Записки научных
семинаров ПОМИ
Том 549, 2025 г.

I. S. Baskov

ISOMORPHIC LODAY FUNCTORS OF NON-HOMEOMORPHIC SPACES

ABSTRACT. Each commutative algebra A gives rise to a representation $\mathcal{L}_A$, which we call the Loday functor of A, of the category Ω of finite sets and surjective maps. In this paper we present two (infinite-dimensional) non-isomorphic algebras over $\mathbb{C}$ with isomorphic Loday functors, namely, the algebras of continuous functions on the Möbius strip and on the cylinder.

§1. INTRODUCTION

Let Ω be the category whose objects are the sets $\langle n\rangle = \{1, \dots, n\}$, $n \in \mathbb{N}$, and whose morphisms are surjective maps. A non-unital commutative algebra A over a field k gives rise to a functor $\mathcal{L}_A \colon \Omega \to \mathrm{Vect}_k$ that takes an object $\langle n\rangle$ to the vector space $A^{\otimes n}$ and a surjection $\sigma \colon \langle n\rangle \to \langle m\rangle$ to the linear map $\sigma^* \colon A^{\otimes n} \to A^{\otimes m}$,

$$a_1 \otimes \cdots \otimes a_n \mapsto b_1 \otimes \cdots \otimes b_m,$$
$$b_j = \prod_{i \in \sigma^{-1}(j)} a_i.$$

Similar functors were considered in [1].

In [2] Podkorytov proved that if for finite-dimensional algebras A and B over an algebraically closed field k the functors $\mathcal{L}_A$ and $\mathcal{L}_B$ are isomorphic, then the algebras A and B are isomorphic.

In this paper we present two non-isomorphic algebras over $\mathbb{C}$ with isomorphic Loday functors. In particular, we show that the $\mathbb{C}$-algebra of continuous functions on the cylinder and the $\mathbb{C}$-algebra of continuous functions on the Möbius strip have isomorphic Loday functors. A related result in the discrete case was obtained in [3].

Key words and phrases: Loday functor, representation of category, category of finite sets and surjections, functor isomorphism.

This work was supported by the Ministry of Science and Higher Education of the Russian Federation (agreement 075-15-2025-344 dated 29/04/2025 for Saint Petersburg Leonhard Euler International Mathematical Institute at PDMI RAS).

It is unclear what information about the space X can be recovered from the functor $\mathcal{L}_{C(X)}$. By Gelfand–Kolmogorov theorem, if for compact Hausdorff spaces X and Y, the algebras $C(X)$ and $C(Y)$ are isomorphic as algebras, then X and Y are homeomorphic. As a result of Theorem 2.10, we see that from an isomorphism of $\mathcal{L}_{C(X)}$ and $\mathcal{L}_{C(Y)}$ one cannot conclude that the spaces X and Y are homeomorphic.

§2. Projections and twists

Let X and Y be topological spaces. Denote by $\langle Y^X\rangle$ the linear span over $\mathbb{C}$ of the set of all continuous maps $X \to Y$. We call an element of $\langle Y^X\rangle$ an *ensemble*. An ensemble

$$A = \sum_{i=1}^{n} u_i a_i \in \langle Y^X\rangle,$$

where $a_i\colon X \to Y$ are continuous maps and $u_i \in \mathbb{C}$, gives rise to a linear map $C(Y) \to C(X)$. Moreover, for each $r \in \mathbb{N}$ such an A induces a linear map

$$A^{(r)}\colon C(Y)^{\otimes r} \to C(X)^{\otimes r}$$

by the rule

$$A^{(r)}(f_1 \otimes \cdots \otimes f_r) = \sum_{i=1}^{n} u_i (f_1 \circ a_i) \otimes \cdots \otimes (f_r \circ a_i).$$

We say that a topological space X is completely Hausdorff if for any two points $x \neq y$ in X there exists a function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.

Definition 2.1. *Let $r \in \mathbb{N}$. An ensemble $A \in \langle Y^X\rangle$ is called r-coherent if $A^{(r)} = 0$.*

There is a simple way to check if A is r-coherent. For any finite set $T \hookrightarrow X$ there is the restriction map

$$\langle Y^X\rangle \to \langle Y^T\rangle, \qquad A \mapsto A|_T = \sum_{i=1}^{n} u_i (a_i|_T).$$

Lemma 2.2. *Let X and Y be topological spaces and $A \in \langle Y^X\rangle$ be an ensemble. If $A|_T = 0$ for all finite sets $T \subset X$ of at most r points, then the ensemble A is r-coherent. If Y is completely Hausdorff, then the converse is true.*

Proof. First, assume $A|_T = 0$ for all finite sets $T \subset X$ of at most r points. Take $T = \{x_1, \dots, x_r\} \subset X$ (some of the points may coincide). In the ensemble $A|_T$ regroup the indices to get

$$A|_T = \sum_{\psi\colon T \to Y} v_\psi \cdot \psi,$$

where the sum is taken over all maps $T \to Y$. Then the condition $A|_T = 0$ means that $v_\psi = 0$ for all functions $\psi\colon T \to Y$. Take $f_1, \dots, f_r \in C(Y)$ and a point $(x_1, \dots, x_r) \in X^r$. With the natural inclusion $C(X)^{\otimes r} \hookrightarrow C(X^r)$ we can consider the element $A^{(r)}(f_1 \otimes \cdots \otimes f_r)$ as a function on X^r. We have

$$A^{(r)}(f_1 \otimes \cdots \otimes f_r)(x_1, \dots, x_r) = \sum_{i=1}^{n} u_i f_1(a_i(x_1)) \dots f_r(a_i(x_r)).$$

Again, regroup the summands to get

$$A^{(r)}(f_1 \otimes \cdots \otimes f_r)(x_1, \dots, x_r) = \sum_{\psi\colon T \to Y} v_\psi \cdot f_1(\psi(x_1)) \dots f_r(\psi(x_r)) = 0.$$

For the other direction, assume that $A^{(r)}(f_1 \otimes \cdots \otimes f_r) = 0$ for all elements $f_1, \dots, f_r \in C(Y)$. Suppose there exists a finite set

$$T = \{x_1, \dots, x_r\} \subset X$$

with $A|_T \neq 0$. Then there is a function $\varphi\colon T \to Y$ such that

$$\sum_{i\colon\ a_i|_T = \varphi} u_i = v_\varphi \neq 0.$$

Define the sets $S_j = \{a_i(x_j) \mid i = 1, \dots, n\}$. Because Y is completely Hausdorff we can choose the continuous functions $f_j\colon Y \to \mathbb{C}$, $j = 1, \dots, r$, such that $f_j(\varphi(x_j)) = 1$ and $f_j = 0$ on $S_j \setminus \{\varphi(x_j)\}$. Then

$$\begin{aligned}
&A^{(r)}(f_1 \otimes \cdots \otimes f_r)(x_1, \dots, x_r) \\
&= \sum_{\substack{\psi\colon T \to Y,\\ \psi \neq \varphi}} v_\psi \cdot f_1(\psi(x_1)) \dots f_r(\psi(x_r)) + v_\varphi \cdot f_1(\varphi(x_1)) \dots f_r(\varphi(x_r)) \\
&= v_\phi \neq 0.
\end{aligned}$$

□

Definition 2.3. *Let X and Y be topological spaces. The following set of functions is called a set of* projections and twists*:*

(1) $p_n\colon X \to X$, $q_n\colon Y \to Y$, $n \in \mathbb{N}$, *satisfying the identities*
  - $p_n^2 = p_n$,

- $p_i \circ p_j = p_j \circ p_i$,
- $q_n^2 = q_n$,
- $q_i \circ q_j = q_j \circ q_i$,
- *the sets* $\operatorname{Mov} p_n = \{x \in X \mid p_n(x) \neq x\}$ *are pairwise disjoint,*
- *the sets* $\operatorname{Mov} q_n = \{y \in Y \mid q_n(y) \neq y\}$ *are pairwise disjoint.*

(2) $h_n \colon X \to Y$, $\bar{h}_n \colon Y \to X$, $n \in \mathbb{N}$, *satisfying the identities*
- $q_j \circ h_i = h_i \circ p_j$,
- $p_j \circ \bar{h}_i = \bar{h}_i \circ q_j$,
- $\bar{h}_j \circ h_i = p_j \circ p_i$,
- $h_i \circ \bar{h}_j = q_i \circ q_j$.

If such a collection of maps exists, we say that X and Y are connected by projections and twists.

In Theorem 2.10 we show that if X and Y are connected by projections and twists then the functors $\mathcal{L}_{C(X)}$ and $\mathcal{L}_{C(Y)}$ are isomorphic.

Define the ensembles

$$Z_n = \prod_{i=1}^{n}(1 - p_i) \in \langle X^X \rangle, \qquad \bar{Z}_n = \prod_{i=1}^{n}(1 - q_i) \in \langle Y^Y \rangle,$$

where the product is the composition and 1 is the identity map. We also set $Z_0 = 1$ and $\bar{Z}_0 = 1$.

As p_n and q_n are idempotent, the elements $1 - p_n$ and $1 - q_n$ are also idempotent and hence Z_n and $\bar{Z}_n$ are idempotent.

Lemma 2.4. *We have the identities*

$$Z_0 p_1 + Z_1 p_2 + \dots + Z_{n-1} p_n = 1 - Z_n,$$
$$\bar{Z}_0 q_1 + \bar{Z}_1 q_2 + \dots + \bar{Z}_{n-1} q_n = 1 - \bar{Z}_n$$

for $n \in \mathbb{N}$.

Proof. We prove the first identity by induction on $n \geqslant 1$. The base case $n = 1$ is trivial. Assume the identity is true for $n - 1$. We have

$$\begin{aligned} &Z_0 p_1 + Z_1 p_2 + \dots + Z_{n-2} p_{n-1} + Z_{n-1} p_n \\ &= 1 - Z_{n-1} + Z_{n-1} p_n = 1 - Z_{n-1}(1 - p_n) = 1 - Z_n. \end{aligned}$$

The second identity is analogous. $\square$

Next, define the ensembles

$$T_n = \sum_{i=1}^{n} \bar{Z}_{i-1} h_i \in \langle Y^X \rangle, \qquad \bar{T}_n = \sum_{i=1}^{n} Z_{i-1} \bar{h}_i \in \langle X^Y \rangle.$$

Lemma 2.5. *For any $n \in \mathbb{N}$ we have the following identities*

$$T_n\bar{T}_n = 1 - \bar{Z}_n, \qquad \bar{T}_nT_n = 1 - Z_n.$$

Proof. We have

$$T_n\bar{T}_n = \left(\sum_{i=1}^{n} \bar{Z}_{i-1}h_i\right)\left(\sum_{i=1}^{n} Z_{i-1}\bar{h}_i\right) = \sum_{i,j=1}^{n} \bar{Z}_{i-1}h_iZ_{j-1}\bar{h}_j.$$

Using the properties of projections and twists we can swap h_iZ_{j-1} to $\bar{Z}_{j-1}h_i$ and simplify to get

$$T_n\bar{T}_n = \sum_{i,j=1}^{n} \bar{Z}_{i-1}\bar{Z}_{j-1}h_i\bar{h}_j = \sum_{i,j=1}^{n} \bar{Z}_{i-1}\bar{Z}_{j-1}q_iq_j.$$

Split this sum into the diagonal and the off-diagonal sums:

$$T_n\bar{T}_n = \sum_{i=1}^{n} \bar{Z}_{i-1}\bar{Z}_{i-1}q_iq_i + \sum_{i,j=1,\ i\neq j}^{n} \bar{Z}_{i-1}\bar{Z}_{j-1}q_iq_j.$$

For the diagonal sum we get

$$\sum_{i=1}^{n} \bar{Z}_{i-1}\bar{Z}_{i-1}q_iq_i = \sum_{i=1}^{n} \bar{Z}_{i-1}q_i = 1 - \bar{Z}_n$$

by Lemma 2.4. To deal with the off-diagonal sum, consider one summand

$$\bar{Z}_{i-1}\bar{Z}_{j-1}q_iq_j,$$

where $i > j$ (the case $i < j$ is similar). Note that the element $\bar{Z}_{i-1}$ contains the factor $1-q_j$. Now, because all of the maps q_i commute with each other, we have $\bar{Z}_{i-1}q_j = 0$ and the whole off-diagonal sum is zero. The second identity is analogous. □

Lemma 2.6. *The ensembles Z_{n+1} and $\bar{Z}_{n+1}$ are n-coherent.*

Proof. We prove that Z_{n+1} is n-coherent. By Lemma 2.2 it is enough to prove that for any finite set $T \subset X$ consisting of at most n points the restriction $Z_{n+1}|_T$ is zero. As the sets $\operatorname{Mov} p_i$ are pairwise disjoint by definition, there is j such that $\operatorname{Mov} p_j \cap T = \emptyset$. Hence, $(1 - p_j)|_T = 0$ and thus $Z_{n+1}|_T = 0$. □

Corollary 2.7. *The ensemble $\bar{T}_{n+1}T_{n+1}$ induces the identity map on the tensor power $C(X)^{\otimes n}$, and the ensemble $T_{n+1}\bar{T}_{n+1}$ induces the identity map on the tensor power $C(Y)^{\otimes n}$.*

Proof. Follows from Lemma 2.5 and Lemma 2.6. □

Lemma 2.8. *For $r \in \mathbb{N}$ we have*

$$T_{r+1}^{(r-1)} = T_r^{(r-1)}.$$

Proof. Consider the ensemble $T_{r+1} - T_r = \bar{Z}_r h_{r+1}$. We have

$$T_{r+1}^{(r-1)} - T_r^{(r-1)} = \bar{Z}_r^{(r-1)} h_{r+1}^{(r-1)} = 0$$

as $\bar{Z}_r^{(r-1)} = 0$ by Lemma 2.6. □

Take any surjection $\sigma\colon \langle r\rangle \to \langle r-1\rangle$. It is clear that for an arbitrary ensemble $A \in \langle Y^X\rangle$ the following diagram commutes

$$\begin{tikzcd} C(Y)^{\otimes r} \arrow[r, "\sigma^*"] \arrow[d, "A^{(r)}"] & C(Y)^{\otimes r-1} \arrow[d, "A^{(r-1)}"] \\ C(X)^{\otimes r} \arrow[r, "\sigma^*"] & C(X)^{\otimes r-1}. \end{tikzcd} \tag{1}$$

Lemma 2.9. *The following square is commutative*

$$\begin{tikzcd} C(Y)^{\otimes r} \arrow[r, "\sigma^*"] \arrow[d, "T_{r+1}^{(r)}"] & C(Y)^{\otimes r-1} \arrow[d, "T_r^{(r-1)}"] \\ C(X)^{\otimes r} \arrow[r, "\sigma^*"] & C(X)^{\otimes r-1}. \end{tikzcd}$$

Proof. By Lemma 2.8 it is enough to prove that the square

$$\begin{tikzcd} C(Y)^{\otimes r} \arrow[r, "\sigma^*"] \arrow[d, "T_{r+1}^{(r)}"] & C(Y)^{\otimes r-1} \arrow[d, "T_{r+1}^{(r-1)}"] \\ C(X)^{\otimes r} \arrow[r, "\sigma^*"] & C(X)^{\otimes r-1} \end{tikzcd}$$

is commutative, which follows from the general case (1). □

Theorem 2.10. *Let X and Y be topological spaces that are connected by projections and twists. Then the Loday functors $\mathcal{L}_{C(X)}$ and $\mathcal{L}_{C(Y)}$ are isomorphic.*

Proof. Define the morphism $\Phi\colon \mathcal{L}_{C(X)} \to \mathcal{L}_{C(Y)}$ with the r-th component $\Phi\colon C(X)^{\otimes r} \to C(Y)^{\otimes r}$ being $\bar{T}_{r+1}^{(r)}$ and $\Psi\colon \mathcal{L}_{C(Y)} \to \mathcal{L}_{C(X)}$ with the r-th component $\Psi\colon C(Y)^{\otimes r} \to C(X)^{\otimes r}$ being $T^{(r)}_{r+1}$. We have a diagram

$$\begin{tikzcd}
C(X) \arrow[d, shift right, "\Phi"'] & C(X)^{\otimes 2} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift right, "\Phi"'] & C(X)^{\otimes 3} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift right, "\Phi"'] & \dots \arrow[l, shift left] \arrow[l, shift right] \\
C(Y) \arrow[u, shift right, "\Psi"'] & C(Y)^{\otimes 2} \arrow[l, shift left] \arrow[l, shift right] \arrow[u, shift right, "\Psi"'] & C(Y)^{\otimes 3} \arrow[l, shift left] \arrow[l, shift right] \arrow[u, shift right, "\Psi"'] & \dots \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

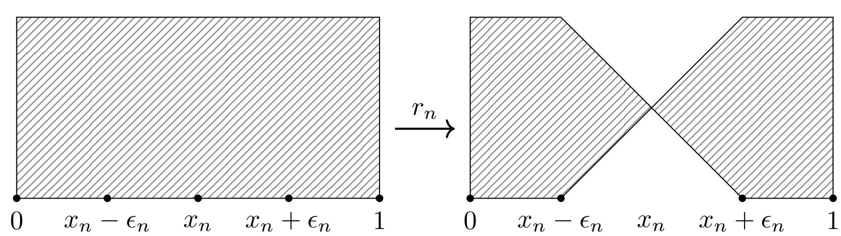
where each square for a chosen surjection $\sigma\colon \langle r\rangle \to \langle r-1\rangle$ is commutative. By Corollary 2.7 the vertical maps are isomorphisms. Hence Φ and Ψ are mutually inverse natural isomorphisms of the Loday functors. $\square$

§3. The cylinder and the Möbius strip

In this section we show that the cylinder and the Möbius strip are connected by projections and twists. By Theorem 2.10 these two non-homeomorphic spaces have isomorphic Loday functors.

Define the cylinder X as the quotient of $[0,1]\times[0,1]$ obtained by identifying the vertical sides via $(0,y)\sim(1,y)$ and the Möbius strip Y as the quotient of $[0,1]\times[0,1]$ obtained by identifying the vertical sides via $(0,y)\sim(1,1-y)$.

Set $x_n = \frac{1}{n+1}$; choose $\epsilon_n > 0$ so that the intervals $(x_n-\epsilon_n, x_n+\epsilon_n)$ and $(x_m-\epsilon_m, x_m+\epsilon_m)$ do not intersect for any $n\neq m$. Looking at the picture below denote by $R_n \subset [0,1]\times[0,1]$ the space on the right and denote by $r_n\colon [0,1]\times[0,1]\to R_n$ some vertical retraction, symmetric about the line $y=\frac{1}{2}$.



The composition

$$[0,1]\times[0,1] \xrightarrow{r_n} R_n \hookrightarrow [0,1]\times[0,1]$$

gives rise to the maps $p_n\colon X\to X$ and $q_n\colon Y\to Y$.

Denote by $\gamma_n\colon R_n \to R_n$ the half-twist of the right half of R_n,

$$\gamma_n(x,y) = \begin{cases} (x,y), & 0 \leqslant x \leqslant x_n, \\ (x,1-y), & x_n \leqslant x \leqslant 1. \end{cases}$$

The composition

$$[0,1]\times[0,1] \xrightarrow{r_n} R_n \xrightarrow{\gamma_n} R_n \hookrightarrow [0,1]\times[0,1]$$

gives rise to the maps $h_n\colon X \to Y$ and $\bar{h}_n\colon Y \to X$.

The reader can easily check that this set of maps defines a set of projections and twists.

Acknowledgments. The proof in this paper is based on ideas from [3].

References

1. J.-L. Loday, *Cyclic homology.* Springer-Verlag, 1992.
2. S. S. Podkorytov, *Commutative algebras and representations of the category of finite sets.* — Zap. Nauchn. Semin. POMI **388** (2011) 189–195.
3. S. S. Podkorytov, *On the algebra of the Möbius crown.* — Zap. Nauchn. Semin. POMI **492** (2020) 149–156.

St. Petersburg Department
of Steklov Institute of Mathematics,
Fontanka 27, St. Petersburg 191011, Russia
E-mail: baskovigor@pdmi.ras.ru

Поступило 15 декабря 2025 г.