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HOMOTOPY SIMILARITY OF MAPS

Abstract. Given based cellular spaces X and Y, X compact, we define a sequence of increasingly fine equivalences on the based-homotopy set $[X, Y]$.

§1. Introduction

Let X and Y be based cellular spaces (i. e., CW-complexes), with X compact. Let Y^X be the set of based continuous maps $X \to Y$ and $\langle Y^X \rangle$ be the free abelian group associated with Y^X. An element $A \in \langle Y^X \rangle$ is called an *ensemble* and has the form

$$A = \sum_i u_i {<} a_i {>}, \tag{1}$$

where $u_i \in \mathbb{Z}$ and $a_i \in Y^X$. A *subspace* of X is a subset containing the basepoint. Let $\mathrm{Sub}_r(X)$ be the set of subspaces $T \subseteq X$ containing at most r points distinct from the basepoint. Introduce the subgroup

$$\langle Y^X \rangle^{(r+1)} = \{ A : A|_T = 0 \text{ in } \langle Y^T \rangle \text{ for all } T \in \mathrm{Sub}_r(X) \} \subseteq \langle Y^X \rangle.$$

We have

$$\langle Y^X \rangle = \langle Y^X \rangle^{(0)} \supseteq \langle Y^X \rangle^{(1)} \supseteq \dots .$$

For ensembles $A, B \in \langle Y^X \rangle$, let

$$A \stackrel{r}{=} B$$

mean that $B - A \in \langle Y^X \rangle^{(r+1)}$.

For maps $a, b \in Y^X$, we say that a is *r-similar* to b, written

$$a \stackrel{r}{\sim} b,$$

if there exists an ensemble $A \in \langle Y^X \rangle$ of the form (1) with all $a_i \sim a$ ('$\sim$' stands for 'based homotopic') and such that $A \stackrel{r}{=} {<}b{>}$. A simple example is given in Section 3.

Our main results state that the relation $\stackrel{r}{\sim}$ is an equivalence (Theorem 8.1) and respects homotopy (Theorem 5.2). It follows that we get a sequence of increasingly fine equivalences on the based-homotopy set $[X, Y]$.

We conjecture that, for 0-connected Y, a map is r-similar to the constant map if and only if it lifts to the classifying space of the $(r+1)$th term of the lower central series of the loop group of Y.

A related notion is that of a homotopy invariant of finite order [4, 5]. A function $f\colon [X,Y] \to L$, where L is an abelian group, is called an invariant of *order* at most r if whenever an ensemble $A \in \langle Y^X \rangle$ of the form (1) satisfies $A \overset{r}{=} 0$ we have

$$\sum_i u_i f([a_i]) = 0.$$

It is clear that $f([a]) = f([b])$ if $a \overset{r}{\sim} b$ and f has order at most r. In §11, we give an example of two maps that are not 2-similar but cannot be distinguished by invariants of order at most 2. In the stable dimension range, invariants of order at most r were characterized in a way similar to our conjecture about r-similarity [4].

The relation between r-similarity and finite-order homotopy invariants is similar to that between n-equivalence and finite-degree invariants in knot theory [1, 2]. The example of §11 is similar to that of [2, Remark 10.8].

§2. Preliminaries

By a *space* we mean a based space (unless the contrary is stated explicitly). The basepoint of a cellular space is a vertex. The basepoint of a space X is denoted by $\daleth_X$ or $\daleth$. A *subspace* is a subset containing the basepoint. A *cover* is a cover by subspaces. A *map* is a based continuous map. The constant map $X \to Y$ is denoted by $\daleth_Y^X$ or $\daleth$. A *homotopy* is a based homotopy.

For a subspace $Z \subseteq X$, $\mathrm{in}\colon Z \to X$ is the inclusion. A wedge of spaces comes with the inclusion maps (i. e., coprojections):

$$\mathrm{in}_k\colon X_k \to X_1 \vee \ldots \vee X_n.$$

Maps $a_k\colon X_k \to Y$ form the map

$$a_1 \mathbin{\underline{\triangledown}} \ldots \mathbin{\underline{\triangledown}} a_n\colon X_1 \vee \ldots \vee X_n \to Y.$$

The same notation is used for homotopy classes.

Write $a \sim|_Z b$ to mean $a|_Z \sim b|_Z$. Similarly, equality of restrictions to a subset C is denoted by the symbol ‘$=|_C$’.

For a set E, the associated abelian group $\langle E \rangle$ is freely generated by the elements $<e>$, $e \in E$. A function $t\colon E \to F$ between two sets induces the homomorphism

$$\langle t\rangle\colon \langle E\rangle \to \langle F\rangle, \qquad {<}e{>} \mapsto {<}t(e){>}.$$

For a cover Γ of a space X, we put

$$\Gamma(r) = \{\, \{\urcorner\} \cup G_1 \cup \ldots \cup G_s \subseteq X : G_1, \ldots, G_s \in \Gamma, 0 \leqslant s \leqslant r \,\}.$$

For ensembles $A, B \in \langle Y^X\rangle$, the formula

$$A \overset{r}{\underset{\Gamma}{=}} B$$

means that $A =|_W B$ in $\langle Y^W\rangle$ for all $W \in \Gamma(r)$.

The symbol '?' denotes a placeholder for functions: for example, the expression $?^2\colon \mathbb{R} \to \mathbb{R}$ designates the function $x \mapsto x^2$.

§3. A simple example

Fix $r \geqslant 0$. For $d = (d_1, \ldots, d_{r+1}) \in \{0,1\}^{r+1} \subseteq \mathbb{Z}^{r+1}$, put

$$|d| = d_1 + \ldots + d_{r+1}.$$

Consider a wedge of spaces

$$W = U_1 \vee \ldots \vee U_{r+1} \vee V.$$

Introduce the maps

$$\Lambda(d) = \lambda_1(d_1) \vee \ldots \vee \lambda_{r+1}(d_{r+1}) \vee \mathrm{id}_V : W \to W, \qquad d \in \{0,1\}^{r+1},$$

where the map $\lambda_k(e)\colon U_k \to U_k$, for $e \in \{0,1\}$, is id if $e = 1$ and $\urcorner$ if $e = 0$.

Lemma 3.1. *Let X and Y be spaces and $p\colon X \to W$ and $q\colon W \to Y$ be maps. Consider the ensemble*

$$A = \sum_{d \in \{0,1\}^{r+1}} (-1)^{|d|} {<}a(d){>} \ \in \langle Y^X\rangle,$$

where $a(d)$ is the composition

$$a(d)\colon X \xrightarrow{p} W \xrightarrow{\Lambda(d)} W \xrightarrow{q} Y.$$

Then $A \overset{r}{=} 0$.

Proof. Take $T \in \mathrm{Sub}_r(X)$. There is an index k such that

$$p(T) \cap \mathrm{in}_k(U_k) = \{\urcorner_W\}.$$

Then $a(d)|_T$ does not depend on d_k. We get

$$A|_T = \sum_{d \in \{0,1\}^{r+1}} (-1)^{|d|} {<}a(d)|_T{>} = 0. \qquad \square$$

Example. Consider the wedge

$$W = S^{n_1} \vee \ldots \vee S^{n_{r+1}}$$

$(n_1, \ldots, n_{r+1} \geqslant 1)$. Put $m = n_1 + \ldots + n_{r+1} - r$ and let $p\colon S^m \to W$ be a map with

$$[p] = \lfloor \ldots \lfloor [\mathrm{in}_1], [\mathrm{in}_2] \rceil, \ldots, [\mathrm{in}_{r+1}] \rceil$$

(the iterated Whitehead product) in $\pi_m(W)$. We show that $\urcorner \overset{r}{\sim} p$. Consider the maps

$$a(d)\colon S^m \xrightarrow{p} W \xrightarrow{\Lambda(d)} W, \qquad d \in \{0,1\}^{r+1}.$$

Put $1_{r+1} = (1, \ldots, 1) \in \{0,1\}^{r+1}$. By Lemma 3.1,

$$\sum_{d \in \{0,1\}^{r+1} \setminus \{1_{r+1}\}} (-1)^{r-|d|} \langle a(d) \rangle \overset{r}{=} \langle a(1_{r+1}) \rangle.$$

All $a(d)$ on the left side are homotopic to $\urcorner$. On the right, $a(1_{r+1}) = p$ because $\Lambda(1_{r+1}) = \mathrm{id}$. Thus $\urcorner \overset{r}{\sim} p$.

§4. Equipment of a cellular space

Let Y be a compact *unbased* cellular space. In this section, we turn off our convention that *maps* and *homotopies* preserve basepoints.

Lemma 4.1. *There exist homotopies*

$$q_t\colon Y^2 \to Y \quad \textit{and} \quad p_t\colon Y^2 \to [0,1], \qquad t \in [0,1],$$

such that

$$q_0(z,y) = y, \qquad q_t(z,z) = z, \qquad p_0(z,y) = 0, \qquad p_t(z,z) = t, \tag{2}$$

and, for any $(z,y) \in Y^2$ *and* $t \in [0,1]$, *one has*

$$p_t(z,y) = 0 \quad \text{or} \quad q_t(z,y) = z. \tag{3}$$

Roughly speaking, the inclusions $\{z\} \to Y$, $z \in Y$, form a parametric cofibration. We say that (q_t, p_t) is an *equipment* of Y.

Proof (after [6, Exemple on p. 490]). By [3, Corollary A.10], Y is an ENR. Embed it to $\mathbb{R}^n$ and choose its neighbourhood $U \subseteq \mathbb{R}^n$ and a retraction $r\colon U \to Y$. Choose $\epsilon > 0$ such that U includes all closed balls of radius ϵ with centres in Y. Consider the homotopy $l_t\colon (\mathbb{R}^n)^2 \to \mathbb{R}^n$, $t \in [0,1]$,

$$l_t(z,y) = y + \min(\epsilon t/|z-y|, 1)(z-y), \qquad z \neq y,$$
$$l_t(z,z) = z.$$

Put

$$q_t(z,y) = r(l_t(z,y)) \quad \text{and} \quad p_t(z,y) = \max(t - |z-y|/\epsilon, 0). \qquad \square$$

Corollary 4.2. *One can continuously associate to each path $v\colon [0,1] \to Y$ a homotopy $E_t(v)\colon Y \to Y$, $t \in [0,1]$, such that*

$$E_0(v) = \mathrm{id} \quad \textit{and} \quad E_t(v)(v(0)) = v(t).$$

Proof. Using Lemma 4.1, put

$$E_t(v)(y) = \begin{cases} q_t(v(0),y) & \text{if } \ p_t(v(0),y) = 0, \\ v(p_t(v(0),y)) & \text{if } \ q_t(v(0),y) = v(0). \end{cases} \qquad \square$$

§5. Coherent homotopies

Let X and Y be cellular spaces, X compact.

Lemma 5.1. *Consider an ensemble*

$$A = \sum_i u_i{<}a_i{>} \ \in \langle Y^X \rangle,$$

and maps $b, \widetilde{b} \in Y^X$, $b \sim \widetilde{b}$. Then there exist maps $\widetilde{a}_i \in Y^X$, $\widetilde{a}_i \sim a_i$, such that the ensemble

$$\widetilde{A} = \sum_i u_i{<}\widetilde{a}_i{>}$$

has the following property: if $A =|_Z {<}b{>}$ for some subspace $Z \subseteq X$, then $\widetilde{A} =|_Z {<}\widetilde{b}{>}$.

Proof. We have a homotopy $h_t \in Y^X$, $t \in [0,1]$, such that $h_0 = b$ and $h_1 = \widetilde{b}$. Replace Y by a compact cellular subspace that includes the images of all a_i and h_t.

For $x \in X$, introduce the path $v_x\colon [0,1] \to Y$, $v_x(t) = h_t(x)$. We have $v_x(0) = b(x)$ and $v_x(1) = \widetilde{b}(x)$. For a subspace $Z \subseteq X$, introduce the functions $e_t^Z\colon Y^Z \to Y^Z$, $t \in [0,1]$,

$$e_t^Z(d)(x) = E_t(v_x)(d(x)), \qquad x \in Z, \quad d \in Y^Z,$$

where E_t is given by Corollary 4.2. For $d \in Y^Z$, we have the homotopy $e_t^Z(d) \in Y^Z$, $t \in [0,1]$. The diagram

$$\begin{array}{ccc} Y^X & \xrightarrow{\;e_t^X\;} & Y^X \\ \downarrow{\scriptstyle ?|_Z} & & \downarrow{\scriptstyle ?|_Z} \\ Y^Z & \xrightarrow{\;e_t^Z\;} & Y^Z \end{array}$$

is commutative. We have $e_0^Z = \mathrm{id}$ because

$$e_0^Z(d)(x) = E_0(v_x)(d(x)) = d(x).$$

We have $e_1^X(b) = \widetilde{b}$ because

$$e_1^X(b)(x) = E_1(v_x)(b(x)) = E_1(v_x)(v_x(0)) = v_x(1) = \widetilde{b}(x).$$

Put $\widetilde{a}_i = e_1^X(a_i)$. Since $a_i = e_0^X(a_i)$, we have $\widetilde{a}_i \sim a_i$. We have

$$(<\widetilde{b}> - \widetilde{A})|_Z = \langle e_1^X \rangle (<b> - A)|_Z = \langle e_1^Z \rangle ((<b> - A)|_Z).$$

Thus $A =|_Z <b>$ implies $\widetilde{A} =|_Z <\widetilde{b}>$. □

Theorem 5.2. *Let maps $a, b, \widetilde{a}, \widetilde{b} \in Y^X$ satisfy*

$$\widetilde{a} \sim a \overset{r}{\sim} b \sim \widetilde{b}.$$

Then $\widetilde{a} \overset{r}{\sim} \widetilde{b}$.

Proof. By the definition of similarity, it suffices to show that $a \overset{r}{\sim} \widetilde{b}$. We have an ensemble

$$A = \sum_i u_i <a_i> \ \in \langle Y^X \rangle,$$

where $a_i \sim a$, such that $A \overset{r}{=} <b>$. By Lemma 5.1, there is an ensemble

$$\widetilde{A} = \sum_i u_i <\widetilde{a}_i> \ \in \langle Y^X \rangle,$$

where $\widetilde{a}_i \sim a_i$, such that $\widetilde{A} \overset{r}{=} <\widetilde{b}>$. Since $a_i \sim a$, we have shown that $a \overset{r}{\sim} \widetilde{b}$. □

§6. Underlaying a cover

Let X and Y be cellular spaces, X compact.

Lemma 6.1. *Consider an ensemble*

$$A = \sum_i u_i{<}a_i{>} \ \in \langle Y^X \rangle.$$

Then there exist maps $\widetilde{a}_i \in Y^X$, $\widetilde{a}_i \sim a_i$, such that the ensemble

$$\widetilde{A} = \sum_i u_i{<}\widetilde{a}_i{>}$$

has the following property: if $A|_Z = 0$ for some subspace $Z \subseteq X$, then $\widetilde{A}|_V = 0$ for some neighbourhood $V \subseteq X$ of Z.

Proof. Replace Y by a compact cellular subspace that includes the images of all a_i. We will use the equipment (q_t, p_t) given by Lemma 4.1.

Let the index i for a_i runs over $1, \dots, n$. Define maps $a_i^k \in Y^X$, $1 \leqslant i \leqslant n$, $0 \leqslant k \leqslant n$, by the rules $a_i^0 = a_i$ and

$$a_i^k(x) = q_1(a_k^{k-1}(x), a_i^{k-1}(x)), \qquad x \in X, \tag{4}$$

for $k \geqslant 1$. Put $\widetilde{a}_i = a_i^n$. We have $a_i^k \sim a_i^{k-1}$ because $a_i^k = h_1$ and $a_i^{k-1} = h_0$ for the homotopy $h_t \in Y^X$, $t \in [0,1]$,

$$h_t(x) = q_t(a_k^{k-1}(x), a_i^{k-1}(x)), \qquad x \in X.$$

Thus $\widetilde{a}_i \sim a_i$.

Claim 1. If $a_i^{k-1} =|_Q a_j^{k-1}$ for some subspace $Q \subseteq X$, then $a_i^k =|_Q a_j^k$.

This follows from (4).

Claim 2. If $a_i^{i-1} =|_Q a_j^{i-1}$ for some subspace $Q \subseteq X$, then there exists a neighbourhood $W \subseteq X$ of Q such that $a_i^i =|_W a_j^i$.

Indeed, if $a_i^{i-1} =|_Q a_j^{i-1}$, then, by (2),

$$p_1(a_i^{i-1}(x), a_j^{i-1}(x)) = 1$$

for $x \in Q$. There exists a neighbourhood $W \subseteq X$ of Q such that

$$p_1(a_i^{i-1}(x), a_j^{i-1}(x)) > 0$$

for $x \in W$. Then, by (3),

$$q_1(a_i^{i-1}(x), a_j^{i-1}(x)) = a_i^{i-1}(x)$$

for $x \in W$. By (4),

$$a_i^i(x) = q_1(a_i^{i-1}(x), a_i^{i-1}(x)) = a_i^{i-1}(x)$$

(because $q_1(z, z) = z$ by (2)) and

$$a^i_j(x) = q_1(a^{i-1}_i(x), a^{i-1}_j(x)).$$

Thus $a^i_i(x) = a^i_j(x)$ for $x \in W$, as required.

Take a subspace $Z \subseteq X$.

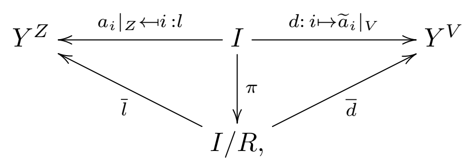
Claim 3. If $a_i =|_Z a_j$, then there exists a neighbourhood $W \subseteq X$ of Z such that $\widetilde{a}_i =|_W \widetilde{a}_j$.

This follows from the construction of $\widetilde{a}_i$ and Claims 1 and 2.

Consider the equivalence

$$R = \{\, (i, j) : a_i =|_Z a_j \,\}$$

on the set $I = \{1, \ldots, n\}$. It follows from Claim 3 that there exists a neighbourhood $V \subseteq X$ of Z such that $\widetilde{a}_i =|_V \widetilde{a}_j$ for all $(i, j) \in R$. We have the commutative diagram



where π is the projection. The function $\bar{l}$ is injective. Consider the elements

$$U = \sum_i u_i{<}i{>} \ \in \langle I\rangle$$

and

$$\overline{U} = \langle \pi\rangle(U) \in \langle I/R\rangle.$$

We have

$$A|_Z = \langle l\rangle(U) = \langle \bar{l}\rangle(\overline{U}) \quad \text{and} \quad \widetilde{A}|_V = \langle d\rangle(U) = \langle \bar{d}\rangle(\overline{U}).$$

If $A|_Z = 0$, then $\overline{U} = 0$ because $\langle \bar{l}\rangle$ is injective. Then $\widetilde{A}|_V = 0$. ☐

Corollary 6.2. *Consider an ensemble*

$$A = \sum_i u_i{<}a_i{>} \ \in \langle Y^X\rangle$$

such that $A \overset{r}{=} 0$. *Then there exist maps* $\widetilde{a}_i \in Y^X$, $\widetilde{a}_i \sim a_i$, *such that the ensemble*

$$\widetilde{A} = \sum_i u_i{<}\widetilde{a}_i{>} \tag{5}$$

satisfies the condition $\widetilde{A} \underset{\Gamma}{\overset{r}{=}} 0$ *for some open cover* Γ *of* X.

Proof. Since $A \overset{r}{=} 0$, we have $A|_T = 0$ for all $T \in \mathrm{Sub}_r(X)$. By Lemma 6.1, there are maps $\widetilde{a}_i \in Y^X$, $\widetilde{a}_i \sim a_i$, such that the ensemble $\widetilde{A}$ given by (5) satisfies the condition $\widetilde{A}|_{V(T)} = 0$ for some neighbourhood $V(T) \subseteq X$ of each $T \in \mathrm{Sub}_r(X)$. There is an open cover Γ of X such that every $W \in \Gamma(r)$ is included in $V(T)$ for some $T \in \mathrm{Sub}_r(X)$. Then $\widetilde{A}|_W = 0$ for all $W \in \Gamma(r)$, that is, $\widetilde{A} \underset{\Gamma}{\overset{r}{=}} 0$. □

Lemma 6.3. *Consider an ensemble* $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<} a_i {>},$$

and a map $b \in Y^X$. *Then there exist maps* $\widetilde{a}_i \in Y^X$, $\widetilde{a}_i \sim a_i$, *such that the ensemble*

$$\widetilde{A} = \sum_i u_i {<} \widetilde{a}_i {>} \tag{6}$$

has the following property: *if* $A =|_Z {<}b{>}$ *for some subspace* $Z \subseteq X$, *then* $\widetilde{A} =|_V {<}b{>}$ *for some neighbourhood* $V \subseteq X$ *of* Z.

Proof. Let Π be the set of subspaces $Z \subseteq X$ such that $A =|_Z {<}b{>}$. By Lemma 6.1, there are maps $\overline{a}_i, \overline{b} \in Y^X$, $\overline{a}_i \sim a_i$ and $\overline{b} \sim b$, such that the ensemble

$$\overline{A} = \sum_i u_i {<} \overline{a}_i {>}$$

satisfies the condition $\overline{A} =|_{V(Z)} {<}\overline{b}{>}$ for some neighbourhood $V(Z) \subseteq X$ of each $Z \in \Pi$. By Lemma 5.1, there are maps $\widetilde{a}_i \in Y^X$, $\widetilde{a}_i \sim \overline{a}_i$, such that the ensemble $\widetilde{A}$ given by (6) satisfies the condition $\widetilde{A} =|_{V(Z)} {<}b{>}$ for all $Z \in \Pi$. □

Corollary 6.4. *Consider an ensemble*

$$A = \sum_i u_i {<} a_i {>} \ \in \langle Y^X \rangle$$

and a map $b \in Y^X$. *Suppose that* $A \overset{r}{=} {<}b{>}$. *Then there exist maps* $\widetilde{a}_i \in Y^X$, $\widetilde{a}_i \sim a_i$, *such that the ensemble*

$$\widetilde{A} = \sum_i u_i {<} \widetilde{a}_i {>} \tag{7}$$

satisfies the condition $\widetilde{A} \underset{\Gamma}{\overset{r}{=}} {<}b{>}$ *for some open cover* Γ *of* X.

Proof. This follows from Lemma 6.3 as Corollary 6.2 does from Lemma 6.1. □

§7. Symmetric characterization of similarity

Let X and Y be cellular spaces, X compact.

Lemma 7.1. *Consider a cover* Γ *of* X*, an open subspace* $G \in \Gamma$*, a closed subspace* $D \subseteq X$*,* $D \subseteq G$*, and maps* $a, b_0, b_1 \in Y^X$ *such that* $a \sim|_G b_0$*,* $b_0 \sim b_1 \operatorname{rel} X \setminus D$*, and* $a \overset{r-1}{\underset{\Gamma}{\sim}} b_0$ *in the following sense: there is an ensemble*

$$A = \sum_i u_i{<}a_i{>} \ \in \langle Y^X \rangle,$$

where $a_i \sim a$*, such that* $A \overset{r-1}{\underset{\Gamma}{\equiv}} {<}b_0{>}$*. Then there exists an ensemble*

$$C = \sum_k w_k{<}c_k{>} \ \in \langle Y^X \rangle,$$

where $c_k \sim a$*, such that* $C \overset{r}{\equiv} {<}b_1{>} - {<}b_0{>}$*.*

Proof. There is a homotopy $h_t \in Y^X$, $t \in [0,1]$, such that $h_s = b_s$, $s = 0, 1$, and $h_t =|_{X \setminus D} b_0$. Choose a continuous function $\phi \colon X \to [0,1]$ such that $\phi|_E = 1$ and $\phi|_{X \setminus F} = 0$ for some subspaces $E, F \subseteq X$, E open, F closed, such that

$$D \subseteq E \subseteq F \subseteq G.$$

Let $p \in Y^G$ be a map such that $p \sim b_0|_G$. Choose a homotopy

$$K_t(p) \in Y^G, \quad t \in [0,1],$$

such that

$$K_0(p) = p, K_1(p) = b_0|_G, \text{ and } K_t(p) = b_0|_G \text{ if } p = b_0|_G.$$

Define a homotopy $L_t(p) \in Y^G$, $t \in [-1,1]$, by the rules

$$L_t(p)(x) = K_{\phi(x)(t+1)}(p)(x), \qquad x \in G,$$

for $t \in [-1, 0]$ and

$$L_t(p)(x) = \begin{cases} h_t(x) & \text{if } x \in E, \\ K_{\phi(x)}(p)(x) & \text{if } x \in G \setminus D \end{cases}$$

for $t \in [0,1]$. We have

$$L_{-1}(p) = p, \qquad L_s(p) =|_E\, b_s,\ s = 0,1,$$
$$L_0(p) =|_{G\setminus D}\, L_1(p), \qquad L_t(p) =|_{G\setminus F}\, p.$$

Moreover, $L_s(b_0|_G) = b_s|_G$, $s = 0,1$.

Let $d \in Y^X$ be a map such that $d \sim|_G b_0$. Define a homotopy $l_t(d) \in Y^X$, $t \in [-1,1]$, by the rules $l_t(d) =|_G L_t(d|_G)$ and $l_t(d) =|_{X\setminus F} d$. We have

$$l_{-1}(d) = d, \qquad l_s(d) =|_E\, b_s,\ s = 0,1,$$
$$l_0(d) =|_{X\setminus D}\, l_1(d), \qquad l_t(d) =|_{X\setminus F}\, d.$$

Since $a_i \sim a \sim|_G b_0$, the homotopies $l_t(a_i)$ are defined. Put

$$C = \sum_i u_i(<l_1(a_i)> - <l_0(a_i)>).$$

We have $l_s(a_i) \sim a_i \sim a$. It remains to show that $C \overset{r}{=} <b_1> - <b_0>$. Take $T \in \mathrm{Sub}_r(X)$. We check that

$$C =|_T <b_1> - <b_0>. \tag{8}$$

We are in one of the following three cases.

Case 1: $T \cap D = \{\triangleleft_X\}$. We have $l_0(a_i) =|_T l_1(a_i)$ and $b_0 =|_T b_1$. Thus both the sides of (8) are zero on T.

Case 2: $T \cap F = \{\triangleleft_X, x_*\}$, where $x_* \in E$ and $x_* \neq \triangleleft_X$. Put $Z = T\setminus\{x_*\}$. We have $Z \in \mathrm{Sub}_{r-1}(X)$ and $Z \cap F = \{\triangleleft_X\}$. Define functions

$$e_s\colon Y^Z \to Y^T, \quad s = 0,1,$$

by the rules $e_s(q)|_Z = q$ and $e_s(q)(x_*) = b_s(x_*)$. We have $e_s(b_0|_Z) = b_s|_T$ and $e_s(a_i|_Z) = l_s(a_i)|_T$. Thus

$$\Big(<b_0> - \sum_i u_i<a_i>\Big)\Big|_Z \xmapsto{\langle e_s\rangle} \Big(<b_s> - \sum_i u_i<l_s(a_i)>\Big)\Big|_T.$$

Since $A \overset{r-1}{=} <b_0>$, the expression on the left is zero. Thus the one on the right is also zero, which implies (8).

Case 3: $T \cap G \notin \mathrm{Sub}_1(X)$. There is a decomposition $T = W \cup Z$ for some subspaces $W, Z \subseteq X$ such that $W \cap Z = \{\triangleleft_X\}$, $W \subseteq G$, $Z \cap F = \{\triangleleft_X\}$, and $Z \in \mathrm{Sub}_{r-2}(X)$. Consider the subspace $M = G \cup Z \subseteq X$. Define functions $f_s\colon Y^M \to Y^T$, $s = 0,1$. Take $q \in Y^M$. If $q \sim|_G b_0$, put $f_s(q) =|_W L_s(q|_G)$

and $f_s(q) = |_Z\, q$. Otherwise, put $f_s(q) = \triangleleft^{T}_{Y}$. We have $f_s(b_0|_M) = b_s|T$ and $f_s(a_i|_M) = l_s(a_i)|_T$. Thus

$$\big({<}b_0{>} - \sum_i u_i{<}a_i{>}\big)\big|_M \xmapsto{\langle f_s\rangle} \big({<}b_s{>} - \sum_i u_i{<}l_s(a_i){>}\big)\big|_T.$$

Since M is included in some element of $\Gamma(r-1)$ and $A \overset{r-1}{\underset{\Gamma}{\equiv}} {<}b_0{>}$, the expression on the left is zero. Thus the one on the right is also zero, which implies (8). □

Lemma 7.2. *Let $a, b, \widetilde{b} \in Y^X$ be maps such that $a \overset{r-1}{\sim} b \sim \widetilde{b}$ and*

$$a \sim|_S\, b \ \text{ for any } S \in \mathrm{Sub}_1(X). \tag{*}$$

Then there exists an ensemble

$$C = \sum_k w_k{<}c_k{>} \ \in \langle Y^X\rangle,$$

where $c_k \sim a$, such that $C \overset{r}{=} {<}\widetilde{b}{>} - {<}b{>}$.

The condition (*) is satisfied automatically if X or Y is 0-connected. It also follows from the condition $a \overset{r-1}{\sim} b$ if $r \geqslant 2$ (cf. the proof of Theorem 7.3).

Proof. There is an ensemble

$$A = \sum_i u_i{<}a_i{>} \ \in \langle Y^X\rangle,$$

where $a_i \sim a$, such that $A \overset{r-1}{=} {<}b{>}$. Using Corollary 6.4, replace each a_i by a homotopic map to get $A \overset{r-1}{\underset{\Gamma}{\equiv}} {<}b{>}$ for some open cover Γ of X.

We say that a subspace $G \subseteq X$ is *primitive* if the map $\mathrm{in}\colon G \to X$ is homotopic to the composition

$$G \xrightarrow{f} S \xrightarrow{\mathrm{in}} X$$

for some subspace $S \in \mathrm{Sub}_1(X)$ and map f. Since X is Hausdorff and locally contractible, for any open subspace $U \subseteq X$ and point $x \in U$, there exists a primitive open subspace $G \subseteq X$ such that $x \in G$ and $G \subseteq U$. We replace the cover Γ by its refinement consisting of primitive open subspaces. Then it follows from (*) that $a \sim|_G\, b$ for each $G \in \Gamma$.

Choose a finite partition of unity subordinate to Γ:

$$\sum_{j=1}^{m} \phi_j = 1,$$

where each $\phi_j \colon X \to [0,1]$ is a continuous function such that $\phi_j|_{X\setminus D_j} = 0$ for some closed subspace $D_j \subseteq X$ such that $D_j \subseteq G_j$ for some $G_j \in \Gamma$. Choose a homotopy $h_t \in Y^X$, $t \in [0,1]$, such that $h_0 = b$ and $h_1 = \widetilde{b}$. Define maps $b_j \in Y^X$, $0 \leqslant j \leqslant m$, by the rule

$$b_j(x) = h_{\phi_1(x)+\ldots+\phi_j(x)}(x).$$

We have $b_0 = b$, $b_m = \widetilde{b}$, and $b_{j-1} \sim b_j \operatorname{rel} X \setminus D_j$.

Take $j \geqslant 1$. Applying Lemma 5.1 to the congruence $A \overset{r-1}{\underset{\Gamma}{\equiv}} {<}b{>}$ and the homotopy $b \sim b_{j-1}$, we get an ensemble

$$A_j = \sum_i u_i {<}a_{ji}{>} \in \langle Y^X \rangle,$$

where $a_{ji} \sim a_i$ $(\sim a)$, such that $A_j \overset{r-1}{\underset{\Gamma}{\equiv}} {<}b_{j-1}{>}$. We have $a \sim|_{G_j} b \sim b_{j-1}$. By Lemma 7.1, there is an ensemble

$$C_j = \sum_k w_{jk} {<}c_{jk}{>} \in \langle Y^X \rangle,$$

where $c_{jk} \sim a$, such that $C_j \overset{r}{=} {<}b_j{>} - {<}b_{j-1}{>}$.

We get

$$\sum_{j=1}^{m} C_j = {<}b_m{>} - {<}b_0{>} = {<}\widetilde{b}{>} - {<}b{>}. \qquad \square$$

Theorem 7.3. *Consider maps $a, b \in Y^X$ and ensembles $A, B \in \langle Y^X \rangle$,*

$$A = \sum_i u_i {<}a_i{>} \quad \text{and} \quad B = \sum_j v_j {<}b_j{>},$$

where

$$\sum_i u_i = \sum_j v_j = 1,$$

$a_i \sim a$, and $b_j \sim b$, such that $A \overset{r}{=} B$. Then $a \overset{r}{\sim} b$.

Proof. Induction on r. If $r \leqslant 0$, the assertion is trivial. Suppose $r \geqslant 1$.

For $S \in \mathrm{Sub}_1(X)$, we have $a \sim|_S b$ because

$$ {<}[a|_S]{>} = \sum_i u_i {<}[a_i|_S]{>} = [\![A|_S]\!] = [\![B|_S]\!] = \sum_j v_j {<}[b_j|_S]{>} = {<}[b|_S]{>} $$

in $\langle [S,Y] \rangle$. Here $[\![?]\!]\colon \langle Y^S \rangle \to \langle [S,Y] \rangle$ is the homomorphism induced by the projection $[?]\colon Y^S \to [S,Y]$.

By induction hypothesis, $a \overset{r-1}{\sim} b$. Take j. Since $b \sim b_j$, Lemma 7.2 gives an ensemble

$$ C_j = \sum_k w_{jk} {<}c_{jk}{>} \ \in \langle Y^X \rangle, $$

where $c_{jk} \sim a$, such that $C_j \overset{r}{=} {<}b_j{>} - {<}b{>}$. We have

$$ A - \sum_j v_j C_j \overset{r}{=} A - \sum_j v_j({<}b_j{>} - {<}b{>}) = A - B + {<}b{>} \overset{r}{=} {<}b{>}, $$

which proves the assertion. □

§8. Similarity is an equivalence

Let X and Y be cellular spaces, X compact.

Theorem 8.1. *The relation $\overset{r}{\sim}$ on Y^X is an equivalence.*

This was conjectured by A. V. Malyutin.

Proof. Reflexivity is trivial. Symmetry follows from Theorem 7.3. It remains to prove transitivity.

Let maps $a, b, c \in Y^X$ satisfy $a \overset{r}{\sim} b \overset{r}{\sim} c$. Then there are ensembles $A, B \in \langle Y^X \rangle$,

$$ A = \sum_i u_i {<}a_i{>} \quad \text{and} \quad B = \sum_j v_j {<}b_j{>}, $$

where $a_i \sim a$ and $b_j \sim b$, such that $A \overset{r}{=} {<}b{>}$ and $B \overset{r}{=} {<}c{>}$. For each j, we have $b \sim b_j$ and, by Lemma 5.1, there is an ensemble

$$ A_j = \sum_i u_i {<}a_{ji}{>} \in \langle Y^X \rangle, $$

where $a_{ji} \sim a_i$ $(\sim a)$, such that $A_j \overset{r}{=} {<}b_j{>}$. We have

$$ \sum_j v_j A_j \overset{r}{=} \sum_j v_j {<}b_j{>} = B \overset{r}{=} {<}c{>}. $$

Thus $a \overset{r}{\sim} c$. □

Using Theorem 5.2, we introduce the relation of r-similarity on $[X, Y]$:

$$[a] \overset{r}{\sim} [b] \quad \Leftrightarrow \quad a \overset{r}{\sim} b.$$

It follows from Theorem 8.1 that it is an equivalence.

§9. The Hopf invariant

Let X and Y be spaces. Let $e \in Z^m(Y)$ and $f \in Z^n(Y)$ $(m, n \geqslant 1)$ be (singular) cocycles and $g \in C^{m+n-1}(Y)$ be a cochain with $\delta g = ef$. Put

$$[X, Y]_{e,f} = \{\, \boldsymbol{a} : \boldsymbol{a}^*([e]) = 0 \text{ and } \boldsymbol{a}^*([f]) = 0 \text{ in } H^\bullet(X) \,\} \subseteq [X, Y]$$

and

$$Y^X_{e,f} = \{\, a : [a] \in [X, Y]_{e,f} \,\} \subseteq Y^X.$$

Given $a \in Y^X_{e,f}$, choose a cochain $p \in C^{m-1}(X)$ such that $\delta p = a^\#(e)$ and put

$$q = pa^\#(f) - a^\#(g) \in C^{m+n-1}(X).$$

Then $\delta q = 0$ and the class $[q] \in H^{m+n-1}(X)$ neither depends on the choice of p nor changes if a is replaced by a homotopic map. Putting $h([a]) = [q]$, we get the function

$$h\colon [X, Y]_{e,f} \to H^{m+n-1}(X),$$

which we call the *Hopf invariant* [7].

Lemma 9.1. *Let X_0 be a space and $t\colon X \to X_0$ be a map. We have the Hopf invariants*

$$h_0\colon [X_0, Y]_{e,f} \to H^{m+n-1}(X_0) \quad \textit{and} \quad h\colon [X, Y]_{e,f} \to H^{m+n-1}(X).$$

Given $a_0 \in Y^{X_0}$, put $a = a_0 \circ t \in Y^X$. If $a_0 \in Y^{X_0}_{e,f}$, then $a \in Y^X_{e,f}$ and $h([a]) = t^*(h_0([a_0]))$ in $H^{m+n-1}(X)$. □

Lemma 9.2. *Take elements $\boldsymbol{u} \in \pi_m(Y)$ and $\boldsymbol{v} \in \pi_n(Y)$. Put*

$$\Delta = \langle \boldsymbol{u}^*([e]), [S^m]\rangle \langle \boldsymbol{v}^*([f]), [S^n]\rangle + (-1)^{mn} \langle \boldsymbol{u}^*([f]), [S^m]\rangle \langle \boldsymbol{v}^*([e]), [S^n]\rangle \in \mathbb{Z}$$

(*the last two Kronecker indices vanish unless $m = n$*). *Consider the Hopf invariant*

$$h\colon [S^{m+n-1}, Y]_{e,f} \to H^{m+n-1}(S^{m+n-1})$$

and the Whitehead product $\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil \in \pi_{m+n-1}(Y) = [S^{m+n-1}, Y]$. Then $\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil \in [S^{m+n-1}, Y]_{e,f}$ and

$$\langle h(\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil), [S^{m+n-1}]\rangle = (-1)^{mn+m+n}\Delta.$$

Caution: the sign in the last equality is sensitive to certain conventions.

Proof (after [7, §19]). We assume that $S^m \vee S^n \subseteq S^m \times S^n$ in the standard way. We have the commutative diagram

$$\begin{array}{ccc} S^{m+n-1} & \xrightarrow{\ \phi\ } & S^m \vee S^n \\ {\scriptstyle \text{in}}\downarrow & & \downarrow{\scriptstyle \text{in}} \\ D^{m+n} & \xrightarrow{\ \chi\ } & S^m \times S^n, \end{array}$$

where $[\phi] = \lfloor [\text{in}_1], [\text{in}_2] \rceil$ in $\pi_{m+n-1}(S^m \vee S^n)$. We have the chain of homomorphisms and sendings

$$\begin{array}{cc} H_{m+n-1}(S^{m+n-1}) & [S^{m+n-1}] \\ {\scriptstyle \partial}\uparrow & \uparrow \\ H_{m+n}(D^{m+n}, S^{m+n-1}) & [D^{m+n}] \\ {\scriptstyle (\chi,\phi)_*}\downarrow & \downarrow \\ H_{m+n}(S^m \times S^n, S^m \vee S^n) & \text{rel}_*([S^m \times S^n]) \\ {\scriptstyle \text{rel}_*}\uparrow & \uparrow \\ H_{m+n}(S^m \times S^n). & [S^m \times S^n] \end{array} \tag{9}$$

Choose representatives $u\colon S^m \to Y$ and $v\colon S^n \to Y$ of $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. Consider the maps

$$a\colon S^{m+n-1} \xrightarrow{\phi} S^m \vee S^n \xrightarrow{w=u\underline{\triangledown}v} Y.$$

Clearly, $[a] = \lfloor \boldsymbol{u}, \boldsymbol{v} \rceil$ in $\pi_{m+n-1}(Y)$.

Choose cocycles $\widehat{e} \in Z^m(S^m \times S^n)$ and $\widehat{f} \in Z^n(S^m \times S^n)$ and a cochain $\widehat{g} \in C^{m+n-1}(S^m \times S^n)$ such that

$$\widehat{e}|_{S^m \vee S^n} = w^{\#}(e), \qquad \widehat{f}|_{S^m \vee S^n} = w^{\#}(f), \quad \text{and} \quad \widehat{g}|_{S^m \vee S^n} = w^{\#}(g).$$

We have

$$a^{\#}(e) = \phi^{\#}(w^{\#}(e)) = \phi^{\#}(\widehat{e}|_{S^m \vee S^n}) = \chi^{\#}(\widehat{e})|_{S^{m+n-1}}$$

in $Z^m(S^{m+n-1})$. It follows that $a^*([e]) = 0$ in $H^m(S^{m+n-1})$ (which is automatic unless $n = 1$). Similarly, $a^*([f]) = 0$ in $H^n(S^{m+n-1})$. Thus $[a] \in [S^{m+n-1}, Y]_{e,f}$.

Let $z_k \in H^k(S^k)$ be the class with $\langle z_k, [S^k] \rangle = 1$. One easily sees that

$$[\widehat{e}] = \langle \boldsymbol{u}^*([e]), [S^m] \rangle (z_m \times 1) + \langle \boldsymbol{v}^*([e]), [S^n] \rangle (1 \times z_n)$$

in $H^m(S^m \times S^n)$ and

$$[\widehat{f}] = \langle \boldsymbol{v}^*([f]), [S^n]\rangle(1 \times z_n) + \langle \boldsymbol{u}^*([f]), [S^m]\rangle(z_m \times 1)$$

in $H^n(S^m \times S^n)$. Thus $[\widehat{e}][\widehat{f}] = \Delta(z_m \times z_n)$ in $H^{m+n}(S^m \times S^n)$ and

$$\langle[\widehat{e}][\widehat{f}], [S^m \times S^n]\rangle = (-1)^{mn}\Delta. \tag{10}$$

Choose a cochain $\widetilde{p} \in C^{m-1}(D^{m+n})$ such that $\delta\widetilde{p} = \chi^\#(\widehat{e})$. Put

$$\widetilde{q} = \widetilde{p}\chi^\#(\widehat{f}) - \chi^\#(\widehat{g}) \in C^{m+n-1}(D^{m+n}).$$

Put

$$p = \widetilde{p}|_{S^{m+n-1}} \in C^{m-1}(S^{m+n-1})$$

and

$$q = \widetilde{q}|_{S^{m+n-1}} \in C^{m+n-1}(S^{m+n-1}).$$

We have

$$\delta p = \delta\widetilde{p}|_{S^{m+n-1}} = \chi^\#(\widehat{e})|_{S^{m+n-1}} = \phi^\#(\widehat{e}|_{S^m \vee S^n}) = \phi^\#(w^\#(e)) = a^\#(e)$$

and

$$\begin{aligned} q &= p\chi^\#(\widehat{f})|_{S^{m+n-1}} - \chi^\#(\widehat{g})|_{S^{m+n-1}} = p\phi^\#(\widehat{f}|_{S^m \vee S^n}) - \phi^\#(\widehat{g}|_{S^m \vee S^n}) \\ &= p\phi^\#(w^\#(f)) - \phi^\#(w^\#(g)) = pa^\#(f) - a^\#(g). \end{aligned}$$

Thus $\delta q = 0$ and $h([a]) = [q]$.

We have

$$\delta\widetilde{q} = \chi^\#(\widehat{e})\chi^\#(\widehat{f}) - \delta\chi^\#(\widehat{g}) = \chi^\#(\widehat{e}\widehat{f} - \delta\widehat{g}).$$

We have the chain of homomorphisms and sendings

$$\begin{array}{ccc}
H^{m+n-1}(S^{m+n-1}) & & [q] \\
\downarrow{\scriptstyle\delta} & & \downarrow \\
H^{m+n}(D^{m+n}, S^{m+n-1}) & & [\chi^\#(\widehat{e}\widehat{f} - \delta\widehat{g})] \\
\uparrow{\scriptstyle(\chi,\phi)^*} & & \uparrow \\
H^{m+n}(S^m \times S^n, S^m \vee S^n) & & [\widehat{e}\widehat{f} - \delta\widehat{g}] \\
\downarrow{\scriptstyle\mathrm{rel}^*} & & \downarrow \\
H^{m+n}(S^m \times S^n). & & [\widehat{e}][\widehat{f}]
\end{array}$$

Collating it with (9) and using (10), we get

$$\langle [q], [S^{m+n-1}] \rangle = (-1)^{m+n} \langle [\widehat{e}][\widehat{f}], [S^m \times S^n] \rangle = (-1)^{mn+m+n} \Delta.$$

This is what we need because $h(\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil) = h([a]) = [q]$. □

Let Γ be an open cover of X. Consider the differential graded ring $C^\bullet(\Gamma)$ of Γ-cochains of X (that is, functions on the set of singular simplices subordinate to Γ). The restriction homomorphism

$$?|_\Gamma \colon C^\bullet(X) \to C^\bullet(\Gamma)$$

is a morphism of differential graded rings; it induces an isomorphism of cohomology rings,

$$?|_\Gamma \colon H^\bullet(X) \to H^\bullet(\Gamma).$$

Lemma 9.3. *Given $a \in Y^X_{e,f}$, choose $\widetilde{p} \in C^{m-1}(\Gamma)$ such that $\delta\widetilde{p} = a^\#(e)|_\Gamma$ and put*

$$\widetilde{q} = \widetilde{p} a^\#(f)|_\Gamma - a^\#(g)|_\Gamma \in C^{m+n-1}(\Gamma).$$

Then $\delta\widetilde{q} = 0$ and $h([a])|_\Gamma = [\widetilde{q}]$ in $H^{m+n-1}(\Gamma)$. □

We suppose that X and Y are cellular spaces and X is compact.

Theorem 9.4. *Consider an ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_i u_i {<} a_i {>},$$

where $a_i \in Y^X_{e,f}$, such that $A \overset{2}{=} 0$. Then

$$\sum_i u_i h([a_i]) = 0$$

in $H^{m+n-1}(X)$.

Thus h may be called a *partial* invariant of order at most 2.

Proof. Using Corollary 6.2, replace a_i by homotopic maps so that $A \underset{\Gamma}{\overset{2}{=}} 0$ for some open cover Γ of X.

Let $B \subseteq C^m(\Gamma)$ be the subgroup generated by the coboundaries $a_i^\#(e)|_\Gamma$. It is free because finitely generated and torsion-free. Thus there is a homorphiam $P \colon B \to C^{m-1}(\Gamma)$ such that $\delta P(b) = b$, $b \in B$. Put

$$\widetilde{q}_i = P(a^\#(e)|_\Gamma) a^\#(f)|_\Gamma - a^\#(g)|_\Gamma \in C^{m+n-1}(\Gamma).$$

By Lemma 9.3, $\delta\widetilde{q}_i = 0$ and

$$h([a_i])|_\Gamma = [\widetilde{q}_i]$$

in $H^{m+n-1}(\Gamma)$.

Take a singular simplex $\sigma\colon \Delta^{m+n-1} \to G$, $G \in \Gamma$. Let

$$\sigma'\colon \Delta^{m-1} \to G \quad \text{and} \quad \sigma''\colon \Delta^{n} \to G$$

be its front and back faces, respectively.

The group $\operatorname{Hom}(B, \mathbb{Q})$ is formed by homomorphisms $\langle ?, T\rangle$, where T runs over $C_m(\Gamma; \mathbb{Q})$, the group of rational Γ-chains in X. Thus there is a chain $T \in C_m(\Gamma; \mathbb{Q})$ such that

$$\langle P(b), \sigma'\rangle = \langle b, T\rangle, \qquad b \in B.$$

We have

$$T = \sum_k c_k \tau_k,$$

where $c_k \in \mathbb{Q}$ and $\tau_k\colon \Delta^m \to G_k$, $G_k \in \Gamma$. Thus

$$\langle P(a_i^{\#}(e)|_\Gamma), \sigma'\rangle = \langle a_i^{\#}(e)|_\Gamma, T\rangle = \sum_k c_k \langle a_i^{\#}(e)|_\Gamma, \tau_k\rangle.$$

We get

$$\begin{aligned}
\langle \widetilde{q}_i, \sigma\rangle &= (-1)^{(m-1)n} \langle P(a_i^{\#}(e)|_\Gamma), \sigma'\rangle \langle a_i^{\#}(f)|_\Gamma, \sigma''\rangle - \langle a_i^{\#}(g)|_\Gamma, \sigma\rangle \\
&= (-1)^{(m-1)n} \sum_k c_k \langle a_i^{\#}(e)|_\Gamma, \tau_k\rangle \langle a_i^{\#}(f)|_\Gamma, \sigma''\rangle - \langle a_i^{\#}(g)|_\Gamma, \sigma\rangle \\
&= (-1)^{(m-1)n} \sum_k c_k \langle (a_i|_{G\cup G_k})^{\#}(e), \tau_k\rangle \langle (a_i|_{G\cup G_k})^{\#}(f), \sigma''\rangle \\
&\quad - \langle (a_i|_G)^{\#}(g), \sigma\rangle.
\end{aligned}$$

We have found functions $R_k\colon Y^{G\cup G_k} \to \mathbb{Q}$ and $S\colon Y^G \to \mathbb{Q}$ such that

$$\langle \widetilde{q}_i, \sigma\rangle = \sum_k R_k(a_i|_{G\cup G_k}) - S(a_i|_G)$$

for all i. Since $A \underset{\Gamma}{\overset{2}{=}} 0$, we have $A|_{G\cup G_k} = 0$ and $A|_G = 0$. Thus

$$\sum_i u_i \langle \widetilde{q}_i, \sigma\rangle = 0.$$

Since σ was taken arbitrarily, we have

$$\sum_i u_i \widetilde{q}_i = 0.$$

We get

$$\sum_i u_i h([a_i])|_\Gamma = \sum_i u_i [\widetilde{q}_i] = 0.$$

Since restriction to Γ here is an isomorphism, we get

$$\sum_i u_i h([a_i]) = 0. \qquad \square$$

Corollary 9.5. *Let* $a, b \in Y^X_{e,f}$ *satisfy* $a \overset{2}{\sim} b$. *Then* $h([a]) = h([b])$.

Proof. There is an ensemble

$$A = \sum_i u_i{<}a_i{>} \ \in \langle Y^X \rangle,$$

where $a_i \sim a$, such that $A \overset{2}{=} {<}b{>}$. Since $A =|_{\{\triangleleft\}} {<}b{>}$, we have

$$\sum_i u_i = 1.$$

By Theorem 9.4,

$$\sum_i u_i h([a_i]) = h([b]).$$

Since $[a_i] = [a]$, we get $h([a]) = h([b])$. $\square$

§10. Maps of $S^p \times S^n$

This section does not depend of the rest of the paper. We recall a theorem of G. W. Whitehead about the fibration of free spheroids (Theorem 10.1) and deduce Lemma 10.3 about certain maps $S^p \times S^n \to Y$ (we need it in §11).

We fix numbers $p, n \geqslant 1$ and a space Y. Let $\Omega^n Y$ be the space of maps $S^n \to Y$, as usual. Let

$$\epsilon\colon S^p \times S^n \to S^p \wedge S^n \to S^{p+n}$$

be the composition of the projection and the standard homeomorphism. For a map $w : S^{p+n} \to Y$, introduce the map

$$\nabla^n(w) : S^p \to \Omega^n Y, \qquad \nabla^n(w)(t)(z) = w(\epsilon(t, z)).$$

Introduce the isomorphism

$$\boldsymbol{\nabla}^n\colon \pi_{p+n}(Y) \to \pi_p(\Omega^n Y), \qquad [w] \mapsto [\nabla^n(w)].$$

Let

$$\mu : S^n \to S^n \vee S^n$$

be the standard comultiplication. Consider the usual multiplication

$$\Omega^n Y \times \Omega^n Y \xrightarrow{*} \Omega^n Y, \qquad v_1 * v_2\colon S^n \xrightarrow{\mu} S^n \vee S^n \xrightarrow{v_1 \underline{\nabla} v_2} Y.$$

For a map $v\colon S^n \to Y$, introduce the map
$$\tau_v\colon \Omega^n Y \xrightarrow{v*?} (\Omega^n Y, v * ⁋),$$
where the target is $\Omega^n Y$ with the specified new basepoint. It induces the isomorphism
$$\tau_{v\,*}\colon \pi_p(\Omega^n Y) \to \pi_p(\Omega^n Y, v * ⁋).$$

Let $\Lambda^n Y$ be the space of *unbased* maps $S^n \to Y$. Consider the fibration
$$\rho\colon \Lambda^n Y \to Y, \qquad v \mapsto v(⁋).$$
We have $\rho^{-1}(⁋) = \Omega^n Y$.

Theorem 10.1 (G. W. Whitehead). *For a map $v\colon S^n \to Y$, the composition*
$$\mathbf{\Gamma}\colon \pi_{p+1}(Y) \xrightarrow{\lfloor ?,[v]\rceil} \pi_{p+n}(Y) \xrightarrow{\boldsymbol{\nabla}^n} \pi_p(\Omega^n Y) \xrightarrow{\tau_{v\,*}} \pi_p(\Omega^n Y, v * ⁋)$$
coincides up to a sign with the connecting homomorphism of the fibration ρ at the point $v * ⁋ \in \Omega^n Y$. Consequently, the composition
$$\pi_{p+1}(Y) \xrightarrow{\mathbf{\Gamma}} \pi_p(\Omega^n Y, v * ⁋) \xrightarrow{\mathrm{in}_*} \pi_p(\Lambda^n Y, v * ⁋)$$
is zero.

Proof. See [8, Theorem (3.2)] and [9, §3]. □

For a map $v\colon S^n \to Y$, introduce the homomorphism
$$\mathbf{\Psi}_v\colon \pi_{p+n}(Y) \xrightarrow{\boldsymbol{\nabla}^n} \pi_p(\Omega^n Y) \xrightarrow{\tau_{v\,*}} \pi_p(\Omega^n Y, v * ⁋) \xrightarrow{\mathrm{in}_*} \pi_p(\Lambda^n Y, v * ⁋).$$
By Theorem 10.1,
$$\mathbf{\Psi}_v(\lfloor \boldsymbol{u}, [v]\rceil) = 0, \qquad \boldsymbol{u} \in \pi_{p+1}(Y). \tag{11}$$
For maps $v\colon S^n \to Y$ and $w\colon S^{p+n} \to Y$, introduce the map
$$\Psi_v(w)\colon S^p \xrightarrow{\nabla^n(w)} \Omega^n Y \xrightarrow{\tau_v} (\Omega^n Y, v * ⁋) \xrightarrow{\mathrm{in}} (\Lambda^n Y, v * ⁋).$$
Clearly,
$$[\Psi_v(w)] = \mathbf{\Psi}_v([w])$$
in $\pi_p(\Lambda^n Y, v * ⁋)$.

Introduce the map
$$\Phi\colon S^p \times S^n \xrightarrow{\mathrm{id}\times\mu} S^p \times (S^n \vee S^n) \xrightarrow{\theta} S^n \vee S^{p+n}, \tag{12}$$
where
$$\theta\colon \quad (t, \mathrm{in}_1(z)) \mapsto \mathrm{in}_1(z), \quad (t, \mathrm{in}_2(z)) \mapsto \mathrm{in}_2(\epsilon(t,z)), \qquad t \in S^p, \quad z \in S^n.$$

For maps $v\colon S^n \to Y$ and $w\colon S^{p+n} \to Y$, introduce the map

$$\Xi(v,w)\colon S^p \times S^n \xrightarrow{\Phi} S^n \vee S^{p+n} \xrightarrow{v \underline{\triangledown} w} Y. \tag{13}$$

For elements $\boldsymbol{v} \in \pi_n(Y)$ and $\boldsymbol{w} \in \pi_{p+n}(Y)$, put

$$\boldsymbol{\Xi}(\boldsymbol{v},\boldsymbol{w}) = [\Xi(v,w)] \in [S^p \times S^n, Y], \tag{14}$$

where v and w are representatives of $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively.

For maps $v_0\colon S^n \to Y$ and $V\colon S^p \to (\Lambda^n Y, v_0)$, introduce the map

$$V^{\times}\colon S^p \times S^n \to Y, \qquad (t,z) \mapsto V(t)(z).$$

For $\boldsymbol{V} \in \pi_p(\Lambda^n Y, v_0)$, put

$$\boldsymbol{V}^{\times} = [V^{\times}] \in [S^p \times S^n, Y],$$

where V is a representative of $\boldsymbol{V}$.

Lemma 10.2. *For maps $v\colon S^n \to Y$ and $w\colon S^{p+n} \to Y$, one has*

$$\Xi(v,w) = \Psi_v(w)^{\times}\colon S^p \times S^n \to Y.$$

Consequently,

$$\boldsymbol{\Xi}([v],[w]) = \boldsymbol{\Psi}_v([w])^{\times}$$

in $[S^p \times S^n, Y]$.

Proof. Take a point $(t,z) \in S^p \times S^n$. We have $\mu(z) = \mathrm{in}_k(\widetilde{z})$ in $S^n \vee S^n$ for some $k \in \{1,2\}$ and $\widetilde{z} \in S^n$. We have

$$\theta(t,\mu(z)) = \theta(t,\mathrm{in}_k(\widetilde{z})) = \begin{cases} \mathrm{in}_1(\widetilde{z}) & \text{if } k=1, \\ \mathrm{in}_2(\epsilon(t,\widetilde{z})) & \text{if } k=2 \end{cases}$$

in $S^n \vee S^{p+n}$. Thus

$$\begin{aligned} \Xi(v,w)(t,z) &= ((v \underline{\triangledown}\, w) \circ \Phi)(t,z) \\ &= ((v \underline{\triangledown}\, w) \circ \theta \circ (\mathrm{id} \times \mu))(t,z) = (v \underline{\triangledown}\, w)(\theta(t,\mu(z))) \\ &= \begin{cases} (v \underline{\triangledown}\, w)(\mathrm{in}_1(\widetilde{z})) = v(\widetilde{z}) & \text{if } k=1, \\ (v \underline{\triangledown}\, w)(\mathrm{in}_1(\widetilde{z})) = v(\widetilde{z}) & \text{if } k=2. \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned}\Psi_v(w)^\times(t,z) &= \Psi_v(w)(t)(z) = \tau_v(\nabla^n(w)(t))(z) = (v * \nabla^n(w)(t))(z) \\ &= (v \,\underline{\vee}\, \nabla^n(w)(t))(\mu(z)) = (v \,\underline{\vee}\, \nabla^n(w)(t))(\mathrm{in}_k(\widetilde{z})) \\ &= \begin{cases} v(\widetilde{z}) & \text{if } \ k=1, \\ \nabla^n(w)(t)(\widetilde{z}) = w(\epsilon(t,\widetilde{z})) & \text{if } \ k=2. \end{cases}\end{aligned}$$

The same. □

Lemma 10.3. *For elements* $\boldsymbol{u} \in \pi_{p+1}(Y)$*,* $\boldsymbol{v} \in \pi_n(Y)$*, and* $\boldsymbol{w} \in \pi_{p+n}(Y)$*, one has*

$$\boldsymbol{\Xi}(\boldsymbol{v}, \lfloor \boldsymbol{u}, \boldsymbol{v} \rceil + \boldsymbol{w}) = \boldsymbol{\Xi}(\boldsymbol{v}, \boldsymbol{w})$$

in $[S^p \times S^n, Y]$*.*

Proof. Choose a representative $v\colon S^n \to Y$ of $\boldsymbol{v}$. By (11),

$$\boldsymbol{\Psi}_v(\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil + \boldsymbol{w}) = \boldsymbol{\Psi}_v(\boldsymbol{w})$$

in $\pi_p(\Lambda^n Y, v * \triangleleft)$. Applying Lemma 10.2 yields the desired equality. □

For a map $w\colon S^{p+n} \to Y$, introduce the map

$$\xi(w)\colon S^p \times S^n \xrightarrow{\epsilon} S^{p+n} \xrightarrow{w} Y.$$

For an element $\boldsymbol{w} \in \pi_{p+n}(Y)$, put

$$\boldsymbol{\xi}(\boldsymbol{w}) = [\xi(w)] \in [S^p \times S^n, Y], \tag{15}$$

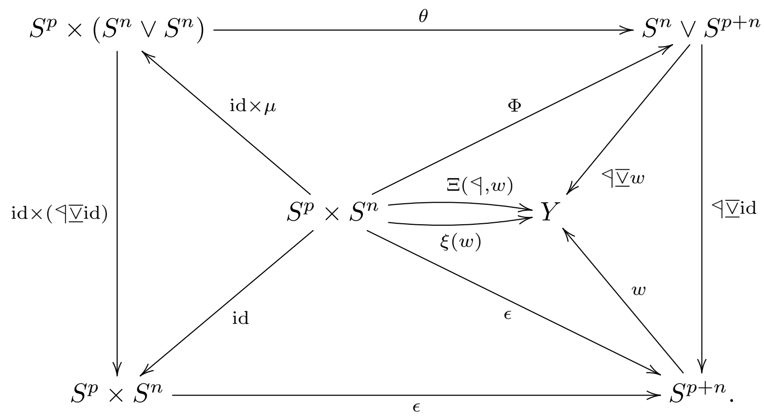
where w is a representative of $\boldsymbol{w}$.

Lemma 10.4. *For en element* $\boldsymbol{w} \in \pi_{p+n}(Y)$*, one has*

$$\boldsymbol{\Xi}(0, \boldsymbol{w}) = \boldsymbol{\xi}(\boldsymbol{w})$$

in $[S^p \times S^n, Y]$*.*

Proof. Choose a representative $w\colon S^{p+n}\to Y$ of $\boldsymbol{w}$. Consider the diagram



Since the map

$$S^n \xrightarrow{\mu} S^n\vee S^n \xrightarrow{\lhd\underline{\nabla}\mathrm{id}} S^n$$

is homotopic to the identity, the left triangle is homotopy commutative. The other empty triangles and the square are commutative. It follows that the parallel curved arrows are homotopic. □

§11. Fineness of 2-similarity

Put $X=S^p\times S^n$ ($p\geqslant 1$, $n\geqslant 2$). Let Y be a space, and let $\boldsymbol{u}\in\pi_{p+1}(Y)$ and $\boldsymbol{v}\in\pi_n(Y)$ be some elements. Consider the Whitehead product

$$\lfloor \boldsymbol{u},\boldsymbol{v}\rceil\in\pi_{p+n}(Y)$$

and the homotopy classes

$$\boldsymbol{k}(t)=\boldsymbol{\xi}(t\lfloor \boldsymbol{u},\boldsymbol{v}\rceil)\in[X,Y],\qquad t\in\mathbb{Z}$$

(see (15)).

Lemma 11.1. *Let L be an abelian group and $f\colon[X,Y]\to L$ be an invariant of order at most r. Then*

$$f(\boldsymbol{k}(r!+t))=f(\boldsymbol{k}(t)),\qquad t\in\mathbb{Z}.$$

Proof (after [5, Lemma 1.5]). We will use the homotopy classes

$$\boldsymbol{K}(s,t)=\boldsymbol{\Xi}(s\boldsymbol{v},t\lfloor \boldsymbol{u},\boldsymbol{v}\rceil)\in[X,Y],\qquad s,t\in\mathbb{Z}$$

(see (14)). By Lemma 10.4,

$$\boldsymbol{K}(0,t)=\boldsymbol{k}(t). \tag{16}$$

We have

$$\boldsymbol{K}(s,m+t)=\boldsymbol{K}(s,t) \qquad \text{if } s\mid m \tag{17}$$

because

$$\begin{aligned}\boldsymbol{\Xi}(s\boldsymbol{v},(m+t)\lfloor\boldsymbol{u},\boldsymbol{v}\rceil)&=\boldsymbol{\Xi}(s\boldsymbol{v},\lfloor(m/s)\boldsymbol{u},s\boldsymbol{v}\rceil+t\lfloor\boldsymbol{u},\boldsymbol{v}\rceil)\\&=\boldsymbol{\Xi}(s\boldsymbol{v},t\lfloor\boldsymbol{u},\boldsymbol{v}\rceil)\quad\text{(by Lemma 10.3)}.\end{aligned}$$

Consider the wedge of r copies of S^n and two copies of S^{p+n}

$$W=S^n\vee\ldots\vee S^n\vee S^{p+n}\vee S^{p+n}$$

and the maps

$$\Lambda(d)=\lambda_1(d_1)\vee\ldots\lambda_r(d_r)\vee\lambda_{r+1}(d_{r+1})\vee\mathrm{id}\colon W\to W,$$

$d=(d_1,\ldots,d_{r+1})\in\{0,1\}^{r+1}\subseteq\mathbb{Z}^{r+1}$, as in §3. Put

$$\mu=\mu_1\vee\mu_2\colon S^n\vee S^{p+n}\to W,$$

where

$$\mu_1\colon S^n\to S^n\vee\ldots\vee S^n \quad\text{and}\quad \mu_2\colon S^{p+n}\to S^{p+n}\vee S^{p+n}$$

are the comultiplications. Choose a map $q\colon W\to Y$ with

$$[q]=\boldsymbol{v}\,\underline{\vee}\ldots\underline{\vee}\,\boldsymbol{v}\,\underline{\vee}\,r!\lfloor\boldsymbol{u},\boldsymbol{v}\rceil\,\underline{\vee}\,t\lfloor\boldsymbol{u},\boldsymbol{v}\rceil.$$

Consider the ensemble $A\in\langle Y^X\rangle$,

$$A=\sum_{d\in\{0,1\}^{r+1}}(-1)^{|d|}{<}a(d){>},$$

where

$$a(d)\colon X\xrightarrow{\Phi}S^n\vee S^{p+n}\xrightarrow{\mu}W\xrightarrow{\Lambda(d)}W\xrightarrow{q}Y,$$

where Φ is as in (13). By Lemma 3.1, $A\overset{r}{=}0$. Clearly,

$$[q\circ\Lambda(d)\circ\mu]=(d_1+\ldots d_r)\boldsymbol{v}\,\underline{\vee}\,(d_{r+1}r!+t)\lfloor\boldsymbol{u},\boldsymbol{v}\rceil$$

in $[S^n\vee S^{p+n},Y]$. Thus, by the construction of $\boldsymbol{K}(s,t)$,

$$[a(d)]=\boldsymbol{K}(d_1+\ldots d_r,d_{r+1}r!+t)$$

in $[X,Y]$. Thus, since f has order at most r,

$$\sum_{d\in\{0,1\}^{r+1}}(-1)^{|d|}f(\boldsymbol{K}(d_1+\ldots d_r,d_{r+1}r!+t))=0.$$

By (17), the class $\boldsymbol{K}(d_1 + \dots d_r, d_{r+1}r! + t)$ does not depend on d_{r+1} if $(d_1, \dots, d_r) \neq (0, \dots, 0)$. Thus the corresponding summands cancel out. We get $f(\boldsymbol{K}(0,t)) - f(\boldsymbol{K}(0, r!+t)) = 0$. By (16), this is what we need. $\square$

Let classes $E \in H^{p+1}(Y)$ and $F \in H^n(Y)$ satisfy $EF = 0$ in $H^{p+n+1}(Y)$. Put, as in Lemma 9.2,

$$\begin{aligned}\Delta = &\langle \boldsymbol{u}^*(E), [S^{p+1}]\rangle\langle \boldsymbol{v}^*(F), [S^n]\rangle \\ &+ (-1)^{(p+1)n}\langle \boldsymbol{u}^*(F), [S^{p+1}]\rangle\langle \boldsymbol{v}^*(E), [S^n]\rangle \in \mathbb{Z}.\end{aligned}$$

If $Y = S^{p+1} \vee S^n$ with $\boldsymbol{u} = [\mathrm{in}_1]$ and $\boldsymbol{v} = [\mathrm{in}_2]$, taking obvious E and F yields $\Delta = 1$. If $p = n-1$ and $Y = S^n$ with $\boldsymbol{u} = \boldsymbol{v} = [\mathrm{id}]$, taking obvious equal E and F yields $\Delta = 1 + (-1)^n$.

Lemma 11.2. *If $\Delta \neq 0$, the classes $\boldsymbol{k}(t)$, $t \in \mathbb{Z}$, are pairwise not 2-similar.*

Proof. Choose cocycles $e \in Z^{p+1}(Y)$ and $f \in Z^n(Y)$ representing E and F, respectively. Choose a cochain $g \in C^{p+n}(Y)$ with $\delta g = ef$. Consider the corresponding Hopf invariants (see §9)

$$h_0\colon \pi_{p+n}(Y) \to H^{p+n}(S^{p+n}) \quad \text{and} \quad h\colon [X,Y]_{e,f} \to H^{p+n}(X).$$

By Lemma 9.2,

$$\langle h_0(\lfloor \boldsymbol{u}, \boldsymbol{v}\rceil), [S^{p+n}]\rangle = (-1)^{pn+p+1}\Delta.$$

We have the decomposition

$$\boldsymbol{k}(t)\colon X \xrightarrow{\ \epsilon\ } S^{p+n} \overset{t[\mathrm{id}]}{\rightsquigarrow} S^{p+n} \overset{\lfloor \boldsymbol{u},\boldsymbol{v}\rceil}{\rightsquigarrow} Y$$

(the wavy arrows denote homotopy classes). Clearly, $\boldsymbol{k}(t) \in [X,Y]_{e,f}$. Since the Brouwer degree of ϵ is 1 and that of $t[\mathrm{id}]$ is t, Lemma 9.1 yields

$$\langle h(\boldsymbol{k}(t)), [X]\rangle = (-1)^{pn+p+1}\Delta t.$$

By Corollary 9.5, the classes $\boldsymbol{k}(t)$, $t \in \mathbb{Z}$, are pairwise not 2-similar if $\Delta \neq 0$. $\square$

Moral. Suppose that $\Delta \neq 0$. The classes $\boldsymbol{k}(0)$ (i. e., $[\urcorner]$) and $\boldsymbol{k}(2)$ in $[X,Y]$, which are not 2-similar by Lemma 11.2, cannot be distinguished by an invariant of order at most 2 by Lemma 11.1. Recall that (X,Y) can be $(S^p \times S^n, S^{p+1} \vee S^n)$ for any $p \geqslant 1$ and $n \geqslant 2$ or $(S^{n-1} \times S^n, S^n)$ for even $n \geqslant 2$.

References

1. M. Gusarov, *On n-equivalence of knots and invariants of finite degree.* In: "Topology of manifolds and varieties", Adv. Sov. Math. **18** (1994), 173–192.
2. M. N. Gusarov, *Variations of knotted graphs. Geometric techniques of n-equivalence.* — St. Petersbg. Math. J. **12** (2001), 569–604.
3. A. Hatcher, *Algebraic topology.* Cambridge University Press, 2002.
4. S. S. Podkorytov, *The order of a homotopy invariant in the stable case.* — Math. Sb. **202** (2011), 1183–1206.
5. S. S. Podkorytov, *On homotopy invariants of finite degree.* — J. Math. Sci., New York **212** (2016), 587–604.
6. J.-P. Serre, *Homologie singulière des espaces fibrés. Applications.* — Ann. Math. (2) **54**, No. 3 (1951), 425–505.
7. N. E. Steenrod, *Cohomology invariants of mappings.* — Ann. Math. (2) **50**, No. 4 (1949), 954–988.
8. G. W. Whitehead, *On products in homotopy groups.* — Ann. Math. (2) **47**, No. 3 (1946), 460–475.
9. J. H. C. Whitehead, *On certain theorems of G. W. Whitehead.* — Ann. Math. (2) **58**, No. 3 (1953), 418–428.

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