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SINGULAR MEANDERS

Abstract. The problem of enumerating meanders – pairs of simple plane curves with transverse intersections – was formulated about forty years ago and is still far from solved. Recently, it was discovered that meanders admit a factorization into prime components. This factorization naturally leads to a broader class of objects, which we call singular meanders, in which tangential intersections between the curves are also allowed. In the present paper we initiate a systematic study of singular meanders: we develop a basic combinatorial framework, point out connections with other combinatorial objects and known integer sequences, and completely enumerate several natural families of singular meanders.

Introduction

A meander is a configuration of a pair of simple curves in a disk. Examples of meanders are shown in Fig. 1. The problem of counting meanders was initially formulated by V. Arnol′d in [1]; a similar problem of counting closed meanders (under the name of planar permutation) was formulated by P. Rosenstiehl in [7]. Meanders appear in different areas of mathematics, and we refer readers to the survey [8] for both connections and historical context. At present, the meander problem is far from being solved: counts are known only for small numbers of intersections; no subexponential-time algorithms for computing these numbers are known; and the asymptotic growth rate remains conjectural.

In a recent paper [3], we introduced the factorization of meanders into prime components, and in this context more general objects naturally arise – singular meanders. These are meanders in which not only transverse intersections but also tangency points are allowed (see the example in Fig. 2). Although singular meanders have not been studied before, preliminary enumerative data suggest links with other combinatorial families (some of which were already mentioned in [3]). In the present note, we

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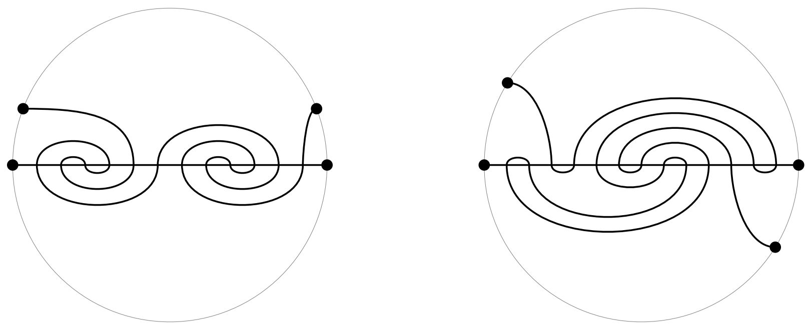


Figure 1. Examples of (non-singular) meanders.

provide a preliminary analysis of singular meanders: we discuss some of their properties, and provide some numerical observations.

The paper is organized as follows. In Sec. 1, we formally define singular meanders and closed singular meanders and introduce the necessary related notions. In Sec. 2, we discuss elementary combinatorial properties of singular meanders. In Sec. 3, we list all families of singular meanders that we are currently able to describe completely.

§1. Definitions

In this section, we present all notation related to singular meanders that we will need later (we refer the reader to [3, Sec. 2] for a more detailed discussion). In Subsec. 1.1, we define closed singular meanders.

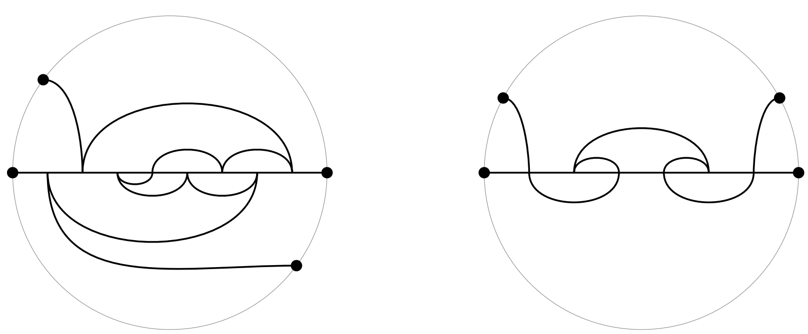


Figure 2. Examples of singular meanders.

Definition 1. *A* singular meander M *is a triple* $(D, (p_1, p_2, p_3, p_4), (l, m))$ *of*

- *a 2-dimensional closed disk* D;
- four distinct points p_1, p_2, p_3, p_4 on the boundary ∂D such that p_3 and p_4 lie in the same connected component of $\partial D \setminus \{p_1, p_2\}$;
- the images m and l of smooth proper embeddings of the segment $[0; 1]$ into D such that $\partial m = \{p_1, p_3\}$, $\partial l = \{p_2, p_4\}$, and $m \cap l$ is a non-empty finite set.

The intersection points of l *and* m *are called* intersections of M.

Definition 2. *We say that two singular meanders*

$$M = (D, (p_1, p_2, p_3, p_4), (l, m)) \quad \text{and} \quad M' = (D', (p'_1, p'_2, p'_3, p'_4), (l', m'))$$

are equivalent *if there exists a homeomorphism* $f \colon D \to D'$ (not necessarily orientation-preserving) such that $f(m) = m'$, $f(l) = l'$, and $f(p_i) = p'_i$ for each $i = 1, \dots, 4$.

Definition 3. *Let* M *be a singular meander and write* $[M]$ *for its equivalence class. For any* $M' \in [M]$, *let* $n_{\mathrm{t}}(M')$ *and* $n_{\mathrm{nt}}(M')$ *be, respectively, the numbers of transverse and non-transverse intersections of the arcs* l *and* m *of* M'.

The order *of* M *is the pair*

$$(n, k) = \left(\max_{M' \in [M]} n_{\mathrm{t}}(M'), \ \min_{M' \in [M]} n_{\mathrm{nt}}(M') \right).$$

If the order of M *is* (n, k), *the* total order *of* M *is* $n + k$. *We denote by* $\mathfrak{M}_{n,k}$ *the set of equivalence classes of singular meanders of order* (n, k) *and by* $\mathcal{M}_{n,k}$ *its cardinality.*

n \ k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	0	1	1	1	1	1	1	1	1
1	1	4	14	48	166	584	2092	7616	28102
2	1	7	36	166	730	3138	13328	56204	235854
3	2	24	188	1224	7202	39808	210992	1085248	5457284

Table 1. Values of $\mathcal{M}_{n,k}$ for small n and k.

Definition 4. *Let*

$$M = (D, (p_1, p_2, p_3, p_4), (l, m)) \quad \textit{and} \quad M' = (D', (p'_1, p'_2, p'_3, p'_4), (l', m'))$$

be two singular meanders. We say that M' is a submeander *of M if*

- $D' \subseteq D$;
- $m' = D' \cap m$;
- $l' = D' \cap l$;
- $p'_1 = \gamma_m(t_1)$, where $\gamma_m\colon [0;\,1] \to D$ is any injective continuous map such that $\gamma_m([0;\,1]) = m$, $\gamma_m(0) = p_1$, and
$$t_1 = \min\{t \in [0;\,1] \mid \gamma_m(t) \in D'\};$$
- Let $S = \gamma_m^{-1}(l) \cap [0;\, t_1]$. If $S \neq \varnothing$, let $t_q = \max S$ and $q = \gamma_m(t_q)$; otherwise, set $q := p_2$. Choose an injective continuous map $\gamma_l\colon [0;\,1] \to D$ with $\gamma_l([0;\,1]) = l$ such that $\gamma_l^{-1}(q) < t$ for all $t \in \gamma_l^{-1}(D')$. Define p'_2 by $p'_2 := \gamma_l(t_2)$, where
$$t_2 = \min\{t \in [0;\,1] \mid \gamma_l(t) \in D'\}.$$

Definition 5. *Let M be a singular meander, and let M' and M'' be two of its submeanders. We say that M' and M'' are* equivalent with respect to M *if they contain exactly the same subset of intersection points of M.*

Definition 6. *Let*

$$M = (D, (p_1, p_2, p_3, p_4), (l, m)) \quad \textit{and} \quad M' = (D', (p'_1, p'_2, p'_3, p'_4), (l', m'))$$

be two singular meanders of order (n, k) and (n', k') respectively, and let

$$M'' = (D'', (p''_1, p''_2, p''_3, p''_4), (l'', m''))$$

be a submeander of M of order (n'', k'') and total order 1 such that $n'' \equiv n' \pmod 2$. Consider a map $f\colon \partial D'' \to \partial D'$ such that $f(p''_i) = p'_i$ for each $i = 1, \ldots, 4$. Then there is a well-defined meander

$$\widetilde{M} = (\widetilde{D}, (p_1, p_2, p_3, p_4), (\widetilde{l}, \widetilde{m}))$$

where

- $\widetilde{D} = \big(D \setminus \operatorname{Int}(D'')\big) \cup_f D'$;
- $\widetilde{l} = \big(l \setminus \operatorname{Int}(D'' \cap l)\big) \cup_f l'$;
- $\widetilde{m} = \big(m \setminus \operatorname{Int}(D'' \cap m)\big) \cup_f m'$.

We say that $\widetilde{M}$ is obtained by the insertion of M' into M at M''.

Definition 7. *Let M be a singular meander of order (n, k). M is said to be* irreducible *if its total order is greater than two and there are precisely $n+k+1$ pairwise non-equivalent submeanders of M. We denote by $\mathcal{M}_{n,k}^{(Ir)}$ the number of equivalence classes of irreducible singular meanders of order (n, k).*
M is called a snake *if its total order is greater than one and there are precisely $\frac{(n+k)(n+k+1)}{2}$ pairwise non-equivalent submeanders of M.*
M is called an iterated snake *if it can be obtained from a snake by finitely many insertions of snakes. We denote by $\mathcal{M}_{n,k}^{(IS)}$ the number of equivalence classes of iterated snakes of order (n, k).*

n \ k	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	0	2	0	8	8	36	72	212	528	1438
2	2	0	12	14	72	162	530	1452	4314	12402
3	0	0	16	48	240	884	3328	11960	42112	145860
4	0	10	36	210	884	3744	14950	57904	218790	809016

Table 2. Values of $\mathcal{M}_{n,k}^{(Ir)}$ for small n and k.

Theorem 1 ([3]). *Each singular meander can be canonically constructed using iterated snakes and irreducible singular meanders.*

Remark 1. The numbers of singular meanders, iterated snakes and irreducible singular meanders of small total order can be found in [2].

Conventions.

(1) Without loss of generality, assume that for every singular meander M of order (n, k), we have $n_{\mathrm{t}}(M) = n$ and $n_{\mathrm{nt}}(M) = k$.

(2) We draw singular meanders in such a way that:
- D is a Euclidean disk;
- l is a horizontal diameter with p_2 at the left end;
- p_1 is placed above p_2.

This allows us to omit the labels l, m, and $p_1, \ldots, p_4$.

1.1. Closed singular meanders.

Definition 8. *A* closed singular meander M *is a tuple* $(D, (p_1, p_2), l, m)$ *of*

- *a 2-dimensional closed disk D;*
- two distinct points p_1, p_2 on the boundary ∂D;
- the image l of a smooth proper embedding of the segment $[0; 1]$ into D such that $\partial l = \{p_1, p_2\}$;
- the image m of a smooth proper embedding of the circle S^1 into D such that m and l intersect (not necessarily transversely) in a non-empty finite set of points.

The intersection points of m and l are called intersections of M*.*

Definition 9. *We say that two closed singular meanders*

$$M = (D, (p_1, p_2), l, m) \quad \textit{and} \quad M' = (D', (p'_1, p'_2), l', m')$$

are equivalent *if there exists an orientation-preserving homeomorphism $f\colon D \to D'$ such that $f(m) = m'$, $f(l) = l'$, and $f(p_i) = p'_i$ for each $i = 1, 2$.*

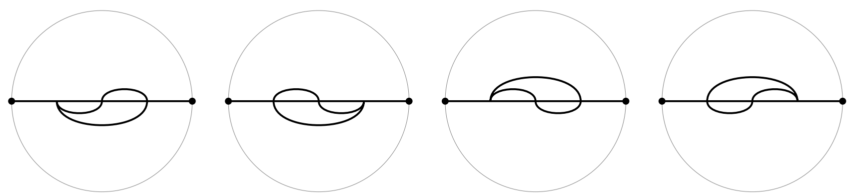


Figure 3. Examples of non-equivalent closed singular meanders.

Remark 2. Note that a homeomorphism in the definition of equivalence for singular meanders is not required to be orientation-preserving, in contrast to the definition of equivalence for closed singular meanders. This difference arises because we want to consider the four closed singular meanders in Fig. 3 as pairwise non-equivalent.

Definition 10. *For closed singular meanders, the* order *and the* total order *are defined verbatim as in Definition* 3. We denote by $\overline{\mathfrak{M}}_{n,k}$ the set of equivalence classes of closed singular meanders of order (n, k) and by $\overline{\mathcal{M}}_{n,k}$ its cardinality.

Remark 3. As in the case of classical meanders, one can introduce singular versions of other related objects: semimeanders (configurations of a ray and a curve, see [5]) and stamp foldings (configurations of a segment and a curve, see [4]). One could also consider more general objects allowing both singular intersection points and multiple connected components.

§2. Combinatorics of singular meanders

Throughout this section, whenever we mention a singular meander, we implicitly choose the geometric representative satisfying our drawing conventions from Sec. 1.

Theorem 2. *Let n be a positive even integer, and k be a non-negative integer. Then*

$$\overline{\mathcal{M}}_{n,k} = \mathcal{M}_{n-1,k} + 2\mathcal{M}_{n,k-1}.$$

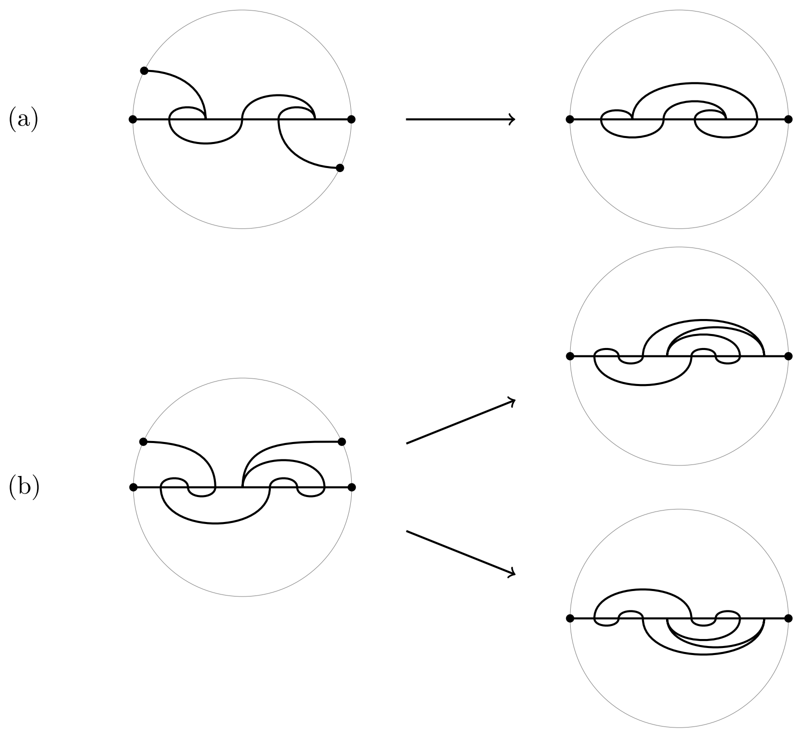


Figure 4. Transforming singular meanders into closed singular meanders.

Proof. Let n and k be as in the statement of the theorem. Note that if $k = 0$ then $\mathcal{M}_{n,-1} = 0$ and we recover a well-known formula, connecting the numbers of open and closed meanders.

For each singular meander of order $(n-1, k)$, we can construct a closed singular meander of order (n, k) by connecting the ends of m and adding one more transverse intersection point on the right; see Fig. 4 (a). For each singular meander of order $(n, k-1)$, there are two different ways to construct closed singular meanders of order (n, k): either connect the ends of m and add one more non-transverse intersection point on the right, or first reflect along l and then add one more non-transverse intersection point on the right; see Fig. 4 (b). (These two closed singular meanders are clearly non-equivalent.) It is clear that non-equivalent singular meanders lead to non-equivalent closed singular meanders. Finally, each closed singular meander can be obtained from a singular meander in this way. □

Theorem 3. *Let k be a positive integer and n be a positive even integer. Then*

$$k\mathcal{M}_{n-1,k} = 2n\mathcal{M}_{n,k-1}.$$

Proof. Let n and k be as in the theorem. The finite group $\mathbb{Z}/(n+k)\mathbb{Z}$ acts on $\overline{\mathfrak{M}}_{n,k}$ by cyclically moving the leftmost intersection point to the right; see Fig. 5. For a closed singular meander M, let O be its orbit under this action. Then O contains $\frac{n+k}{d_O}$ elements, where $d_O \geqslant 1$ denotes the cardinality of the stabilizer of any element in O (for instance, in Fig. 5 we have $d_O = 2$). Moreover, among these elements, exactly $\frac{n}{d_O}$ are obtained from singular meanders of order $(n-1, k)$ and $\frac{k}{d_O}$ are obtained from singular meanders of order $(n, k-1)$. Thus

$$\mathcal{M}_{n-1,k} = \sum_O \frac{n}{d_O},$$
$$2\mathcal{M}_{n,k-1} = \sum_O \frac{k}{d_O},$$
$$\frac{\mathcal{M}_{n-1,k}}{n} = \frac{2\mathcal{M}_{n,k-1}}{k},$$

where the sums are taken over all orbits O. Hence $k\mathcal{M}_{n-1,k} = 2n\mathcal{M}_{n,k-1}$, as claimed. □

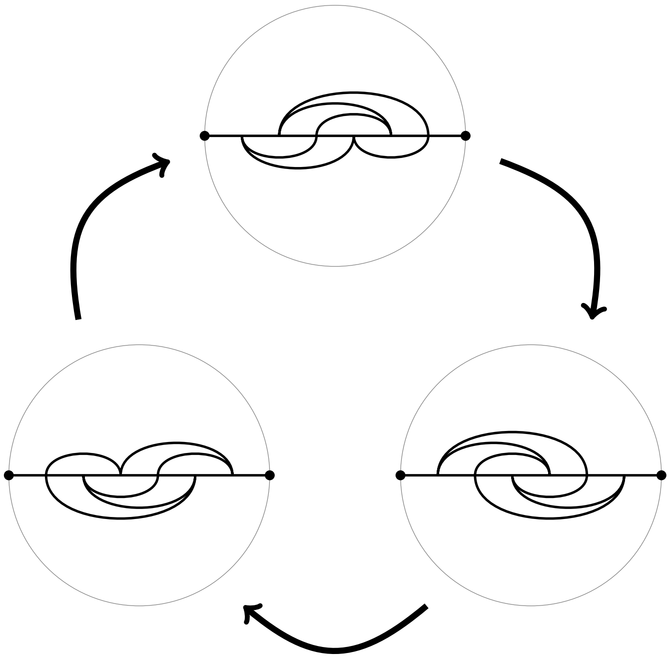


Figure 5. Action of the group $\mathbb{Z}/(n+k)\mathbb{Z}$ on closed singular meanders.

Remark 4. The statement of Theorem 3 can be reformulated as follows. Let

$$\phi(x,t) := \sum_{n,k\geqslant 0} \mathcal{M}_{n,k} x^n t^k$$

be the generating function for $\{\mathcal{M}_{n,k}\}_{n,k\geqslant 0}$, and let

$$\phi_{(o)}(x,t) := \sum_{n,k\geqslant 0} \mathcal{M}_{2n+1,k} x^{2n+1} t^k \quad \text{and} \quad \phi_{(e)}(x,t) := \sum_{n,k\geqslant 0} \mathcal{M}_{2n,k} x^{2n} t^k$$

be its odd and even parts with respect to the variable x. Then

$$2\,\partial_x \phi_{(e)}(x,t) = \partial_t \phi_{(o)}(x,t).$$

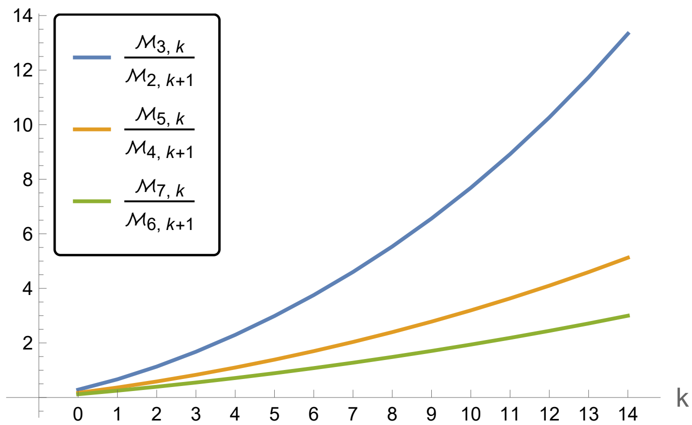


Figure 6. Numerical behavior of the quotient $\mathcal{M}_{n,k}/\mathcal{M}_{n-1,k+1}$ for a fixed odd n.

Remark 5. Although for any fixed odd n the quotient $\frac{\mathcal{M}_{n,k-1}}{\mathcal{M}_{n-1,k}}$ is well approximated by a third-degree polynomial in k (see Fig. 6), we were not able to find an exact relation.

Corollary 1. *Let k be a positive integer and n be a positive even integer. Then*

$$k\mathcal{M}^{(Ir)}_{n-1,k} = 2n\mathcal{M}^{(Ir)}_{n,k-1}.$$

Proof. It is easy to see that the group action described in the proof of Theorem 3 preserves irreducibility in the following sense: if a closed singular meander M is an image of an irreducible singular meander, then all other closed singular meanders in the orbit of M are also images of irreducible singular meanders (see the example in Fig. 5). □

Corollary 2. *Let n be a positive even integer and k a positive integer coprime to n. Then*

$$\begin{aligned} \mathcal{M}_{n-1,k} &\equiv 0 \pmod{2n}, \\ \mathcal{M}_{n,k-1} &\equiv 0 \pmod{k}, \\ \mathcal{M}^{(Ir)}_{n-1,k} &\equiv 0 \pmod{2n}, \\ \mathcal{M}^{(Ir)}_{n,k-1} &\equiv 0 \pmod{k}. \end{aligned}$$

§3. Some families of singular meanders with a complete enumeration

3.1. Singular meanders with a small number of transverse intersections.

Theorem 4.

$$\sum_{k\geqslant 0}\mathcal{M}_{1,k}t^k=\frac{1}{(1-2t)\sqrt{1-4t}}.$$

Proof. Let $M=(D,(p_1,p_2,p_3,p_4),(l,m))$ be a singular meander of order $(1,k)$. Without loss of generality, we may assume that (see Fig. 7 (a))

(1) D is the Euclidean disk $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leqslant 1\}$;
(2) $p_1=(0,1)$;
(3) $p_2=(-1,0)$;
(4) $p_3=(0,-1)$;
(5) $p_4=(1,0)$.

After performing an isotopy, we may assume that $l\cup m$ is contained in a small neighborhood of

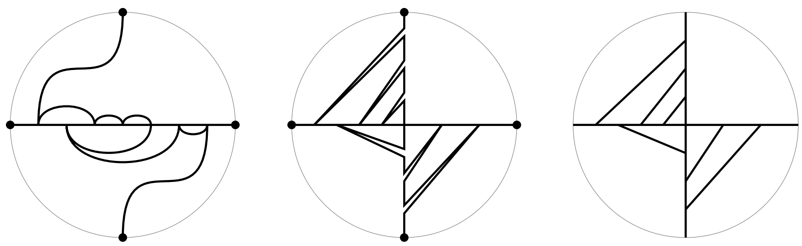
$$C_M:=I_X\cup I_Y\cup\bigcup_{i=1}^{k}I_i,$$

where $I_X=\{(t,0)\mid t\in[-1;1]\}$, $I_Y=\{(0,t)\mid t\in[-1;1]\}$, and $\{I_i\}$ is a set of pairwise disjoint segments (which we call *chords*) with one endpoint on I_X and the other endpoint on I_Y; see Fig. 7 (b). We call such C_M a *carcass* (with k chords); see Fig. 7 (c).

The carcass is uniquely defined up to isotopy of D that keeps $I_X\cup I_Y$ setwise fixed. Non-equivalent singular meanders lead to non-isotopic carcasses. Thus the number of non-equivalent singular meanders of order $(1,k)$ equals the number of non-isotopic carcasses with k chords. The number of carcasses with k chords is given by

$$\sum_{\substack{x_1,x_2,x_3,x_4\geqslant 0\\ x_1+x_2+x_3+x_4=k}}\binom{x_1+x_2}{x_1}\binom{x_2+x_3}{x_2}\binom{x_3+x_4}{x_3}\binom{x_4+x_1}{x_4},\qquad(1)$$

where x_i is the number of chords lying in the i-th quadrant (for example, in Fig. 7 (c) we have $x_1=0$, $x_2=3$, $x_3=1$, and $x_4=2$), and $\binom{x_1+x_2}{x_1}$ corresponds to all possible arrangements of the endpoints of chords on the



(a) (b) (c)

Figure 7. Example of a singular meander of order $(1,6)$ and its corresponding carcass.

segment $\{(0,t) \mid t \in (0,1)\}$, with the other binomial coefficients interpreted analogously. Thus, summing the coefficients given by Eq. (1) we get

$$\sum_{k\geqslant 0} \mathcal{M}_{1,k} t^k = \frac{1}{(1-2t)\sqrt{1-4t}}. \qquad \square$$

Remark 6. The sequence $\{\mathcal{M}_{1,k}\}_{k\geqslant 0}$ coincides with the OEIS sequence A082590 [9], which counts the number of k-letter words over the alphabet $\{1,2,3,4\}$ having as many occurrences of the substring (consecutive subword) $[1,2]$ as of $[1,3]$. An explicit bijection between equivalence classes of singular meanders with a single transverse intersection and such words will be given in a subsequent paper (together with several other interpretations of singular meanders with a single transverse intersection).

Remark 7. A technique similar to that in the proof of Theorem 4 can be applied to obtain a formula for $\mathcal{M}_{n,k}$ for other values of n. However, the formulas quickly become too cumbersome and cannot be easily expressed via generating functions. For example,

$$\mathcal{M}_{3,k} = 2 \sum_{\substack{x_0,\dots,x_3\geqslant 0\\ y_0,\dots,y_3\geqslant 0\\ u_0,\dots,u_3\geqslant 0\\ \sum_{i=0}^{3}(x_i+y_i+u_i)=k}} \left(\prod_{i=0}^{3} \binom{x_i+x_{i+1}+u_i}{u_i} \prod_{i=0}^{3} \binom{y_i+y_{i+1}+u_i}{u_i} \right),$$

where the indices of x_i and y_i are considered modulo 4.

Theorem 5.
$$\sum_{k\geqslant 0}\mathcal{M}_{2,k}t^k = \frac{1-3t}{(1-2t)^2\sqrt{(1-4t)^3}}.$$

Proof. From Theorem 3 with $n = 2$ it follows that
$$k\,\mathcal{M}_{1,k} = 4\,\mathcal{M}_{2,k-1},$$
and hence
$$\mathcal{M}_{2,k} = \frac{k+1}{4}\,\mathcal{M}_{1,k+1}.$$
Using Theorem 4, we obtain
$$\sum_{k\geqslant 0}\mathcal{M}_{2,k}t^k = \frac{1}{4}\,\partial_t\left(\frac{1}{(1-2t)\sqrt{1-4t}}\right) = \frac{1-3t}{(1-2t)^2\sqrt{(1-4t)^3}}. \qquad \square$$

3.2. Iterated snakes. Iterated snakes form a rather simple class of singular meanders, allowing a complete enumeration. Let
$$F(x,t) := \sum_{n,k\geqslant 0}\mathcal{M}^{(IS)}_{n,k}x^n t^k$$
be the generating function for iterated snakes. Then $F(x,t)$ can be found from the following equation (see [3]):
$$v^3 + xv^2 + 2v - 4(t-1)^2(v+x) = 0 \quad \text{where} \quad v = 2F(x,t) + x + 2. \tag{2}$$
Some subsequences of the array $\left\{\mathcal{M}^{(IS)}_{n,k}\right\}_{n,k\geq 0}$ coincide with other combinatorial sequences that can be found in the OEIS [9][1]:

- $\left\{\mathcal{M}^{(IS)}_{0,k}\right\}_{k>0}$ is A000012 (this is just the sequence of all ones);
- $\left\{\mathcal{M}^{(IS)}_{1,k}\right\}_{k\geqslant 0}$ is A007070;
- $\left\{\mathcal{M}^{(IS)}_{2,k}\right\}_{k\geqslant 0}$ is A181292;
- $\left\{\mathcal{M}^{(IS)}_{n,0}\right\}_{n\geqslant 0}$ is A007165.

The first three sequences coincide since their first terms agree and they satisfy the same recurrence relation (which follows from Eq. (2)). The sequence A007165 counts the number of P-graphs, which are just graph-theoretic reformulations of iterated snakes; see [6].

[1]For an exact match, singular meanders of total order 1 should also be considered as snakes.

3.3. Irreducible singular meanders.

Theorem 6. *Let $\varphi(x)$ be Euler's totient function. Then, for all integers $n, k \geqslant 0$,*

(1) $\mathcal{M}^{(Ir)}_{n,k} = 0$ *for* $k < 3$ *or* $n < 1$;
(2) $\mathcal{M}^{(Ir)}_{2n+1,3} = 0$;
(3) $\mathcal{M}^{(Ir)}_{2n,3} = \varphi(n+4) - 2$;
(4) $\mathcal{M}^{(Ir)}_{2n-1,4} = n\big(\varphi(n+4) - 2\big)$;
(5) $\displaystyle\sum_{k\geqslant 0} \mathcal{M}^{(Ir)}_{1,k} t^k = \frac{1}{8}\sqrt{\left(\frac{1-3t}{1+t}\right)^4 - 1}$;
(6) $\displaystyle\sum_{k\geqslant 0} \mathcal{M}^{(Ir)}_{2,k} t^k = \frac{2t - 1 + \sqrt{\frac{1-3t}{1+t}}}{\sqrt{(1-3t)(1+t)^5}}$.

Proof. Cases (1)–(3) were proved in [3]. Case (4) follows from case (3) and Theorem 3. Analogously, case (6) follows from case (5) and Theorem 3. The only thing that remains to prove is case (5).

To prove case (5), we introduce two additional classes of singular meanders. A singular meander is said to be *arborescent*[2] if it can be obtained from an irreducible singular meander by finitely many insertions of irreducible singular meanders. The generating function for arborescent singular meanders of order $(1, k)$ can be easily expressed in terms of the generating function for irreducible singular meanders of order $(1, k)$. Let $\mathcal{M}^{(IIr)}_{n,k}$ be the number of non-equivalent arborescent singular meanders of order (n, k), and set

$$F(t) := \sum_{k\geqslant 0} \mathcal{M}^{(Ir)}_{1,k} t^k,$$

$$G(t) := \sum_{k\geqslant 0} \mathcal{M}^{(IIr)}_{1,k} t^k = \sum_{r\geqslant 0} F(t)^r = \frac{1}{1 - F(t)}.$$

A singular meander is called a *B-meander* if it can be obtained by inserting singular meanders of order $(0, p)$ into arborescent singular meanders. Since for each $p > 0$ there is a unique equivalence class of singular meanders of order $(0, p)$, each arborescent singular meander of order (n, r) leads

[2]We use this term because the poset of its submeanders (see [3]) is a tree.

to $\binom{k-1}{r-1}$ non-equivalent B-meanders of order (n,k). Thus the generating function $B(t)$ for the numbers of B-meanders of order $(1,k)$ is given by

$$B(t) = \sum_{k\geqslant 0}\sum_{r=1}^{k} \mathcal{M}_{1,r}^{(IIr)} \binom{k-1}{r-1} t^k = \sum_{r\geqslant 1} \mathcal{M}_{1,r}^{(IIr)} \left(\frac{t}{1-t}\right)^r = G\left(\frac{t}{1-t}\right).$$

From Theorem 1 it follows that each singular meander of order $(1,k)$ is uniquely obtained via a sequential insertion of B-meanders and iterated snakes, where iterated snakes are not inserted into iterated snakes and B-meanders are not inserted into B-meanders. At the level of generating functions this leads to the following equation:

$$C(t) = \frac{A(t) + B(t) + 2A(t)B(t)}{1 - A(t)B(t)}, \tag{3}$$

where

$$C(t) := \sum_{k\geqslant 0} \mathcal{M}_{1,k} t^k = \frac{1}{(1-2t)\sqrt{1-4t}},$$

$$A(t) := \sum_{k\geqslant 0} \mathcal{M}_{1,k}^{(IS)} t^k = \frac{1}{2t^2 - 4t + 1}.$$

From Eq. (3) one finds $B(t)$, then $G(t)$, and finally $F(t)$, which completes the proof of case (5) and hence of the theorem. □

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