Записки научных
семинаров ПОМИ
Том 549, 2025 г.

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PING-PONG FOR BASIS-CONJUGATING HNN-EXTENSION OF FREE GROUP

Abstract. We isolate a tractable class of HNN-extensions of a free group – namely, multiple HNN-extensions by basis-conjugating embeddings. For this class, we construct a normal form and establish a practical version of the ping-pong lemma that provides verifiable sufficient conditions for a set of elements to generate a free subgroup.

We then apply these results to the pure braid group P_{n+1}, exploiting its well-known decomposition as a semidirect product of free groups. Our approach yields new families of free subgroups within the first two factors $F_n \rtimes F_{n-1}$ of this decomposition.

§1. Introduction

In this paper, we study the small subclass of (multiple) partial ascending HNN-extensions of finitely generated free groups that originates from the notion of basis-conjugating automorphisms. We find sufficient conditions for a set of elements in such a group to generate a free subgroup. We specialize our results to the pure braid group in order to produce new families of free subgroups.

We begin with a review of basic concepts involved in our work, in order to introduce notation and clarify our motivation.

1.1. Ascending HNN-extension of free group. Let F be a free group of finite rank. If $\varphi\colon F \to F$ is an automorphism, one may form its *mapping torus*

$$M_\varphi = F \rtimes_\varphi \mathbb{Z} = \langle F, t \mid a^t = \varphi(a),\ a \in F\rangle.$$

The group M_φ has been studied extensively. Its structure is similar to the fundamental group of a closed three-dimensional manifold fibered over the

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Key words and phrases: group theory, ping-pong lemma, HNN-extension, braid group.

This work was supported by the Ministry of Science and Higher Education of the Russian Federation (agreement 075-15-2025-344 dated 29/04/2025 for Saint Petersburg Leonhard Euler International Mathematical Institute at PDMI RAS). The work was supported by the Theoretical Physics and Mathematics Advancement Foundation "BASIS", project no. 24-7-1-26-3.

circle; see [7]. If $\varphi\colon F \to F$ is a monomorphism, we have an *ascending HNN-extension*

$$F{*}_{\varphi} = \langle F, t \mid a^t = \varphi(a),\ a \in F\rangle.$$

An ascending HNN-extension of a free group is a common generalization of the mapping tori of free group automorphisms and the Baumslag–Solitar group

$$BS(1,n) = \langle a, t \mid a^t = a^n\rangle.$$

The ascending HNN-extensions of free groups are of particular importance in the theory of one-relator groups; see an overview in [14]. There are some interesting structural results concerning ascending HNN-extensions of free groups. For example, in [1], it is proved that these groups are residually finite, and in [11], it is shown that most of these groups are word-hyperbolic.

1.2. Partial ascending HNN-extension of free group. Let $F(Y)$ be a free group of finite rank with a fixed basis Y. If $Y' \subset Y$ is a subset and $\varphi\colon F(Y') \to F(Y)$ is an embedding of groups, we have a *partial ascending HNN-extension*

$$\langle F(Y), t \mid y^t = \varphi(y),\ y \in Y'\rangle.$$

This generalization arises naturally in the study of the ordinary ascending HNN-extension of a free group; see [2, 7].

1.3. Multiple HNN-extension. If K is a group and $\{\varphi_i \mid 1 \leqslant i \leqslant n\}$ is a collection of embeddings of subgroups $\{L_i \mid 1 \leqslant i \leqslant n\}$ of K into K, then the group

$$\langle K, x_1, \ldots, x_n \mid y^{x_i} = \varphi_i(y),\ 1 \leqslant i \leqslant n,\ y \in L_i\rangle$$

is called a *multiple HNN-extension* of K with stable letters x_i, $1 \leqslant i \leqslant n$. The structure of subgroups of a multiple HNN-extension is studied in [12]. However, in the context of ascending HNN-extensions of free groups, the case of multiple extensions has been studied considerably less. For example, it is unknown whether multiple ascending HNN-extensions of free groups are residually finite; see Problem 4.4 in [1].

1.4. Basis-conjugating embedding. Given a subset $Y' \subset Y$ and a choice of an element $w_y \in F(Y)$ for each $y \in Y'$, we can form a homomorphism

$$\varphi\colon F(Y') \to F(Y),\ \varphi(y) := y^{w_y},\ y \in Y'.$$

We call such a homomorphism a *basis-conjugating embedding*. Clearly, this homomorphism is injective, since it induces a monomorphism on the abelianizations. This notion is strongly related to basis-conjugating automorphisms of free groups. The subgroup $\mathrm{P}\Sigma_{|Y|} \subset \mathrm{Aut}(F(Y))$ consisting of basis-conjugating automorphisms has been studied extensively in works [3, 10, 16].

1.5. Overview of results. Let $n \geqslant 1$ and for every index $1 \leqslant i \leqslant n$ we are given a subset $Y_i = \{y_{i1}, \dots, y_{im_i}\}$ of Y and two basis-conjugating embeddings $\varphi_i, \psi_i \colon F(Y_i) \to F(Y)$. Then, for every i and $1 \leqslant j \leqslant m_i$ there are unique elements $w_{ij}, v_{ij} \in F(Y)$ such that

$$\varphi_i(y_{ij}) = y_{ij}^{w_{ij}},\ \psi_i(y_{ij}) = y_{ij}^{v_{ij}},$$

and the words $y_{ij}^{w_{ij}} = w_{ij}^{-1} y_{ij} w_{ij}$ and $y_{ij}^{v_{ij}} = v_{ij}^{-1} y_{ij} v_{ij}$ are reduced. We can form a multiple HNN-extension of $F(Y)$ by identifying the images of $F(Y_i)$ in $F(Y)$:

$$\begin{aligned} G &= \langle F(Y), x_1, \dots, x_n \mid \varphi_i(y)^{x_i} = \psi_i(y),\ 1 \leqslant i \leqslant n,\ y \in Y_i \rangle \\ &= \langle Y, x_1, \dots, x_n \mid (y_{ij}^{w_{ij}})^{x_i} = y_{ij}^{v_{ij}},\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i \rangle. \end{aligned} \tag{1}$$

The basic facts about HNN-extensions say that the natural homomorphism $F(Y) \to G$ is injective and that the group G is torsion-free; see [8]. Also note that the abelianization of G is free abelian of rank $|Y| + n$.

Let $X = \{x_1, \dots, x_n\}$. We call the set $Y^{\pm} \cup X^{\pm}$ an *extended alphabet*. Here are our main results.

Theorem 1.1 (Normal form for G)**.** *Every element of G has a unique reduced expression in the extended alphabet with no subwords of the form*

$$x_i v_{ij}^{-1} y_{ij}^{\pm 1} \text{ and } x_i^{-1} w_{ij}^{-1} y_{ij}^{\pm 1}.$$

Theorem 1.2 (Ping-pong lemma for G)**.** *Let $T_1, \dots, T_k$ be pairwise disjoint nonempty subsets of X. If subgroups $A_1, \dots, A_k \subset G$ satisfy the conditions $A_t \subset \langle Y, T_t \rangle$ and $A_t \cap \langle Y \rangle = 1$ for every $1 \leqslant t \leqslant k$, then the natural homomorphism $A_1 * \ldots * A_k \to G$ is injective.*

Recall that the pure braid group P_n is generated by the braids $A_{i,j}$ with $1 \leqslant i < j \leqslant n$. There is a decomposition

$$P_{n+1} = F_n \rtimes F_{n-1} \rtimes \ldots \rtimes \mathbb{Z}.$$

The summand $F_n \rtimes F_{n-1}$ corresponds to a subgroup $P_{n+1}^{(2)} \subset P_{n+1}$ consisting of braids that become trivial after the last two strands were removed. We write

$$P_{n+1}^{(2)} = F(x_1, \dots, x_n) \rtimes F(y_1, \dots, y_{n-1}),$$

where $x_i := A_{i,n+1}$ and $y_i := A_{i,n}$. Note that the abelianization of this group is free abelian of rank $2n-1$, generated by the images of x_i's and y_j's. Roughly speaking, the abelianization homomorphism

$$P_{n+1}^{(2)} \to (P_{n+1}^{(2)})_{\mathrm{ab}}$$

assigns to a pure braid w, given by a word in the alphabet $Y^{\pm} \cup X^{\pm}$, the tuple of exponent sums of the letters. We denote these sums by $\exp_{x_i}(w)$ and $\exp_{y_j}(w)$.

The subgroup of $P_{n+1}^{(2)}$ generated by the elements $\{x_1, \ldots, x_n\}$ is free of rank n. We use our above results to show that one can replace the generators $\{x_1, \ldots, x_{n-1}\}$ by suitable braids $\{w_1, \ldots, w_{n-1}\}$ while keeping the resulting subgroup

$$\langle w_1, \ldots, w_{n-1}, x_n \rangle$$

free.

Theorem 1.3. *Let $w_1, w_2, \ldots, w_{n-1}$ be braids from $P_{n+1}^{(2)}$ such that for each $1 \leqslant i < n$ one has*:

(1) $\exp_{x_j}(w_i) \neq 0$ *if and only if* $j = i$;
(2) $[w_i, x_n] \neq 1$.

Then the subgroup $\langle w_1, \ldots, w_{n-1}, x_n \rangle$ of P_{n+1} is free of rank n.

Remark 1.4. Condition (2) is crucial. For example, for $w_i = y_i x_i$ one has

$$\langle y_1 x_1, \ldots, y_{n-1} x_{n-1}, x_n \rangle \cong F_{n-1} \times \mathbb{Z}.$$

It is easy to show that the elements $w_1, \ldots, w_{n-1}$ generate a free group of rank $n-1$, and that for every i the subgroup $\langle w_i, x_n \rangle$ is free of rank 2. We use our method to show that the full subgroup is also free. We also show that, despite the fact that the lower central series Lie ring of $P_{n+1}^{(2)}$ behaves in a controlled way, our criterion implies the freeness of some subgroups that cannot be deduced using the Lie-theoretic approach.

Example 1.5. Let $n \geqslant 1$ and m, k be nonzero integers.

(1) The subgroup

$$\langle A_{1,n+1}^k A_{1,n}^m, A_{2,n+1}^k A_{2,n}^m, \ldots, A_{n-1,n+1}^k A_{n-1,n}^m, A_{n,n+1}^k \rangle$$

of P_{n+1} is free if and only if $(m, k) \neq (-1, -1)$.

(2) The subgroup

$$\langle A_{1,n}^m A_{1,n+1}^k, A_{2,n}^m A_{2,n+1}^k, \ldots, A_{n-1,n}^m A_{n-1,n+1}^k, A_{n,n+1}^k \rangle$$

of P_{n+1} is free if and only if $(m, k) \neq (1, 1)$.

1.6. Organization. In Section 2, we construct a normal form for a multiple HNN-extension of a free group by basis-conjugating embeddings (Theorem 1.1) and give sufficient conditions to fulfill the requirements of the ping-pong lemma (Theorem 1.2). In Section 3, we recall what is known about freeness of subgroups in the pure braid group and use results from Section 2 to produce new families of free subgroups in the pure braid group (Theorem 1.3). We have relegated to Appendix A the routine verification of the local confluence of the rewriting system that appears in the proof of Theorem 1.1.

§2. Proof of main results

Let Σ be a set, and let Σ^* denote the free monoid generated by Σ. We call a relation on Σ^* a *congruence* if it is an equivalence relation such that $w_1uw_2 \sim w_1u'w_2$ whenever $u \sim u'$. For $R \subset \Sigma^* \times \Sigma^*$, denote by $\rho(R)$ the congruence on Σ^* generated by R. Let

$$M := \mathrm{Mon}\langle \Sigma \mid R\rangle \overset{\mathrm{def}}{=} \Sigma^*/\rho(R)$$

be the monoid given by the presentation $\langle \Sigma \mid R\rangle$. We have a natural quotient homomorphism

$$\pi\colon \Sigma^* \to M.$$

Since R consists of ordered pairs, we can interpret it as a rewriting system: each $(r, r') \in R$ corresponds to the rewriting rule $w_1rw_2 \mapsto w_1r'w_2$. We say that a rewriting system is *locally confluent* if any two one-step applications of rules to the same word can be completed to chains of rule applications ending at a common word:

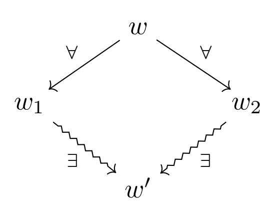

Figure 1. Here, a straight arrow corresponds to a single application of a rule, whereas a squiggly arrow corresponds to a finite sequence of applications.

We say that a rewriting system is *Noetherian* if one cannot apply given rules to a word infinitely many times (typically, this means that there is a bounded discrete parameter which is optimized after every application). We say that a subset $N \subset \Sigma^*$ is a *normal form* for M if the restriction $\pi\mid_N : N \to M$ is a bijection. A version of the diamond lemma (see [18, Theorem 2]) says that if a rewriting system $\{r_j \mapsto r'_j\}$ is locally confluent and Noetherian, then the set $\{w \in \Sigma^* \mid w \not\supset r_j \ \forall j\}$ is a normal form for M.

The group G given by the presentation (1) is generated as a monoid by the extended alphabet $\Sigma = Y^\pm \cup X^\pm$. Consider the following rewriting system for G:

(1) $y_j^\varepsilon y_j^{-\varepsilon} \mapsto \varnothing$, where $1 \leqslant j \leqslant m$, and $\varepsilon \in \{\pm 1\}$;
(2) $x_i^\varepsilon x_i^{-\varepsilon} \mapsto \varnothing$, where $1 \leqslant i \leqslant n$, and $\varepsilon \in \{\pm 1\}$;
(3) $x_i v_{ij}^{-1} y_{ij}^\varepsilon \mapsto w_{ij}^{-1} y_{ij}^\varepsilon w_{ij} x_i v_{ij}^{-1}$, where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$, and $\varepsilon \in \{\pm 1\}$;
(4) $x_i^{-1} w_{ij}^{-1} y_{ij}^\varepsilon \mapsto v_{ij}^{-1} y_{ij}^\varepsilon v_{ij} x_i^{-1} w_{ij}^{-1}$, where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$, and $\varepsilon \in \{\pm 1\}$.

Proof of Theorem 1.1. Clearly, the above set (1)–(4) is a complete set of relations for G. One can verify that this system is locally confluent. The verification of this is technical and has been deferred to Appendix A.

In order to show that it is Noetherian, we introduce the following parameter. Write a word $w \in \Sigma^*$ as a product

$$w = u_0 x_{i_1}^{\pm 1} u_1 x_{i_2}^{\pm 1} \dots x_{i_k}^{\pm 1} u_k, \quad k \geqslant 0,\ u_i \in (Y^\pm)^*.$$

We can associate with the word w the vector of non-negative numbers

$$\nu(w) = (|u_0|, \dots, |u_k|) \in \mathbb{Z}_{\geqslant 0}^{k+1}.$$

Observe that each application of the rule either decreases k, or decreases one of the coordinates of $\nu(w)$ without touching the ones following it (however, it can increase the previous ones). Define an order on the set of tuples of non-negative integers by the rule $(a_0, \dots, a_k) < (b_0, \dots, b_m)$ if $k < m$ or $k = m$ and for some i one has $a_i < b_i$ and $a_j = b_j$ for $j > i$. This order is known as the *right shortlex order*. The result follows, since this order is well-founded, hence there are no infinite decreasing chains. □

Remark 2.1. Let

$$w = u_0 x_{i_1}^{\varepsilon_1} u_1 x_{i_2}^{\varepsilon_2} \dots x_{i_k}^{\varepsilon_k} u_k$$

be a word in Σ^* and

$$w' = u'_0 x_{j_1}^{\eta_1} u'_1 x_{j_2}^{\eta_2} \dots x_{j_l}^{\eta_l} u'_l$$

be its normal form, where $u_i, u'_j \in (Y^{\pm})^*$. Then the sequence $(x_{j_1}^{\eta_1}, \dots, x_{j_l}^{\eta_l})$ is a subsequence of $(x_{i_1}^{\varepsilon_1}, \dots, x_{i_k}^{\varepsilon_k})$.

The following result is folklore.

Lemma 2.2 (Ping-pong lemma, [19, Lemma 4])**.** *Let G be a group acting on a set X, and let $A_1, A_2, \dots, A_k \subset G$ be nontrivial subgroups such that $|A_1| \geqslant 3$. If there are disjoint nonempty subsets $U_1, \dots, U_k \subset X$ such that for each pair of distinct indices $t \neq s$ and each nontrivial $a \in A_t$ one has $a(U_s) \subset U_t$, then $\langle A_1, \dots, A_k \rangle = A_1 * \dots * A_k$.*

Let $k \geqslant 1$, and let $T_1, \dots, T_k$ be pairwise disjoint nonempty subsets of X. Let G act on itself by left shifts. For every $1 \leqslant t \leqslant k$ set

$$U_t = \{u \in G \mid \text{the normal form of } u \text{ looks like } u' x_i^{\pm 1} u'', \\ \text{where } x_i \in T_t \text{ and } u' \in (Y^{\pm})^*\}.$$

The sets U_t are nonempty and pairwise disjoint. We say that a subset S of G is *greedy* if it possesses the following property: if the concatenation of two words uu' is already in the normal form, and $u \in S$, then $uu' \in S$. Clearly, each U_t is greedy.

Proof of Theorem 1.2. Fix $1 \leqslant t \neq s \leqslant k$ and an element $a \in A_t \setminus \{1\}$. We aim to show that $au \in U_t$ for each $u \in U_s$.

Let $u' x_i^{\varepsilon} u''$ be the normal form of u, where $x_i \in T_s$ and $u' \in (Y^{\pm})^*$. Let p be the normal form of au'. Since $a \in \langle Y, T_t \rangle$ and $u' \in \langle Y \rangle$, we have $p \in (Y^{\pm} \cup T_t^{\pm})^*$ by Remark 2.1. Hence the last letter of p is not $x_i^{-\varepsilon}$, and the word $p x_i^{\varepsilon} u''$ is the normal form of au.

The word p contains $x_l^{\pm 1}$ for some $x_l \in T_t$, since if this is not the case, then $a = p{u'}^{-1} \in \langle Y \rangle$, which contradicts the assumptions. Hence $p \in U_t$. The result follows since U_t is greedy. □

Consider the special case where $\varphi_i = \psi_i$ for each $1 \leqslant i \leqslant n$. In this case, we formulate a condition that ensures the second condition from Theorem 1.2. The presentation (1) reads now as

$$G = \langle y_1, \dots, y_m, x_1, \dots, x_n \mid [x_i, y_{ij}^{w_{ij}}] = 1,\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i \rangle.$$

If we add the relations $[x_i, y_j] = 1$ for all i and j, then we obtain a map $\pi\colon G \twoheadrightarrow F(X) \times F(Y)$. Consider the following commutative diagram of projections:

$$\begin{array}{ccccc} F(X) & & & \overset{\pi_X}{\twoheadleftarrow} & \\ {\scriptstyle \pi_X}\uparrow & & & & \\ F(X \cup Y) & \xrightarrow{[-]} & G & \xrightarrow{\pi} & F(X) \times F(Y). \\ {\scriptstyle \pi_Y}\downarrow & & & & \\ F(Y) & & & \underset{\pi_Y}{\twoheadleftarrow} & \end{array}$$

Proposition 2.3. *Let φ be an automorphism of a free group $F(X \cup Y)$ that descends as the identity map on $F(X) \times F(Y)$, let $w \in F(X \cup Y)$ be an element with $\pi_X(w) \neq 1$, and let A be a subgroup of G generated by the set $\{[\varphi^k(w)] \mid k \in \mathbb{Z}\}$. Then the intersection $A \cap \langle Y \rangle$ is trivial.*

Proof. Note that the restriction $[-]\,|_{F(Y)} \colon F(Y) \to G$ is injective, since G is an HNN-extension of $F(Y)$. Hence the restriction

$$\pi\,|_{\langle Y \rangle} \colon \langle Y \rangle \to F(X) \times F(Y)$$

is injective. Therefore, it is enough to show that

$$\pi(A) \cap \pi(\langle Y \rangle) = 1.$$

Note that

$$\begin{aligned} \pi(A) &= \pi(\langle [\varphi^k(w)] \mid k \in \mathbb{Z} \rangle) = \langle (\pi_X(\varphi^k(w)), \pi_Y(\varphi^k(w))) \mid k \in \mathbb{Z} \rangle \\ &= \langle (\pi_X(w), \pi_Y(w)) \rangle \end{aligned}$$

and $\pi(\langle Y \rangle) = 1 \times F(Y)$. The result follows since the restriction of π_X to $\langle (\pi_X(w), \pi_Y(w)) \rangle$ is injective while $\pi_X(1 \times F(Y)) = 1$. □

§3. Free subgroups in pure braid group

We now apply the general machinery developed above to the pure braid group. The pure braid group contains many free subgroups. We list some of them below.

(1) For every $1 \leqslant i \leqslant n + 1$ there is a strand-removing homomorphism $d_i \colon P_{n+1} \to P_n$. It is well known that its kernel $\ker(d_i)$ is free of rank n. The proof is homotopy-theoretic.

(2) There is a rank n free subgroup of P_{n+1} given by an embedding of Milnor's construction $F[S^1]$ into a simplicial group AP of pure braids. The classical proof [4, 5] relies on the Lie rings $\mathrm{gr}_*(F_n)$ and $\mathrm{gr}_*(P_{n+1})$. There is also a group-theoretic proof [9].

(3) The images of the above subgroups under automorphisms of the pure braid group are also free. These include, for example, the free subgroup of rank n generated by the braids[1] $A_{0,1}, \dots, A_{0,n}$, where

$$A_{0,i} = (A_{1,i}A_{2,i} \dots A_{i-1,i}A_{i,i+1} \dots A_{i,n+1})^{-1}.$$

Note that $\langle A_{0,1}, \dots, A_{0,n+1} \rangle$ is a subgroup of braids that become trivial after being transplanted onto a two-sphere. It is worth noting that by the main result of [6] we have an isomorphism

$$\ker(P_{n+1} \to P_{n+1}(S^2)) \cong F_n \times \mathbb{Z}.$$

(4) In [13], it is proved that each pair of non-commuting braids x and y in P_{n+1} generate a free subgroup of rank 2. The proof involves the theory of 3-manifolds and the theory of group actions on trees. This result suggests that there might be a lot of free subgroups in general.

We refer to [17] for standard notions of braid theory. The group $P_{n+1}^{(2)}$ decomposes as a semidirect product

$$P_{n+1}^{(2)} = F(x_1, \dots, x_n) \rtimes F(y_1, \dots, y_{n-1}), \tag{2}$$

and the action of $y_j = A_{j,n}$ on $x_i = A_{i,n+1}$ is given by the following relations:

$$x_i^{y_j} = \begin{cases} x_i, & i < j; \quad \text{(R1)} \\ x_i^{(x_i x_n)^{-1}}, & i = j; \quad \text{(R2)} \\ x_i^{[x_n, x_j]}, & j < i < n; \quad \text{(R3)} \\ x_n^{x_j^{-1}}, & j < i = n. \quad \text{(R4)} \end{cases}$$

We start with some manipulations with relations in the group $P_{n+1}^{(2)}$.

- The relation (R1) reads as

$$[x_i, y_j] = 1. \tag{R1'}$$

- The relation (R2) reads as $x_i^{y_i x_i x_n} = x_i$, which is equivalent by (R4) to

$$x_i^{x_n} = x_i^{y_i^{-1}}. \tag{R2'}$$

[1] See [9, Fig. 1] for a picture of the braid $A_{0,i}$.

- The relation (R3) reads as $x_i^{y_j x_j x_n} = x_i^{x_n x_j}$, which is equivalent to $x_i^{x_n} = x_i^{x_n y_j^{-1}}$ by (R4), which is equivalent to $x_i^{x_n} = x_i^{y_i^{-1} y_j}$ by (R2'). By (R4), one can replace this relation with

$$[x_i, y_j^{y_i}] = 1. \tag{R3'}$$

- By (R2'), the relation (R4) is equivalent to

$$y_i^{x_n} = y_i[x_i, y_i]. \tag{R4'}$$

Hence

$$\begin{aligned} P_{n+1}^{(2)} &= \langle x_i, y_i \mid \text{(R1)}, \text{(R2)}, \text{(R3)}, \text{(R4)}\rangle \\ &= \langle x_i, y_i \mid \text{(R1')}, \text{(R2')}, \text{(R3')}, \text{(R4')}\rangle. \end{aligned}$$

Let G_n be the group given by the following presentation

$$G_n := \langle x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} \mid [x_i, y_j] = 1,\ i < j;\ [x_i, y_j^{y_i}] = 1,\ i > j\rangle.$$

The group $P_{n+1}^{(2)}$ admits a semidirect product decomposition

$$P_{n+1}^{(2)} = G_n \rtimes \langle t\rangle,$$

where the letter $t := x_n$ acts on G_n via the automorphism $\varphi\colon G_n \to G_n$ given by

$$\varphi(x_i) = x_i^{y_i^{-1}},\ \varphi(y_i) = y_i[x_i, y_i]. \tag{3}$$

Let $X := \{x_1, \ldots, x_{n-1}\}$ and $Y := \{y_1, \ldots, y_{n-1}\}$.

Corollary 3.1 (Normal form for G_n)**.** *Each element of G_n can be uniquely written as a reduced word w in the extended alphabet $X^{\pm} \cup Y^{\pm}$ such that w does not contain subwords $x_i^{\varepsilon} y_j^{\eta}$ and $x_j^{\varepsilon} y_j^{-1} y_i^{\eta}$ with $i < j$ and $\varepsilon, \eta \in \{\pm 1\}$.*

Example 3.2. The group P_4 decomposes as the semidirect product

$$F(x_1, x_2, x_3) \rtimes F(y_1, y_2) \rtimes \langle A_{1,2}\rangle.$$

If we replace $A_{1,2}$ with the generator of the center

$$z = (A_{1,2}A_{1,3}A_{1,4}) \cdot (A_{2,3}A_{2,4}) \cdot A_{3,4} = (A_{1,2}y_1x_1) \cdot (y_2x_2) \cdot x_3,$$

then we can rewrite this decomposition as

$$\begin{aligned} P_4 &= P_4^{(2)} \oplus \mathbb{Z} = (G_3 \rtimes \mathbb{Z}) \oplus \mathbb{Z} \\ &= \langle y_1, y_2, x_1, x_2 \mid [x_1, y_2] = 1, [x_2, y_1^{y_2}] = 1\rangle \rtimes (\mathbb{Z} \oplus \mathbb{Z}), \end{aligned}$$

where the generators of $\mathbb{Z} \oplus \mathbb{Z}$ are x_3 and z. The letter x_3 acts on the normal subgroup $\langle y_1, y_2, x_1, x_2\rangle$ of P_4 by the automorphism φ given by the formula (3), and z is central.

Lemma 3.3. *Let G be a group, φ an automorphism of G, and H a subgroup of G. Consider the semidirect product $G \rtimes_\varphi \mathbb{Z}$ of G with the infinite cyclic group $\mathbb{Z} = \langle t \rangle$. The following are equivalent*:

(1) *the natural map $\coprod_{k\in\mathbb{Z}} \varphi^k(H) \to G$ is injective*;
(2) *the subgroup $\langle H, t\rangle$ of $G \rtimes_\varphi \mathbb{Z}$ decomposes as the free product $H * \langle t\rangle$.*

Proof. (1) $\Rightarrow$ (2). Let $\Phi\colon H * \mathbb{Z} \to G \rtimes \mathbb{Z}$ be the natural map. Let

$$w = t^{n_1} h_1 t^{n_2} h_2 \dots h_{k-1} t^{n_k}$$

be an element in the free product $H * \mathbb{Z}$ written in the normal form, i.e., we assume that $n_i \neq 0$ for all $i \notin \{1, k\}$, and $h_i \neq 1$ for all i. One has

$$\begin{aligned} \Phi(w) &= t^{n_1} h_1 t^{-n_1} t^{n_1+n_2} h_2 t^{-(n_1+n_2)} \dots h_{k-1} t^{-(n_1+\dots+n_{k-1})} \cdot t^{n_1+n_2+\dots+n_k} \\ &= \varphi^{-n_1}(h_1)\varphi^{-(n_1+n_2)}(h_2)\dots\varphi^{-(n_1+\dots+n_{k-1})}(h_{k-1}) \cdot t^{n_1+\dots+n_k}. \end{aligned}$$

In the above expansion, all adjacent powers of φ are distinct since $n_i \neq 0$. Hence if $k > 1$, then $\Phi(w) \neq 1$. If $k = 1$, then $\Phi(w) = t^{n_1}$, which is trivial if and only if $n_1 = 0$, i.e., if $w = 1$.

(2) $\Rightarrow$ (1). This follows since we have the natural embedding

$$\coprod_{k\in\mathbb{Z}} t^{-k} H t^k \hookrightarrow H * \langle t\rangle. \qquad \square$$

Proof of Theorem 1.3. Let $B_i = \langle w_i\rangle$ for $1 \leqslant i < n$. We aim to show that the natural map

$$B_1 * \dots * B_{n-1} * \langle t\rangle \to G_n \rtimes \langle t\rangle = P^{(2)}_{n+1}$$

is injective.

Let $A_i = \langle \varphi^k(w_i) \mid k \in \mathbb{Z}\rangle$ for $1 \leqslant i < n$. In the light of Lemma 3.3, it is enough to show that:

(1) for each $1 \leqslant i < n$, the set $\{\varphi^k(w_i)\}_{k\in\mathbb{Z}}$ is a free basis of some subgroup in G_n;
(2) the natural map $A_1 * \dots * A_{n-1} \to G_n$ is injective.

By the main result of [13], the condition $[w_i, t] \neq 1$ implies that the subgroup $\langle w_i, t\rangle$ is free of rank two. The other direction of Lemma 3.3 implies the statement (1).

The formula (3) lifts the automorphism $\varphi\colon G_n \to G_n$ to an automorphism of $F(X \cup Y)$. The latter induces the identity map on the product $F(X) \times F(Y)$. In G_n we have $A_i \cap \langle Y\rangle = 1$ by Proposition 2.3. Then, the statement (2) follows from Theorem 1.2. $\square$

3.1. Possible generalization. It is natural to ask whether the statement of Theorem 1.3 still hold true if the last generator $t = x_n$ is replaced by a word w_n such that

$$w_n \in \langle Y, x_n \rangle, \ \exp_{x_n}(w_n) \neq 0, \ \text{and } [w_i, w_n] \neq 1 \text{ for every } i < n.$$

It seems that in this case, the subgroup $\langle w_1, \dots, w_n \rangle$ would also be free, but the argument has to be modified: the presented proof uses the fact that the conjugation by the last generator induces the identity automorphism on $F(X) \times F(Y)$ in order to demonstrate that $\langle w_n^{-k} w_i w_n^k \mid k \in \mathbb{Z} \rangle \cap \langle Y \rangle = 1$. Our argument easily generalizes to words w_n from the normal closure of x_n in the group $\langle Y, x_n \rangle$. However, for the case of three strands we can ensure that no restrictions are needed. Note that $G_2 = \langle y_1, x_1 \rangle$ and $P_3^{(2)} = P_3$.

Proposition 3.4. *Let $w_1, w_2 \in P_3$ be braids such that $w_1 \in \langle y_1, x_1 \rangle$ and $\exp_{x_1}(w_1) \neq 0$. Then the intersection $\langle w_2^{-k} w_1 w_2^k \mid k \in \mathbb{Z} \rangle \cap \langle y_1 \rangle$ is trivial.*

Proof. The center of P_3 is generated by the braid $z = y_1 x_1 t$, where $t = x_2$ as before. Hence we can assume that w_2 is from $\langle y_1, x_1 \rangle$. It is sufficient to demonstrate that the normal closure of w_1 in $\langle y_1, x_1 \rangle$ does not intersect $\langle y_1 \rangle$. If $w_1 = (u^k)^v$, then the normal closure of w_1 is contained in the normal closure of u. Hence we can assume that the cyclic reduction of w_1 is not a proper power.

Note that $\langle y_1, x_1 \rangle$ is free of rank two. The one-relator group

$$\langle x_1, y_1 \mid w_1 \rangle$$

is torsion-free by [15, Theorem 4.12]. It follows that any nontrivial intersection should give $y_1 = 1$ in this group. Hence this group would be generated by x_1. But since $\exp_{x_1}(w_1) \neq 0$, this would imply that this group is finite cyclic, which is impossible. □

3.2. Lie-theoretic approach. With a group G filtered by the lower central series one can associate a graded Lie ring

$$\mathrm{gr}_*(G) := \bigoplus_{k \geqslant 1} \frac{\gamma_k(G)}{\gamma_{k+1}(G)}$$

with the Lie bracket given by the commutator. This construction is functorial. Moreover, if G is residually nilpotent, then injectivity of a homomorphism $f \colon G \to H$ follows from injectivity of the homomorphism $\mathrm{gr}_*(f)$.

This approach was taken in [4, 5] to show injectivity of the natural simplicial map

$$F[S^1] \to \mathrm{AP}.$$

In degree n, it corresponds to the homomorphism $F_n \to P_{n+1}$.

Note that if two homomorphisms $f, g\colon G \to H$ induce the same homomorphism of abelianizations $G_{\mathrm{ab}} \to H_{\mathrm{ab}}$, then they induce the same homomorphism of graded Lie rings $\mathrm{gr}_*(G) \to \mathrm{gr}_*(H)$. Since the subgroup

$$\langle y_1x_1, \dots, y_{n-1}x_{n-1}, t\rangle \subset P_{n+1}$$

is isomorphic to $F_{n-1} \times \mathbb{Z}$, it follows that the corresponding homomorphism $F_n \to P_{n+1}$ is not injective. Therefore, the induced homomorphism $\mathrm{gr}_*(F_n) \to \mathrm{gr}_*(P_{n+1})$ is not injective. On the other hand, Theorem 1.3 implies that the subgroup

$$\langle x_1y_1, \dots, x_{n-1}y_{n-1}, t\rangle \subset P_{n+1}$$

is free of rank n. But this clearly could not be deduced via Lie-theoretic approach, since $y_ix_i = [y_i, x_i]x_iy_i$ and the two homomorphisms $F_n \to P_{n+1}$ induce the same map of abelianizations.

Acknowledgment. I am grateful to Leonid Danilevich for the statement and the proof of Lemma 3.3 and to Artem Semidetnov for kindly introducing me to Stallings foldings. I sincerely thank Andrei Malyutin for his careful reading of a draft of this paper.

Appendix §A. Proof of local confluence

In this section, we fill a gap in the proof of Theorem 1.1 and show that the rewriting system is locally confluent.

In order to demonstrate that a rewriting system is locally confluent, one must inspect all possible ways to apply two rules in some overlapping fragments of a given word. It is easy to see that the left hand sides of the rules (3) and (4) do not intersect and there are no self-intersections for the rules (3) and (4). The case of self-overlappings for the rules (1) and (2) is analysed in Fig. 2. The cases of overlappings of the rules (1) and (3), and of the rules (2) and (3), are analysed in Fig. 3 and Fig. 4 respectively. The cases of overlappings of the rules (1) and (4), and of the rules (2) and (4), are completely analogous.

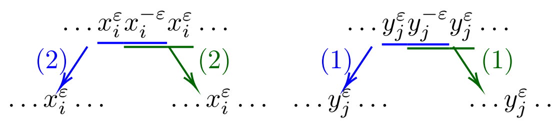

Figure 2. Self-overlapping for rules (1) and (2).

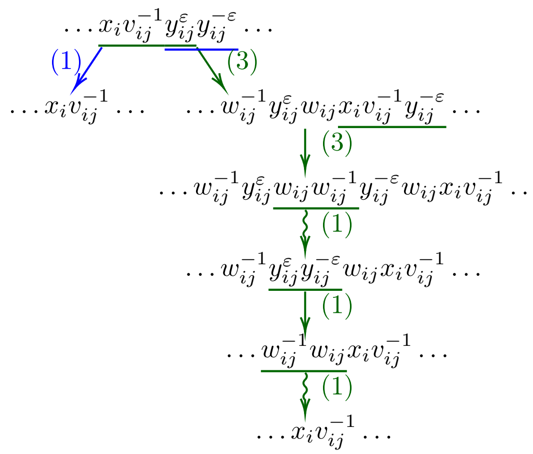

Figure 3. Overlapping of rules (1) and (3).

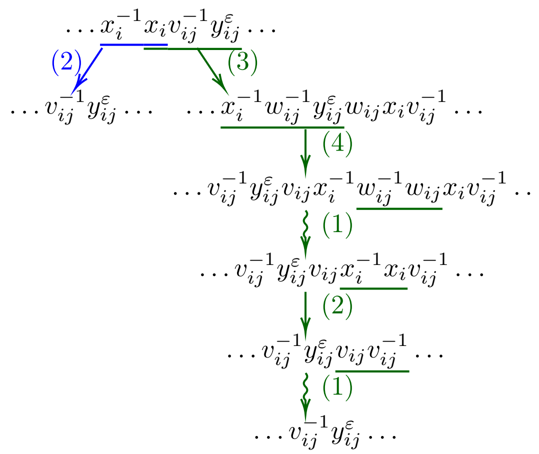

Figure 4. Overlapping of rules (2) and (3).

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Поступило 15 декабря 2025 г.