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THE GORDIAN GRAPH IS ONE-ENDED

Abstract. In 2005, Gambaudo and Ghys posed the question of studying the behavior at infinity of the Gordian graph by analyzing the "ends" – the unbounded connected components of the complements of bounded subsets. We completely answer this question by proving that the Gordian graph has exactly one end.

§1. Introduction

The *Gordian distance* $d_G(K, Q)$ between two tame knots K and Q is defined as the minimal number of crossing changes required to transform K into Q, where the minimum is taken over all diagrams. The *Gordian graph* is the graph whose vertices are ambient isotopy classes of unoriented tame knots in S^3, with two vertices connected by an edge if and only if the corresponding knots are related by a single crossing change. We denote the Gordian graph by G.

In 2005, Gambaudo and Ghys discovered a remarkable geometric property of the Gordian graph [1, Theorem C]. They proved that for every integer $d \geqslant 1$, there exists a quasi-isometric embedding $\mu\colon \mathbb{Z}^d \hookrightarrow G$. This finding was part of their broader study of the global geometric structure of the Gordian graph, with emphasis on its behavior at infinity. Gambaudo and Ghys further proposed several open questions in this direction (see [1, p. 547]). In particular, they highlighted the importance of investigating the space of ends of the Gordian graph, defined as the unbounded connected components that remain after removing large balls centered at the unknot.

In this paper, we answer the question about the behavior of the Gordian graph at infinity using the proposed concept of BU-ends (remove a *bounded*

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set, and count the resulting *unbounded* components). Specifically, we prove that the Gordian graph has exactly one BU-end.

The *number of BU-ends* of the Gordian graph, denoted $\mathrm{BU}(G)$, is defined as an element of the set $\mathbb{N} \cup \{\infty\}$ given by

$$\mathrm{BU}(G) = \sup_{V \in \mathcal{B}(G)} \mathrm{UCC}(G \setminus V),$$

where $\mathrm{UCC}(G \setminus V)$ is the cardinality of the set of unbounded connected components of $G \setminus V$, and $\mathcal{B}(G)$ is the set of all bounded subsets of $\mathrm{Vert}(G)$ (with respect to the Gordian distance metric), and $G \setminus V$ denotes the graph obtained by removing from G the vertex set V and all edges incident to vertices in V.

Theorem 1. *The number of* BU*-ends of the Gordian graph is equal to one.*

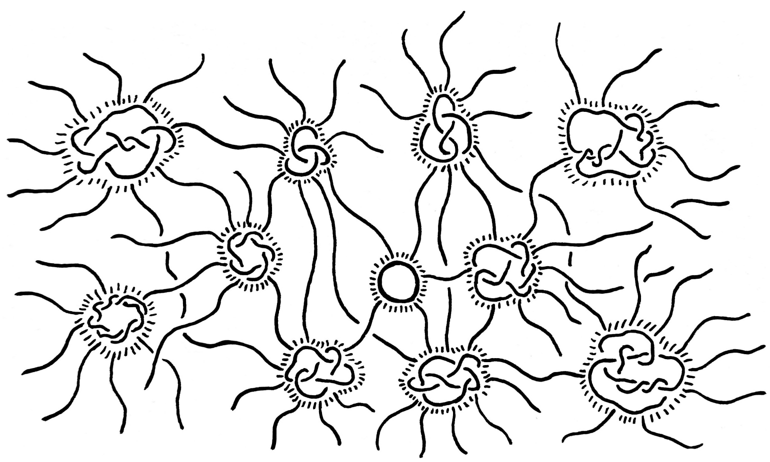

Figure 1. The Gordian graph.

In 2005, Julien Marché proved that the Gordian graph contains an isometrically embedded infinite countable tree with infinite valency. His approach also yields a result that is a direct precursor to our main theorem: the Gordian graph remains connected after the removal of any finite set of vertices. In the Appendix, we provide an elementary proof of this connectivity result that applies not only to crossing changes but also to any local move whose associated graph has vertices of infinite valency in every connected component.

§2. Proof of the main theorem

For a knot K, let $\Sigma_2(K)$ denote the *double branched cover* of S^3 over K, and let $e_2(K)$ denote the *minimal number of generators* of the first integer homology group $\mathrm{H}_1(\Sigma_2(K))$.

Lemma 1 (Montesinos trick)**.** *For arbitrary knots K and Q, we have the lower bound:*

$$\mathrm{d}_G(K, Q) \geqslant |e_2(K) - e_2(Q)|.$$

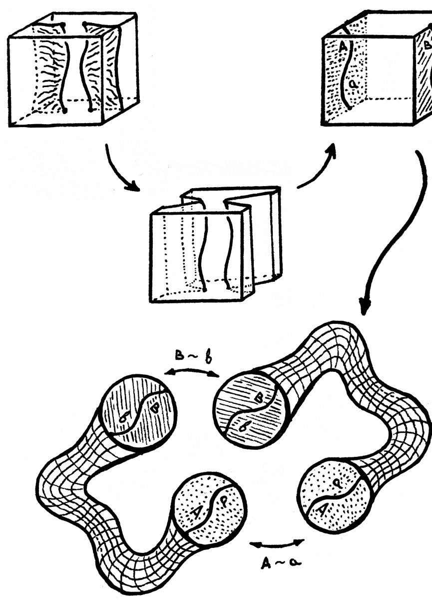

Figure 2. The double branched cover of any rational tangle is a solid torus.

Proof of Lemma 1. Since the double branched cover of any rational tangle is a solid torus (see Fig. 2), rational tangle replacement in a knot $X \subset S^3$ (with crossing changes as a special case) corresponds to Dehn surgery on the double branched cover of S^3 over X. As a single Dehn surgery changes the minimal number of generators of $\mathrm{H}_1(\Sigma_2(K))$ by at most one (Lemma 3, [2]), we obtain the desired lower bound on the Gordian distance. □

Definition 1. *Let K and Q be knots in S^3. A knot W is called a* connected sum *of K and Q if there exists a 2-sphere $S \subset S^3$ that intersects W transversely in two points, such that the two tangles obtained by cutting S^3 along S have closures ambient isotopic to K and Q, respectively.*

Remark 1. For unoriented knots, the connected sum is not well-defined: there may be two non-isotopic knots that are both connected sums of K and Q. This ambiguity is typically resolved by choosing orientations on K and Q. However, due to the specifics of our further reasoning and our desire to work with unoriented knots, we are satisfied with this definition despite its ambiguity.

We denote by U the trivial knot and by Q the right-handed trefoil. For $n \in \mathbb{N}$, let $B_n(U)$ be the set of vertices in the Gordian graph at distance at most n from U.

Proof of Theorem 1. We prove that the Gordian graph has exactly one BU-end. This follows from the fact that for any $r \in \mathbb{N}$ and any knots K and S with $d_G(K,U) \geqslant 2r$ and $d_G(S,U) \geqslant 2r$, there exists a path in the Gordian graph connecting K to S that avoids the ball $B_{r-1}(U)$. Consequently, the complement $G \setminus B_{r-1}(U)$ has a unique unbounded connected component containing $G \setminus B_{2r}(U)$, which implies the result.

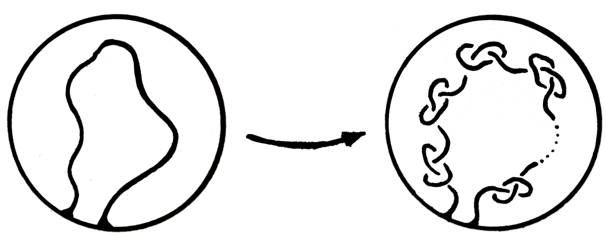

Figure 3. The Replacement.

It suffices to show that any such knot K can be connected to the connected sum of r trefoils by a path in $G \setminus B_{r-1}(U)$. We now present the construction of this path and verify that it avoids $B_{r-1}(U)$. Start with an arbitrary diagram of K. Select a closed disk that intersects K in a simple arc and replace it with a disk containing a diagram of a 1-tangle whose strand is knotted as the connected sum of r trefoils (see Fig. 3). This operation yields a diagram of a knot that is the connected sum of K and Q^r, where Q^r denotes the closure of this 1-tangle, that is the connected sum of r trefoils. We denote this knot by $Q^r\#K$.

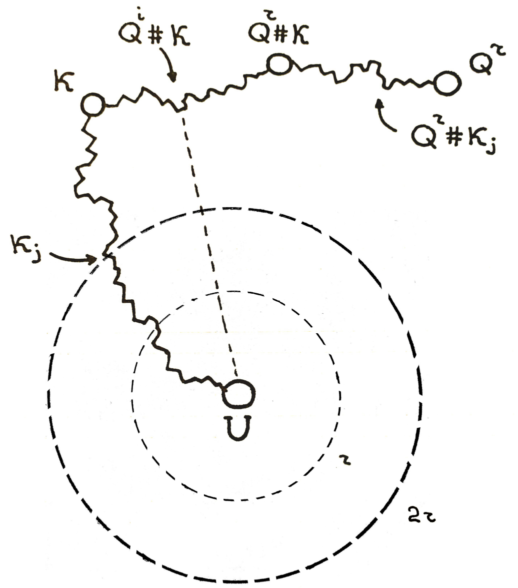

Figure 4. The Build Process.

A path of length r from $Q^r\#K$ to K in the Gordian graph is obtained by sequentially unknotting each trefoil summand via one crossing change per trefoil. The intermediate knot on this path, which is the connected sum of K and i trefoils, is denoted by $Q^i\#K$.

Similarly, it is well known that any diagram of a knot can be transformed into a diagram of the unknot by a finite sequence of crossing changes.

Hence, by applying corresponding crossing changes to the part of the diagram of $Q^r \# K$ that corresponds to K, we can transform it into a diagram of Q^r. Denote the intermediate knots on this path by $Q^r \# K_j$, where K_j ranges from K to U.

Combining this with the path from K to $Q^r \# K$ constructed earlier, we obtain a path connecting K to Q^r (see Fig. 4). It remains to verify that this path lies in $G \setminus B_{r-1}(U)$.

By the properties of double branched covers, we have

$$\mathrm{H}_1(\Sigma_2(Q^r \# K_j)) \cong \underbrace{\mathrm{H}_1(\Sigma_2(Q)) \oplus \cdots \oplus \mathrm{H}_1(\Sigma_2(Q))}_{r \text{ times}} \oplus \mathrm{H}_1(\Sigma_2(K_j)).$$

Since $\mathrm{H}_1(\Sigma_2(Q)) \cong \mathbb{Z}_3$ (see [4, p. 304]), it follows that $e_2(Q^r \# K_j) \geqslant r$. Therefore, by Lemma 1

$$\mathrm{d}_G(Q^r \# K_j, U) \geqslant |e_2(Q^r \# K_j) - e_2(U)| \geqslant r.$$

On the other hand, suppose that $i < r$ and $\mathrm{d}_G(Q^i \# K, U) < r$. By construction of the path, we have $\mathrm{d}_G(K, Q^i \# K) \leqslant i$, and by the triangle inequality,

$$2r \leqslant \mathrm{d}_G(K, U) \leqslant \mathrm{d}_G(K, Q^i \# K) + \mathrm{d}_G(Q^i \# K, U) < i + r < 2r,$$

which leads to a contradiction. Thus, $\mathrm{d}_G(Q^i \# K, U) \geqslant r$ and hence the entire path lies in $G \setminus B_{r-1}(U)$. This completes the proof. □

Appendix

Theorem 2. *The Gordian graph remains connected after the removal of any finite set of vertices.*

Proof of Theorem 2. Let W be a finite set of vertices in the Gordian graph G, and let K and S be two knots not contained in W. We show that the vertices corresponding to K and S remain connected by a path in $G \setminus W$. Let γ be a path in G from K to S with vertex sequence

$$K = V_0,\, V_1,\, \ldots,\, V_n = S.$$

For any knot H, we associate to γ a path γ_H whose i-th vertex V_i^H is defined to be the connected sum of V_i and H. Informally, γ_H is obtained by "shifting" γ with H. As noted in Remark 1, this construction is not unique; however, one can always perform the connected summation in a consistent manner along the entire path. For our argument, the existence of at least one such path is sufficient.

Since the set $S_1(U)$ of knots Q with $d_G(U, Q) = 1$ is infinite (for example, it includes all twisted knots), there must exist a knot Q for which every vertex of the path γ_Q lies outside W. Assume, for contradiction, that for every $Q \in S_1(U)$ at least one vertex V_i^Q of the path γ_Q lies in W. Let $W_s \subset W$ be the set of all such vertices. By Schubert's Theorem (see [5]), each knot has only finitely many connected summands, so the set of all connected summands of knots in W_s is finite. However, for each $Q \in S_1(U)$, there exists a knot in W_s that has Q as a connected summand. Since $S_1(U)$ is infinite, this yields a contradiction.

Let $Q \in S_1(U)$ be a knot such that all vertices of γ_Q lie outside W. Then the first vertex V_0^Q of γ_Q is the connected sum of K and Q, and the last vertex V_n^Q is the connected sum of S and Q. Since $d_G(U, Q) = 1$, we have $d_G(K, V_0^Q) = 1$ and $d_G(S, V_n^Q) = 1$. Therefore, adding these two edges to γ_Q produces a path connecting K and S in $G \setminus W$. □

Remark 2. For any other local move where the vertex corresponding to the trivial knot has infinite valency in the associated graph, the proof is carried out in precisely the same manner for each connected component of the graph.

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