(\mathcal{N}V)(x,y) = \sum_{m=-M}^{M} e^{i\tau_m x} B(x,y,\partial_x+i\tau_m,\partial_y)V(\tau_m,x,y).$$

Therefore, if

$$\begin{aligned} &A(x,y,\partial_x,\partial_y)(\mathcal{N}V)(x,y) = 0, \ \text{ in } D\\ &B(x,y,\partial_x,\partial_y)(\mathcal{N}V)(x,y) = 0 \end{aligned}$$

then due to Theorem 1.3 $V(\tau_m,x,y)=0$ for all $m\neq 0$. If $m=0$ then using (2.27) we get that the function $V(0,x,y)$ is even and again the reference to Theorem 1.3 gives $V(0,x,y)=0$. This implies $\mathcal{N}V=0$. Thus the operator

$$(A,B)\,:\,C^{2,\alpha}_{0,e,Mp}(D)\to C^{0,\alpha}_{0,e,Mp}(D)\times C^{0,\alpha}_{e,Mp}(\mathbb{R})$$

is invertible. Here the index Mp indicate that we are dealing here with $(2M+1)\Lambda$–periodic functions. Now small perturbation arguments lead to invertibility of the corresponding nonlinear problem, which leads to the required result.

2.8. Generalized eigenfunctions for the eigenvalue $z = 0$. The eigenvalue $z = 0$ of the problem (1.23) is simple and has the eigenfunction $w = \psi_x$. To find generalized eigenfunctions is equivalent to finding of the Floque solutions to the problem (1.24). Then the generalized eigenfunctions are coefficients in the Floquet solutions.

The first generalized eigenfunction u_1 is the coefficient in the Floquet solution

$$u = xu_0 + u_1, \quad u_0 = \psi_x.$$

Then

$$\begin{aligned}&(\Delta + \omega_*)u_1 + 2u_{0x} = 0 \;\text{ in } D,\\ &\nabla\psi \cdot \nabla u_1 - \sigma u_1 + \psi_x u_0 = 0 \;\text{ for } y = \xi(x).\end{aligned} \tag{2.28}$$

The function u_1 is even, Λ-periodic solution to (2.28) satisfying $u_1(x,-d)=0$. Since $\mu_1 < 0$ and $\mu_2 > 0$ this problem is uniquely solvable.

Let us find u_1. We assume that we have a branch of solutions parameterized by the period Λ, which depends on t now. We introduce

$$\lambda = \frac{\Lambda(0)}{\Lambda(t)}$$

and make the change of variables

$$X = \lambda x, \quad \partial_x = \lambda \partial_X.$$

Then the problem can be rewritten as a problem with the same period Λ_0 as

$$\begin{aligned}&(\lambda^2\partial_X^2 + \partial_y^2 + \omega_*)u_1 + 2\lambda u_{0X} = 0 \;\text{ in } \widehat{D},\\ &(\lambda^2\psi_X u_{1X} + \psi_y u_{1y} - \sigma u_1 + 2\lambda\psi_X u_0 = 0 \;\text{ for } y = \xi(X)\end{aligned} \tag{2.29}$$

Consider the problem for ψ in (X, y) variables:

$$\begin{aligned}&(\lambda^2\partial_X^2 + \partial_y^2)\psi + \omega(\psi) = 0 \;\text{ in } \widehat{D}\\ &\frac{1}{2}(\lambda^2\psi_x^2 + \psi_y^2) + \xi = R \;\text{ for } y = \xi(X,t)\\ &\psi(X,\xi(X),t) = 0, \;\; \psi(X,-d) = p_0.\end{aligned}$$

Differentiating these relations with respect to t, we get

$$\begin{aligned}
&(\lambda^2\partial_X^2+\partial_y^2)\psi_t+\omega'(\psi)\psi_t=-2\lambda\lambda'\psi_{XX} \text{ in } \widehat{D}\\
&(\lambda^2\psi_x\psi_{tx}+\psi_y\psi_{ty})+(\lambda^2\psi_x\psi_{xy}+\psi_y\psi_{yy})\xi_t+\xi_t=-\lambda\lambda'\psi_X^2\\
&\qquad\qquad \text{for } y=\xi(X,t)\\
&\psi_t(X,\xi(X),t)+\psi_y\xi_t=0,\ \ \psi_t(X,-d)=0.
\end{aligned}$$

It can be transformed to

$$\begin{aligned}
&(\lambda^2\partial_X^2+\partial_y^2)\psi_t+\omega'(\psi)\psi_t=-2\lambda\lambda'\psi_{XX} \text{ in } \widehat{D}\\
&(\lambda^2\psi_x\psi_{tx}+\psi_y\psi_{ty})-\sigma_\lambda\psi_t=-\lambda\lambda'\psi_x^2 \text{ for } y=\xi(X,t)\\
&\psi_t(X,-d)=0,
\end{aligned}$$

where

$$\sigma_\lambda=-\frac{\lambda^2\psi_x\psi_{xy}+\psi_y\psi_{yy}+1}{\psi_y}.$$

Therefore

$$u_1=\frac{1}{\lambda\lambda'}\psi_t.$$

The second generalized eigenvector u_2 is the last term in possible solution

$$\frac{x^2}{2}u_0+xu_1+u_2.$$

Then u_2 must satisfy

$$\begin{aligned}
&(\Delta+\omega_*)u_2+u_0+2u_{1x}=0 \text{ in } D,\\
&\nabla\psi\cdot\nabla u_2-\sigma u_2+\psi_x u_1=0 \text{ for } y=\xi(x), && (2.30)\\
&u_2(x,-d)=0. && (2.31)
\end{aligned}$$

The function must be u_2 is odd and Λ-periodic. The solvability condition for (2.30) is the following

$$\int\limits_\Omega (u_0+2u_{1x})u_0\,dxdy-\int\limits_S \psi_x u_1 u_0\frac{dx}{\psi_y}=0. \tag{2.32}$$

Consider an analytic branch of water waves starting from a laminar flow, where the period is considered as a parameter. Near a laminar flow $(\widehat{psi}(y),0)$ the branch is described by

$$\psi(x,y,t)=\widehat{psi}(y)+t\cos(\tau_*x)\gamma(y,\tau_*)+O(t^2),$$

$$\xi(x,t)=t\frac{\tau_*\sin(\tau_*x)}{\widehat{\psi}_y(y)}\gamma(0,\tau_*)+O(t^2)$$

and

$$\lambda = 1 - ct^2 + O(t^4).$$

Therefore

$$u_0 = -t\tau_* \sin(\tau_* x)\gamma(y, \tau_*) + O(t^2),$$
$$u_1 = -\frac{1}{2ct}\cos(\tau_* x)\gamma(y, \tau_*) + O(1)$$

and

$$\lambda' = -2ct + O(t^3).$$

Since the left-hand side (LHS) of (2.32) is

$$LHS = -\frac{\tau_*}{2c}\int\limits_\omega \sin^2(\tau_* x)\gamma^2)y, \tau_*)dxdy + O(t),$$

it is different from zero for small t. Since the function $LHS(t)$ is analytic, it does not vanish except some isolated t-points.

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