Записки научных
семинаров ПОМИ
Том 549, 2025 г.

I. Nasonov, G. Panina

EACH SIMPLE CONVEX POLYTOPE FROM $\mathbb{R}^n$ HAS A POINT WITH $2n+4$ NORMALS TO THE BOUNDARY

Abstract. We prove that for $n > 3$ each generic simple polytope in $\mathbb{R}^n$ contains a point with at least $2n+4$ emanating normals to the boundary. This result is a piecewise-linear counterpart of a long-standing problem about normals to smooth convex bodies.

§1. Introduction

It has long been conjectured that for any smooth convex body $\mathbf{P} \subset \mathbb{R}^n$ there exists a point in its interior which belongs to at least $2n$ normals emanating from distinct boundary points of $\mathbf{P}$. The proof of the conjecture is a simple exercise for $n = 2$. There are two known proofs for $n = 3$: a geometrical one by E. Heil [2], and a topological one by J. Pardon [3]. For $n = 4$ the conjecture was proved by J. Pardon [3], and (to the best of our knowledge) nothing is known for higher dimensions. In this paper we solve a similar problem for simple convex polytopes.

Let $\mathbf{P} \subset \mathbb{R}^n$ be a compact convex polytope with non-empty interior and let $y \in Int\ \mathbf{P}$. A *normal* to the boundary of $\mathbf{P}$ emanating from y is a line yz such that $z \in \partial\mathbf{P}$, and yz is orthogonal to some supporting hyperplane containing z. The point z is called the *base* of this normal.

Our main result is:

Theorem 1. *For $n > 2$, each generic simple polytope $\mathbf{P} \in \mathbb{R}^n$ has a point with (at least) $2n+4$ emanating normals to the boundary.*

Important remarks are:

- Each polytope has a point with $2n+2$ normals. This follows directly from the Morse count, see [4] for details.

Key words and phrases: Morse theory, bifurcation, critical point.

The work of I. Nasonov was performed at the Saint Petersburg Leonhard Euler International Mathematical Institute and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement No. 075-15-2025-343).

- The statement of the theorem is not true for $n = 2$: a triangle has no point with eight normals.
- For $n = 3$ the bound is exact: there exists a tetrahedron with at most 10 normals emanating from its inner points.
- There is already a proof for $n = 3$ for all generic polytopes, not only simple ones (see [5], and a slightly different proof in [6]) which is quite involved.
- The proof for $n = 4$ is also involved, but that for $n > 4$ is simpler. For this reason we conjecture that for large n there is a point with $2n + 6$ normals to the boundary of $\mathbf{P}$.

The **main idea of the proof in a nutshell** looks like this:

(1) In the smooth case the bases of the normals emanating from y are the critical points of the *squared-distance function*

$$SQD_y^{\mathbf{P}}(x) = |x - y|^2 \colon \partial\mathbf{P} \to \mathbb{R}.$$

Since $SQD_y^{\mathbf{P}}$ is generically a Morse function, Morse theory is applicable. The Morse-theoretic machinery extends also to polytopes [4], so counting normals to the boundary is equivalent to counting critical points of $SQD_y^{\mathbf{P}}$.

(2) We choose a very special point $y \in \mathbf{P}$, force it to travel along some carefully chosen trajectories and watch the bifurcations of critical points of $SQD_y^{\mathbf{P}}$. This allows us to prove that if a polytope $\mathbf{P}$ has no point with $2n+4$ normals, then all its $(n-3)$-faces are *skew*, that is, the adjacent cones have a very specific shape.

(3) For $n = 4$ we need to once again look at bifurcations, in the same spirit of the item (2), but using other trajectories.

(4) For $n > 4$ we arrive at some non-realizable combinatorics.

The paper relies very much on ideas developed in [5] and [4]. So we refer the reader to these papers for more detailed explanations of the backgrounds and also for more figures.

§2. Definitions and preliminaries

Let $\mathbf{P} \subset \mathbb{R}^n$ be a compact convex polytope with non-empty interior. *A face* of $\mathbf{P}$ is the intersection of $\mathbf{P}$ with a support hyperplane. The faces of dimension k will be called *k-faces* for short. 0-faces and 1-faces are called vertices and edges, as usual. The faces of dimension $n-1$ are called *facets*.

Throughout the paper we assume that $\mathbf{P}$ is *simple*, that is, each vertex has exactly n incident edges (equivalently, meets n facets).

The *cone* of a face F (denoted by CF) is defined as the positive span of $\mathbf{P}$ assuming that the origin of the ambient space $\mathbb{R}^n$ lies in the relative interior of F. In plain words, the cone CF is bounded by extensions of the facets containing F.

Let F be a face of $\mathbf{P}$. The *active region* $\mathcal{AR}(F)$ [4] is the set of all points $y \in Int\,\mathbf{P}$ such that F contains the base of some normal emanating from y.

The *bifurcation set* $\mathcal{B}(\mathbf{P})$ [4] is defined as the union of the boundaries of active regions. It is the piecewise linear counterpart of the focal set that appears in the smooth case.

The bifurcation set cuts $\mathbf{P}$ into a number of (open) cameras. For points in one and the same camera, the number of emanating normals is one and the same, and the bases of the normals lie in one and the same faces.

Although the function $SQD_y^{\mathbf{P}}$ is non-smooth, it can be treated as a Morse function. So counting critical points of $SQD_y^{\mathbf{P}}$ amounts to counting normals, exactly as it is in the smooth case.

Lemma 1 ([4])**.** *For $y \notin \mathcal{B}(\mathbf{P})$, we have*:

(1) *Local maxima of $SQD_y^{\mathbf{P}}$ are attained at some vertices of $\mathbf{P}$.*
(2) *Local minima of $SQD_y^{\mathbf{P}}$ are attained at some facets of $\mathbf{P}$.*
(3) *Critical points of Morse index m are attained at some $(n-m-1)$-faces of $\mathbf{P}$.*
(4) *If y crosses transversally one sheet of $\mathcal{B}(\mathbf{P})$, a pair of critical points of $SQD_y^{\mathbf{P}}$ either annihilates or is born. The Morse indices of this pair of points differ by one.*
(5) *The number of critical points is even.*

Convention: From now on, we call local maxima (local minima) just maxima and minima for short.

We say that a point $y \in Int(\mathbf{P})$ *projects* onto a face F if the orthogonal projection of y to the affine hull of F lies in the interior of the face F.

The function $SQD_y^{\mathbf{P}}$ has a critical point at a face F iff two conditions hold:

(1) y projects to F.
(2) $\overrightarrow{zy}$ makes angles non-greater than $\pi/2$ with zx, where z is the projection of y, and x ranges over $\mathbf{P}$.

Genericity convention ([5]): A polytope $\mathbf{P}$ with the vertex set $\mathbf{V}$ is *generic* if affine hulls of any two subsets of $\mathbf{V}$ are neither parallel nor orthogonal, unless this is dictated by unavoidable reasons (for example, an edge is always parallel to a face containing the edge).

Important observations are: (1) For a generic polytope, the sheets of $\mathcal{B}(\mathbf{P})$ intersect transversally. (2) Any polytope can be turned to a generic one by a close to *id* projective transform.

§3. Some necessary tools from spherical geometry

A *spherical polygon* is a polygon with geodesic edges lying in the standard sphere S^2. It is always supposed to fit in an open hemisphere. A *spherical polytope* in the standard sphere S^k is defined analogously.

A vertex V of a polytope $\mathbf{P} \subset \mathbb{R}^n$ yields a convex spherical polytope $Spher(V) \subset S^{n-1}$ which is the intersection of a small sphere centered at V with the polytope $\mathbf{P}$. Although the sphere is small, we assume that it is equipped with the metric of the unit sphere.

Analogously, an m-face F of a polytope $\mathbf{P}$ yields a convex spherical polytope $Spher(F) \subset S^{n-m-1}$ which is the intersection of a small sphere centered at any point x in the relative interior of F with the polytope $\mathbf{P}$ and with the $(n-m)$-plane passing through x and orthogonal to F.

We borrow the definition and properties of *skew spherical triangles* from [4].

Definition 1 ([4]). *A spherical triangle $V_1V_2V_3$ is nice if there is a point X in its interior such that*

(1) *X projects to all the three edges of the triangle, and*
(2) *$|XV_i| < \pi/2$ for each i.*

Otherwise, a spherical triangle is called skew.

Lemma 2 ([4]). *For a skew triangle, up to renumbering,*

(1) *the angles at V_1 and V_3 are acute, the angle at V_2 is obtuse,*
(2) *the edges V_1V_2 and V_1V_3 are longer than $\pi/2$, the edge V_2V_3 is shorter than $\pi/2$,*
(3) *if $Z \in V_1V_2V_3$ and $|V_1Z| < \pi/2, |V_3Z| < \pi/2$ then $|V_2Z| < \pi/2$.*

For a spherical convex polytope P and a point $Y \in P$ denote by

$$SQD_Y^P\colon \partial P \to \mathbb{R}$$

the spherical analog of $SQD^{\mathbf{P}}$. Spherical squared-distance functions inherits the properties of the Euclidean squared-distance function only if the distance from Y to the base of the corresponding normal are less than $\pi/2$.

§4. Skew and nice faces

For an $(n-3)$-face F, the polytope $Spher(F)$ is a spherical triangle.

Definition 2. *An $(n-3)$-face F of $\mathbf{P}$ is called nice (respectively, skew) if $Spher(F)$ is nice (respectively, skew).*

Let us extend this notion to faces of other dimensions. The motivation comes from counting normals to the faces of a cone: for $y \in CF$ let

$$SQD_y^{CF} \colon \partial CF \to \mathbb{R}$$

be the squared-distance function. If F is a nice $(n-3)$-face of $\mathbf{P}$, then there exists a point $y \in CF$ such that SQD_y^{CF} has at least 7 critical points. By genericity we may assume that the set of points with at least seven normals has a non-empty interior.

Definition 3. *For $k = 3, 4, \ldots$, a $(n-k)$-face F of $\mathbf{P}$ is called nice if there exists a point $y \in CF$ such that SQD_y^{CF} has at least $2k+1$ critical points on the faces of CF.*

The first observations are following:

Lemma 3. *In the above notation,*

1. *The number of critical points of SQD_y^{CF} is odd.*
2. *If an $(n-k)$-face F is nice, there is a point $y \in \mathbf{P}$ with at least $2k+1$ normals to the faces of $\mathbf{P}$ containing F.*
3. *Let F, F' be some faces such that F' is a facet of F. Then the cone F' equals CF intersected with a halfspace h^+ bounded by some hyperplane h.*

Proof. (1) This follows from "attaching handles" Morse count [1]: on the one hand, the preimage $\left(SQD_y^{CF}\right)^{-1}[0, r]$ for a fixed y and sufficiently large r is a ball, so its Euler characteristic is 1. On the other hand, the preimage decomposes into handles of different dimensions whose Euler characteristics sum up to 1. Each handle corresponds to a critical point, so the number of critical points is odd.

(2) For this purpose choose y lying close to F. This ensures that the bases of the normals to CF lie on the faces of $\mathbf{P}$.

(3) Follows from simplicity of $\mathbf{P}$. □

Proposition 1. *Each face of a nice face of* $\mathbf{P}$ *is nice.*

Proof. We shall prove that if $F' \subset F$ are some faces, $\dim F = n-k$, $\dim F' = n-k-1$, and F is nice, then F' is also nice. Let h be a hyperplane given by Lemma 3. Pick a point $y \in CF$ for which SQD_y^{CF} has at least $2k+1$ critical points. Denote these points by $x_1, x_2, \ldots \in \partial CF$.

We can assume that $y \in CF'$, and $x_1, x_2, \ldots \in \partial CF'$. Indeed, otherwise, one translates y by a vector parallel to F.

Case 1. If SQD_y^{CF} already has at least $2k+3$ critical points, we are done since the points $x_1, x_2, \ldots, x_{2k+3}$ are the critical points of $SQD_y^{CF'}$.

Case 2. Otherwise, the number of critical points SQD_y^{CF} is exactly $2k+1$. Now $y \in CF'$ and $x_1, \ldots, x_{2k+1} \in \partial CF \cap h^+ \subset \partial CF'$ are all the critical points of $SQD_y^{CF'}$.

Let the point y travel by a straight line parallel to F so that its distance to the hyperplane h decreases. Consider the *first event*, which is (by definition)

(a) either a bifurcation of $SQD_y^{CF'}$, or

(b) the point y reaches h.

Case (a). By genericity convention we may assume that y crosses transversally exactly one sheet of $\mathcal{B}(\mathbf{P})$ at a time. During the travel, the pairwise distances between the points $x_1, \ldots, x_{2k+1}$ and y persist. So, they never collide with each other. The only possible bifurcation is a bifurcation with some extra critical point, not from this list.

Case (b). Just before reaching h, $SQD_y^{CF'}$ has additional local minima on h, which were not on the list.

Hence we have at least $2k+2$ critical points; since Lemma 3 implies the count is odd, there must be at least $2k+3$. □

Corollary 1. *If* $\mathbf{P}$ *has a nice* $(n-3)$*-face* F*, then there exists* $V \in Vert(\mathbf{P})$ *and a point* $y \in \mathbf{P}$ *such that* $SQD_y^{\mathbf{P}}$ *has at least* $2n+1$ *critical points on the faces of* $\mathbf{P}$ *containing* V*.*

Proof. Follows from Lemma 3 and Proposition 1. □

Proposition 2. *If* $\mathbf{P}$ *has at least one nice face, then there is a point in* Int $\mathbf{P}$ *with at least* $2n+4$ *normals to the boundary.*

Proof. By Corollary 1, there is a vertex V and a point $y \in \mathbf{P}$, such that $SQD_y^{\mathbf{P}}$ has at least $2n+1$ critical points on the faces of $\mathbf{P}$, containing V. We can assume that y lies sufficiently close to V. Besides we have a global maximum, so altogether there are at least $2n+2$ critical points. If there are $2n+4$ critical points, we are done. Otherwise, there are exactly $2n+2$ critical points since the number is always even. So, we don't have any critical points except those listed above.

Let y travel straight away from V until the first event occurs. If y reaches $\partial\mathbf{P}$, an extra minimum appears just beforehand. In the case of a bifurcation, the old $2n+1$ critical points cannot collide and annihilate because their pairwise distances increase; they also cannot collide with the global maximum. Hence the first bifurcation is a birth of two new critical points. (Here again by genericity convention we assume that y crosses transversally exactly one sheet of $\mathcal{B}(\mathbf{P})$ at a time.) □

§5. Proof of Theorem 1 for $n = 4$

Due to Proposition 2 it remains to prove the theorem for the case when all the edges of $\mathbf{P}$ are skew.

Take any vertex V of $\mathbf{P}$. The associated spherical polytope $Spher(V) \subset S^3$ is a convex spherical tetrahedron Δ. The vertices of Δ correspond to edges of $\mathbf{P}$ that emanate from V, so all the vertices of Δ are skew.

Lemma 4. *The tetrahedron Δ contains a point Y in its interior such that*

$$SQD_Y^{\Delta} \colon \partial\Delta \to \mathbb{R}$$

has at least 8 critical points, and the distances from Y to these critical points are less than $\pi/2$.

Proof. Take $Y \in \Delta$ which minimizes the Hausdorff distance to $\partial\Delta$. Then automatically the distance from Y to any other point of Δ is less than $\pi/2$. In particular this means that minima, saddle points, and maxima of SQD_Y^{Δ} are attained respectively at facets, edges, and vertices of Δ (which is not always true for spherical polytopes).

Observe that the edges of Δ correspond to 2-faces of $\mathbf{P}$. Therefore, by Lemma 2, the spherical simplex Δ has exactly four acute edges. These edges form a closed broken line $\mathcal{A}$. Denote by $SQD_Y^{\mathcal{A}} \colon \mathcal{A} \to \mathbb{R}$ the spherical distance from Y to the points on $\mathcal{A}$.

The following fact is a spherical counterpart of Lemma 6 from [5].

- Each minimum of $SQD_Y^{\mathcal{A}}$ is a saddle point of SQD_Y^{Δ}.

- Each maximum of $SQD_Y^{\mathcal{A}}$ is a maximum of SQD_Y^{Δ}.

Let us comment on this fact a little. $SQD_Y^{\mathcal{A}}$ has a minimum at an edge e iff Y projects to e and distance to that projection is less than $\pi/2$. This is already sufficient for SQD_Y^{Δ} to have a saddle point at e since e is acute.

Besides, $SQD_Y^{\mathcal{A}}$ never attains a minimum at a vertex of $\mathcal{A}$. Indeed, if $V_1V_2V_3$ is a skew triangle (as in Lemma 2) then there is no point $Z \in V_1V_2V_3$ that $|V_1Z| > \pi/2$ and $|V_3Z| > \pi/2$.

For the second statement, we need the following observation. Assume that $V_1V_2V_3$ is a skew triangle, and a point X in it is such that $|XV_1|$ and $|XV_3|$ are less than $\pi/2$. Then $|XV_2| < \pi/2$, see Lemma 2.

Denote by d the Hausdorff distance from Y to $\partial\Delta$ and consider three cases.

(1) d is attained at four vertices of Δ. Then SQD_Y^{Δ} has four maxima, therefore at least three saddle points and one minimum.

(2) d is attained at three vertices of Δ. Then SQD_Y^{Δ} has three maxima, hence $SQD_Y^{\mathcal{A}}$ has three maxima, hence $SQD_Y^{\mathcal{A}}$ has three minima, hence SQD_Y^{Δ} has three saddle points. There is also at least one minimum, and since the number of critical points is even, we get 8.

(3) d is attained at two vertices of Δ, say, at U and V. Then Y belongs to the edge UV. Otherwise, one can shift Y and decrease the Hausdorff distance. But none of U, V is a maximum for a point lying on UV by the properties of skew triangles, see Lemma 2. □

Now prove the theorem. Take any vertex V of the polytope $\mathbf{P}$. By Lemma 4, V is a nice face. Indeed, 8 "short" critical points of SQD_Y^{Δ} correspond to 8 critical points of SQD_Y^{CV}.

So to complete the theorem it remains to apply Proposition 2.

§6. Proof of Theorem 1 for $n > 4$

Assume the contrary, that a polytope $\mathbf{P}$ has no point with $2n+4$ normals.

By Proposition 2, all the $(n-3)$-faces of $\mathbf{P}$ are skew. Hence by Lemma 2 each $(n-3)$-face is contained in two acute $(n-2)$-faces and in one obtuse $(n-2)$-face. Color each acute $(n-2)$-face *red* and each obtuse $(n-2)$-face *blue*. We arrive at a coloring of all $(n-2)$-faces of $\mathbf{P}$ with the key combinatorial property: *each* $(n-3)$*-face has exactly two adjacent red* $(n-2)$*-faces and one blue* $(n-2)$*-face.*

We show that no such coloring exists, yielding a contradiction. Choose any vertex V and cut it off from $\mathbf{P}$ by a hyperplane h. Then $h \cap \mathbf{P}$ is a simplex Δ^{n-1}, which inherits the coloring and the key property: the $(n-3)$-faces of Δ^{n-1} are colored red and blue such that each of the $(n-4)$-face has exactly two adjacent red faces and one blue face. Such a coloring does not exist. It is an easy fact; one of the ways to prove it is to further cut the vertices and to reduce the dimensions of simplices until we get Δ^4. A simple case analysis shows that Δ^4 is not colorable in this sense. □

Remark. This proof breaks down for four-dimensional polytopes. Moreover, there exist colorable polytopes in $\mathbb{R}^4$. For instance, a cube is colorable.

References

1. J. Milnor, *Morse Theory.* Ann. of Math. Stud. **51**, Princeton University Press, Princeton, 1963.
2. E. Heil, *Concurrent normals and critical points under weak smoothness assumptions.* In: Discrete geometry and convexity (New York, 1982), volume 440 of Ann. New York Acad. Sci., 170–178. New York Acad. Sci., New York, 1985.
3. J. Pardon, *Concurrent normals to convex bodies and spaces of Morse functions.* — Math. Ann. **352**, No. 1 (2012) 55–71.
4. I. Nasonov, G. Panina, D. Siersma, *Concurrent normals problem for convex polytopes and Euclidean distance degree.* — Acta Math. Hungar. **174** (2024) 522–538.
5. I. Nasonov, G. Panina, *Each generic polytope in $\mathbb{R}^3$ has a point with ten normals to the boundary*, — arXiv:2411.12745 (2025).
6. R. Angst, *On skew triangulations of the sphere and the 10 normals conjecture for convex polytopes*, Master Thesis, ETH Zurich, 2025, `https://doi.org/10.3929/ethz-c-000783475`.

Saint Petersburg University,
7/9 Universitetskaya nab.,
St. Petersburg, 199034 Russia
E-mail: `wanua-nasonov-i04@yandex.ru`

Поступило 1 декабря 2025 г.

St. Petersburg department of
Steklov institute of mathematics RAS,
Fontanka 27,
St. Petersburg, 191023 Russia
E-mail: `gaiane-panina@rambler.ru`