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INFINITE MIDWAY CLIQUES IN THE GORDIAN GRAPH

Abstract. Inspired by results of Hirasawa, Uchida, and Baader, we reveal a new geometric pattern in the Gordian complex of knots. We prove that for any two vertices at Gordian distance 2, the intersection of their 1-neighborhoods contains an infinite-dimensional simplex. The proof relies on a new geometric sufficient condition of the non-splittability of links, based on an iterative construction of gropes from unknotted one-holed tori. As a corollary, the Gordian graph remains connected after removing any induced locally finite subgraph.

§1. Introduction

The *Gordian distance* $d_G(K, Q)$ between two tame knots K and Q is defined as the minimal number of crossing changes required to transform K into Q, where the minimum is taken over all diagrams. The *Gordian complex* $\mathcal{G}$ is defined as follows: its vertices are isotopy classes of tame knots in S^3, and $n+1$ vertices $\{K_0, \dots, K_n\}$ form an n-simplex precisely when all pairwise Gordian distances between them equal 1. Ubiquitous geometric patterns in the *Gordian graph*, the 1-skeleton of the Gordian complex, constitute the main subject of our investigation.

In 2002, Hirasawa and Uchida [2] discovered a nontrivial geometric pattern in the Gordian graph. They showed that every edge in this graph is contained in an infinite complete subgraph (see Fig. 1 left). In the language of the Gordian complex, this means that for any 1-simplex e, there exists an infinitely high-dimensional simplex σ such that e is a face of σ.

In 2006, Baader [1] revealed another geometric pattern: for every pair of knots K and Q with Gordian distance two, there exist infinitely many non-equivalent knots whose Gordian distance to both K and Q is one (see Fig. 1 right).

Key words and phrases: knot theory, Gordian graph, Gordian complex, crossing change, non-splittability, satellite knots, incompressible tori.

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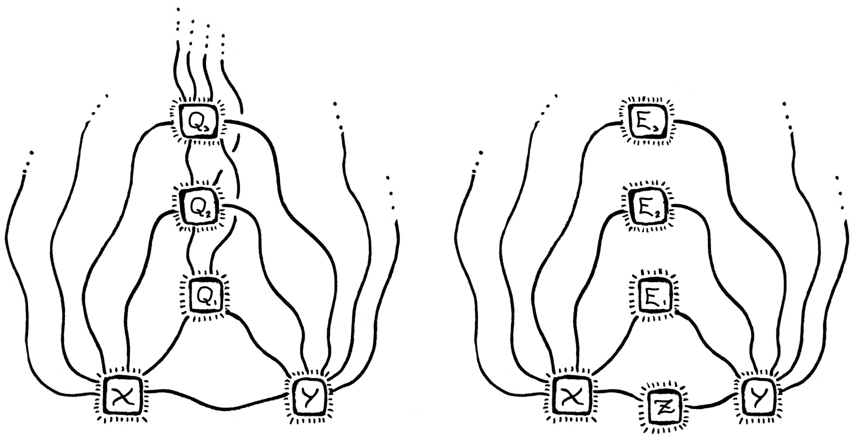

Figure 1. Hirasawa–Uchida pattern and Baader pattern.

In this work, we present a new pattern that conceptually unifies these two previous patterns (see Fig. 2).

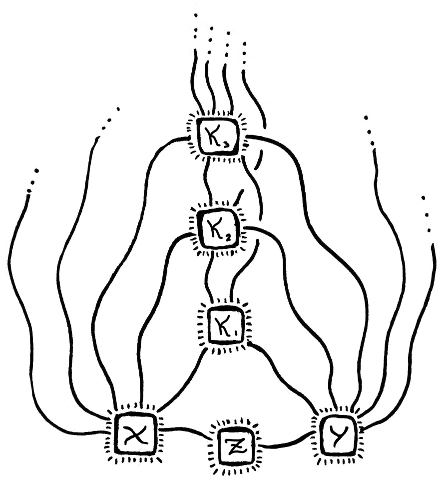

Figure 2. Our pattern.

Theorem 1. *In the Gordian complex of knots, for any pair of vertices at Gordian distance* 2*, the intersection of their* 1*-neighborhoods contains an infinitely high dimensional simplex.*

Corollary 1. *In the Gordian complex of knots, the intersection of* 1*-neighborhoods of any pair of vertices is either empty or contains an infinitely high dimensional simplex.*

A subgraph F of a graph G is *induced* if every edge in G whose endpoints belong to F is also an edge in F. A subgraph is *locally finite* if every vertex of the subgraph has finite degree (in the subgraph).

Corollary 2. *The connectivity of the Gordian graph is preserved under the removal of any induced locally finite subgraph. Moreover, the distance between any two vertices in the complement is equal to their Gordian distance.*

§2. Proof of the main theorem

Definition 1. *Let $W = N(K)$ be a closed tubular neighborhood of a nontrivial knot $K \subset S^3$, and let Q be a knot in an unknotted solid torus $V \subset S^3$ such that Q is not contained in any* 3*-ball within V and is not isotopic to the core of V. A* satellite *of K is the image $K^* = \varphi(Q) \subset W \subset S^3$ under a homeomorphism $\varphi\colon V \to W$. We say K is a* companion *of any knot K^* constructed* (*up to knot type*) *in this manner.*

Remark 1. One can see that if T is an incompressible non-boundary-parallel torus in $S^3 \setminus \mathrm{Int}(N(X))$ for a knot $X \subset S^3$, and if the core of the open solid torus component V of $S^3 \setminus T$ is knotted into a nontrivial knot Y, then Y is a companion of X.

Recall that an essential simple closed curve in the torus $\partial(D^2 \times S^1)$ that bounds a disk in the solid torus $D^2 \times S^1$ is called a *meridian curve*.

Let $K \cup L \subset S^3$ be a link. We say that K is *unlinked to L* if there exists a 2-sphere $S \subset S^3 \setminus (K \cup L)$ that *separates* K and L; that is, K and L lie in distinct connected components of $S^3 \setminus S$. Otherwise, we say that K is *linked to L*. If K is an unknotted circle, then it is unlinked to L if and only if K bounds a disk in $S^3 \setminus L$.

The following lemma gives an equivalent characterization of the companion property (see Remark 1).

Lemma 1. *Let W be a non-trivially knotted solid torus in S^3, and let K be a knot in* $\mathrm{Int}(W)$*. Then the following conditions are equivalent*:

(1) *a meridian curve of the solid torus W is linked to K*;
(2) *the torus ∂W is incompressible in $S^3 \setminus K$.*

Proof of Lemma 1. If ∂W is compressible in $S^3 \setminus K$ then there exists an essential simple closed curve $\gamma \subset \partial W$ that bounds a disk $D \subset S^3 \setminus K$ such that $D \cap \partial W = \gamma$. Since ∂W is incompressible in $S^3 \setminus \mathrm{Int}(W)$, it follows that the disk D must lie in W. Since there is only one isotopy class of essential simple closed curves on ∂W that bound a disk in W, the curve γ must be a meridian one. Hence, no meridian curve is linked to K.

Conversely, if a meridian curve c of W is unlinked to K, then there exists a sphere $S \subset S^3$ that separates c from K and intersects ∂W transversely. Since both c and K lie in the (connected) solid torus W, it follows that S intersects W. If $\partial W \cap S = \varnothing$, then S bounds a ball E in W, and E contains K. In this case, it follows immediately that ∂W is compressible in $S^3 \setminus K$. If $\partial W \cap S \neq \varnothing$, then at least two components of $S \setminus \partial W$ are open disks and we denote by B the closure of such a component. If ∂B is an essential (simple closed) curve on ∂W then B is a compressing disk for ∂W in $S^3 \setminus K$. If ∂B bounds a disk $B' \subset \partial W$, then $B \cup B'$ forms a 2-sphere that splits S^3 into two open balls. If the 2-sphere $B \cup B'$ separates c and K, then we have a 3-ball in W that contains K, and we proceed as above. If the 2-sphere $B \cup B'$ does not separate c and K, let E be the closed ball bounded by $B \cup B'$ that is disjoint from $K \cup c$. An obvious isotopy of S in a small neighborhood of E yields a 2-sphere S' that separates K and c and has fewer intersection circles with ∂W than S does. We then proceed by induction. □

Definition 2. *We say that a surface $H \subset S^3$ is an* unknotted torus with a hole (*or* unknotted one-holed torus) *if there exists an unknotted solid torus $V \subset S^3$ such that $H \subset \partial V$ and $\partial V \setminus H$ is an open disk.*

We say that a pair of simple closed curves in an unknotted one-holed torus H is a *meridian-longitude pair* of H if this pair is a meridian-longitude pair of some (and hence any) unknotted torus containing H.

Remark 2. Note that elements of any meridian-longitude pair of an unknotted torus with a hole $H \subset S^3$ represent a unique pair of isotopy classes of non-boundary parallel essential simple closed curves on H that bound disks in $S^3 \setminus H$. Indeed, let a simple closed curve on H bounds a disk $B \subset S^3 \setminus H$, and let T be a unknotted torus containing H. By a standard cut-and-paste argument (performing surgery along subdisks of $T \setminus H$), we can assume that B is disjoint from $T \setminus H$, so that $\mathrm{Int}(B)$ lies in one of the solid tori $S^3 \setminus T$, from which the required claim follows.

Lemma 2 (Tomsk 2023)**.** *Let K be a knot in S^3, let H be an unknotted one-holed torus in $S^3 \setminus K$, and let $\{m_H, l_H\}$ be a meridian-longitude pair of H. If each of m_H and l_H is linked to K, then ∂H is linked to K.*

Proof of Lemma 2. Assume to the contrary that each of m_H and l_H is linked to K, while ∂H is not. Then there exists a sphere $S \subset S^3$ that separates ∂H from K and intersects H transversely. If $H \cap S = \varnothing$, then S separates $H \supset \partial H$ from K, and hence neither m_H nor l_H is linked to K. If $H \cap S \neq \varnothing$, then at least two components of $S \setminus H$ are open disks and we denote by B the closure of such a component. By Remark 2, one of the following four possibilities holds: ∂B is isotopic (in H) to m_H, ∂B is isotopic (in H) to l_H, ∂B is isotopic (in H) to ∂H, or ∂B bounds a disk $B' \subset H$.

The cases where ∂B is isotopic to m_H or l_H contradict the assumption that m_H and l_H are linked to K.

If ∂B is isotopic to ∂H, then we have an unknotted torus $T \subset H \cup B$. This torus gives a decomposition $S^3 = V \cup W$ of the three-sphere into two solid tori such that $V \cap W = T$. Since the knot $K \subset S^3 \setminus T$ is connected, we have either $K \subset \mathrm{Int}\, V$ or $K \subset \mathrm{Int}\, W$. In any case, the boundary circle of a meridian disk of the solid torus not containing the knot is unlinked to K. This implies that either m_H or l_H is unlinked to K.

Finally, if ∂B bounds a disk $B' \subset H$, we surger H along B (by replacing a neighborhood of ∂B in H with two parallel copies of B) to obtain a new unknotted one-holed torus H' in $S^3 \setminus K$ with $\partial H' = \partial H$. Without loss of generality we can assume that neither m_H nor l_H intersects B', so that H' contains m_H and l_H (and $\{m_H, l_H\}$ is a meridian-longitude pair of H'). We notice that the number of intersection circles in $H' \cap S$ is less than that of $H \cap S$ and proceed by induction. □

Proof of Theorem 1. If two knots X and Y have Gordian distance two, then there exists a knot Z such that both X and Y can be obtained from Z by a single crossing change. One can readily draw a diagram of Z with two specified crossings, $\boxed{\text{A}}$ and $\boxed{\text{B}}$, such that a crossing change at A or B transforms Z into X or Y, respectively. Furthermore, these two crossings can be brought adjacent to each other within the diagram of Z via an isotopy. The relevant part of such a diagram for Z is shown in Fig. 3.

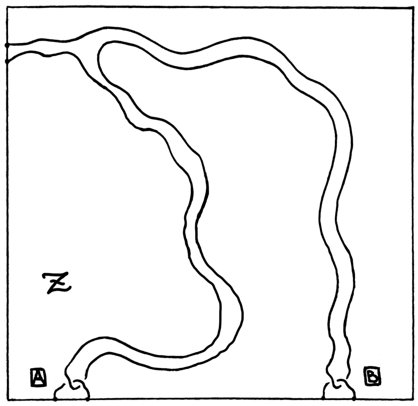

Figure 3. A part of a diagram of Z.

Now we define the family of knots $\{K_n\}_{n\in\mathbb{N}}$ diagrammatically. The relevant part of the knot K_n diagram we are now constructing is shown in Fig. 5. Outside the indicated region, the diagram of K_n is assumed to be identical to the initial diagram of Z.

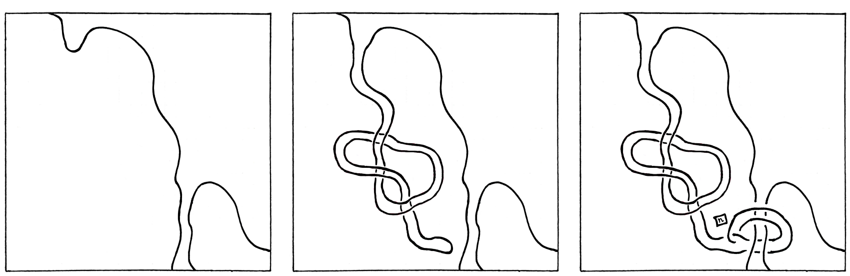

Figure 4. Tying and clasping.

The knots in this family are constructed iteratively; specifically, K_n is obtained from K_{n-1} via the following modification. We proceed as shown in Fig. 4. Starting from a strand of K_{n-1} in the upper left part of the diagram, we form a small loop, tie it into some knot P_n (the specific choice of which is explained below), and wrap it around an analogous loop from the K_{n-1} construction (see Figs. 4 and 5). Finally, a crossing change at the label $\boxed{\text{n}}$ attaches the resulting clasper to itself.

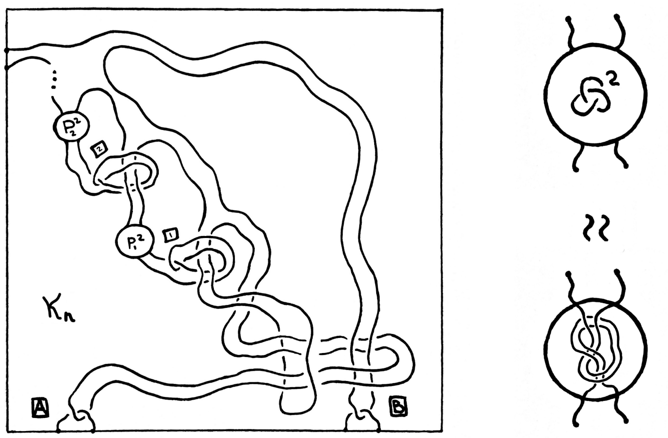

Figure 5. A part of a diagram of K_n and an example of a doubled tangle obtained from the trefoil knot.

This construction admits a more formal description. The diagram of K_n differs from that of K_{n-1} only by the addition of a block containing the 2-tangle P_n^2 (see Fig. 5). In order to construct this 2-tangle, we first obtain a 1-tangle from P_n by cutting it at a point and placing the resulting P_n-knotted arc in a 3-ball so that only its endpoints lie on the boundary. We then double the strand of this tangle by performing a parallel shift along a continuous normal vector field defined on it, which yields P_n^2. Although the choice of vector field may introduce additional twists, this ambiguity is irrelevant for our purposes. As an example, Fig. 5 shows this doubling construction applied to the trefoil knot.

The specification of P_n requires some preliminaries. According to Schubert's Theorem [3] (Satz 6), the set $\mathrm{C}(Q)$ of companions of any knot Q is finite. We take P_n to be any prime knot not in $\bigcup_{i=1}^{n-1} \mathrm{C}(K_i)$, with P_1 chosen arbitrarily.

Note that any two knots K_i and K_j with $i \neq j$ differ by a single crossing change. Without loss of generality, assume $i > j$; then changing the crossing at the label $\boxed{j+1}$ in the diagram of K_i yields K_j. Moreover, crossing change at labels $\boxed{A}$ or $\boxed{B}$ transforms the knot K_i into knots X or Y, respectively.

We now show that the family $\{K_n\}_{n \in \mathbb{N}}$ contains infinitely many distinct knots. By construction, there exists a solid torus $W \subset S^3$ that is knotted into the non-trivial knot P_n and contains K_n (see Fig. 6). It is straightforward to observe that ∂W is a non-boundary-parallel torus in $S^3 \setminus \mathrm{Int}(N(K_n))$, where $N(K_n)$ is a tubular neighborhood of K_n. Indeed, if ∂W is boundary-parallel, then the linking number of each of its meridian curves with K_n must be ± 1, but Fig. 6 shows that this linking number is actually zero. If ∂W is incompressible in $S^3 \setminus K_n$, then by Remark 1, P_n is a companion of K_n. Since P_n is not a companion of any K_i with $i < n$, it follows that K_n cannot be isotopic to any such K_i. Therefore, proving incompressibility suffices.

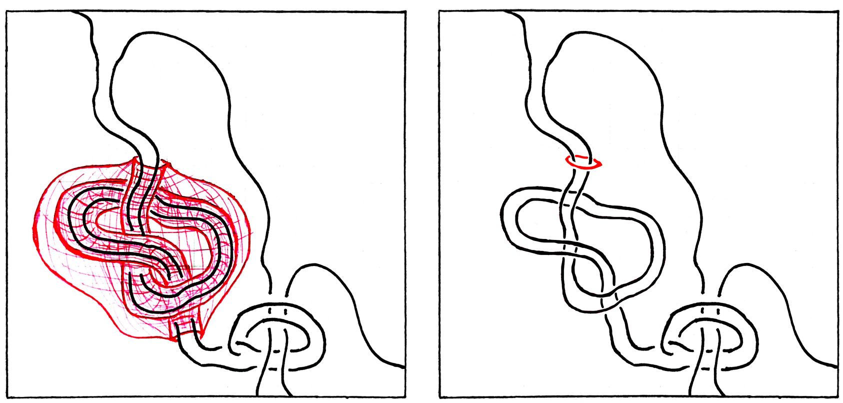

Figure 6. The red knotted torus is incompressible in $S^3 \setminus N(K_n)$ if and only if the red meridian curve is linked to K_n.

By Lemma 1, ∂W is incompressible in $S^3 \setminus K_n$ if and only if a meridian curve of ∂W is linked to K_n. We verify that a meridian curve is linked to K_n by iteratively applying Lemma 2. First, we construct an unknotted one-holed torus H in $S^3 \backslash K_n$ such that its boundary ∂H is a meridian curve of ∂W. This reduces the original problem to verifying that each element of a meridian-longitude pair of H is linked to K_n. For each of these curves, we construct its own one-holed torus and continue the process recursively. The iteration continues until at some step we can verify that the required curves are linked to K_n simply by using the linking number. As illustrated in Fig. 7, this process always terminates. We eventually reach a stage where for the next constructed torus, both curves have linking number ± 1 with K_n. For a more detailed step-by-step construction, see Appendix. The constructed grope of unknotted one-holed tori establishes that a meridian curve of ∂W is linked to K_n, as required. This completes the proof. □

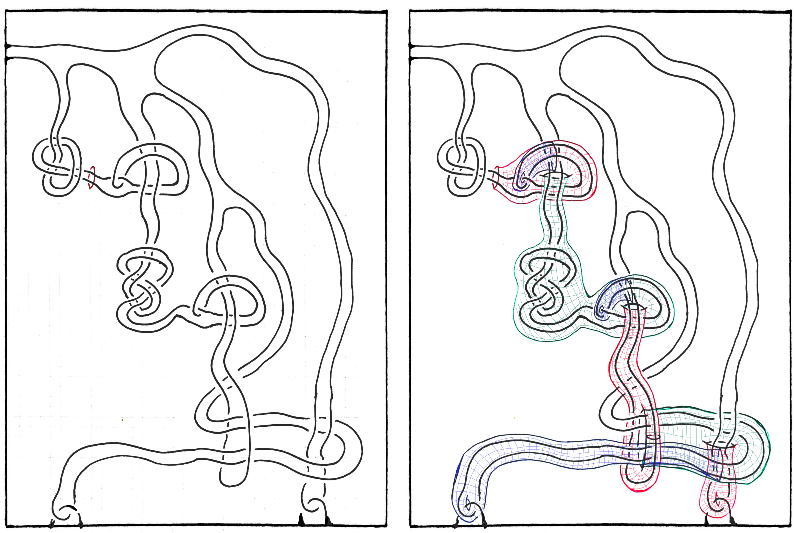

Figure 7. The grope of unknotted one-holed tori on the right side proves that the red meridian curve on the left side is linked to K_2.

Proof of Corollary 2. Suppose vertices A and B lie in the complement of an induced locally finite subgraph. Let γ be a shortest path between A and B in the Gordian graph. We show that γ can be replaced by a path of the same length lying entirely in the complement.

Suppose some vertex x on γ does not belong to the complement, and let v and w be its neighbors in γ. By Theorem 1, there exists an infinite family of vertices $\{k_n\}_{n\in\mathbb{N}}$ such that each k_n is adjacent to both v and w, and any two distinct members of the family $\{k_n\}_{n\in\mathbb{N}}$ are adjacent. Since the removed subgraph is locally finite, it cannot contain all vertices from $\{k_n\}_{n\in\mathbb{N}}$, therefore, there exists some $x' \in \{k_n\}_{n\in\mathbb{N}}$ in the complement.

We replace the edges $[v, x]$ and $[x, w]$ in γ with $[v, x']$ and $[x', w]$, respectively, and iterate this process until all vertices of γ lie in the complement. The length of γ remains unchanged throughout this process. □

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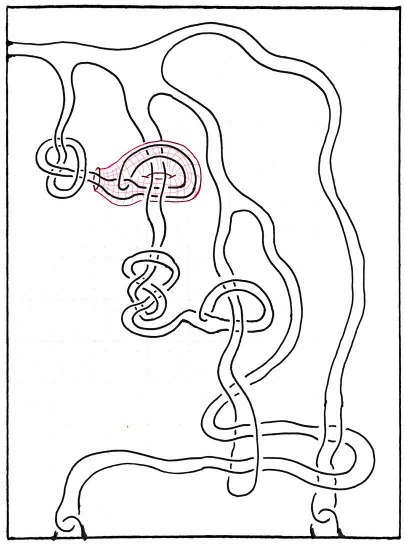
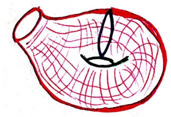
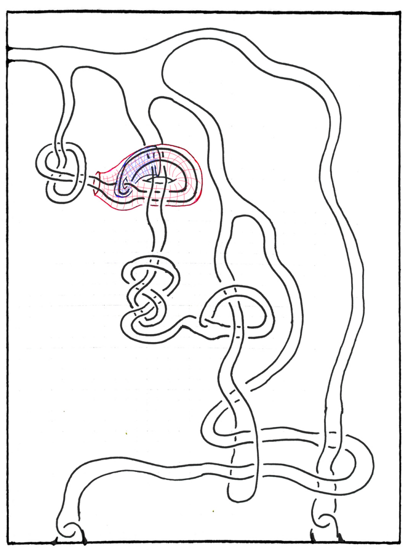
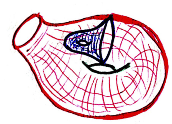
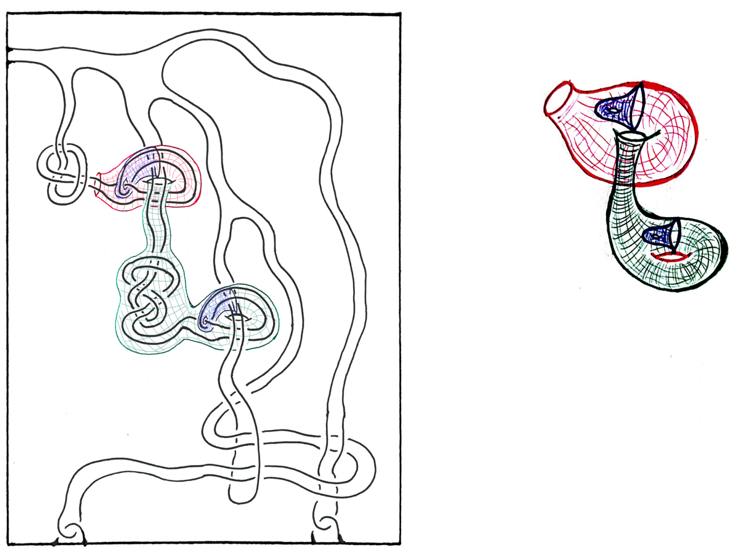
Appendix

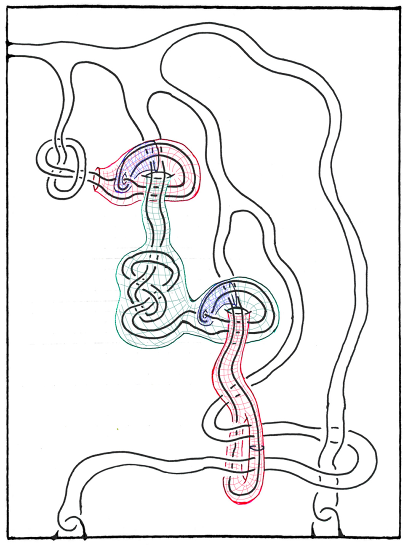


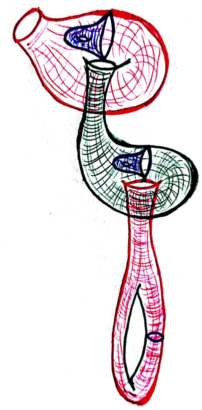


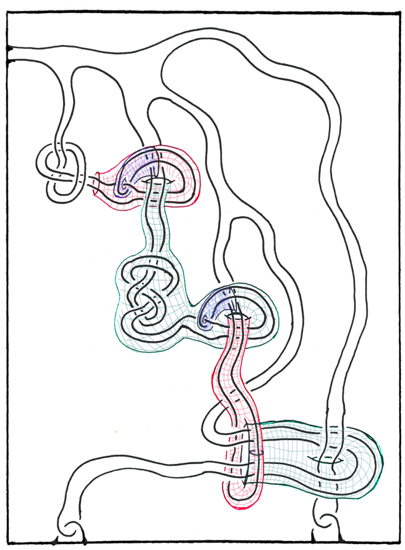


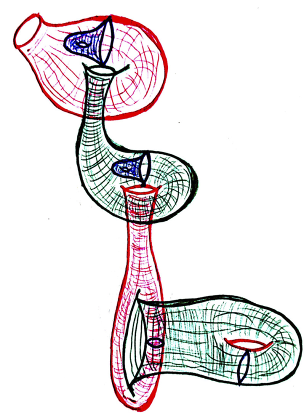


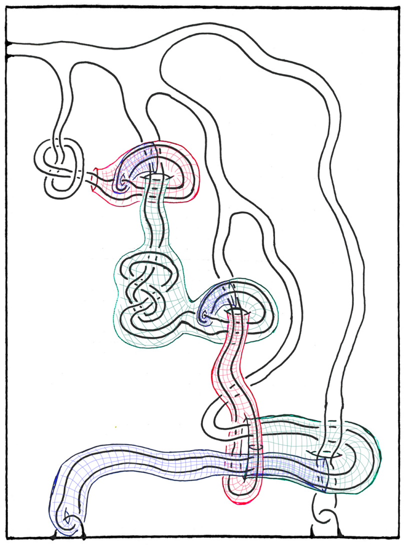


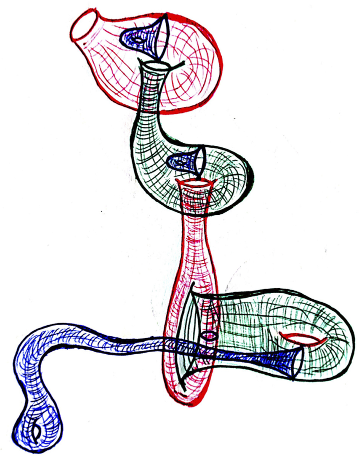


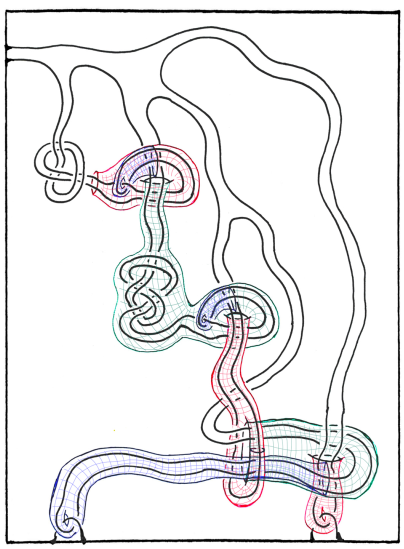


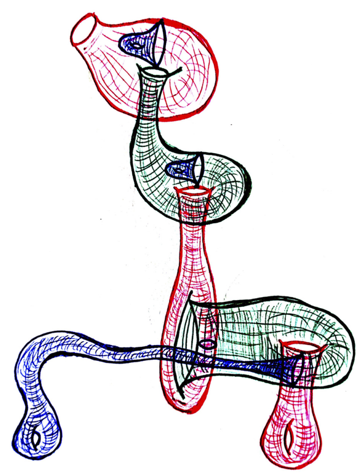












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