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O. V. Postnova, N. Yu. Reshetikhin, V. V. Serganova

CHARACTER MEASURES IN LARGE TENSOR PRODUCTS OF REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

Abstract. We describe the asymptotics of the character measure on irreducible components in large tensor products of the finite dimensional representations of simple Lie algebras with a non-generic argument of characters.

**Dedicated to N. M. Bogoliubov
on the occasion of his 75th birthday**

§1. Introduction

1.1. An overview and the setup. Asymptotical behavior of measures on irreducible components in large tensor powers has been studied before. For example, in [1] the statistics of irreducible components of large tensor powers of the vector representation of sl_n with respect to the Plancherel measure was considered. The Plancherel statistics on irreducible components of representations of S_N for large N was described in [2–8]. The asymptotic of multiplicities of irreducible representations in large tensor products of irreducible finite dimensional representations of a finite dimensional simple Lie algebras was described in [9]. Essentially, this gives the asymptotic of the Plancherel measure on irreducible components of such representations.

Key words and phrases: Lie algebras, characters, irreducible representations, tensor power decomposition.

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This paper is the continuation of [10] where first two authors described the asymptotic of the character measure when the argument of characters lies strictly inside the principal Weyl chamber. We use some of the results of [11] where the asymptotic of multiplicities in such tensor products were computed for all values of the highest weights, i.e., strictly inside the principal Weyl chamber and when it lies on one of its walls.

1.1.1. *Notations.* Let $\mathfrak{g}$ be a simple Lie algebra. Choose a Borel subalgebra $\mathfrak{b} \in \mathfrak{g}$ and let $\mathfrak{h}$ be the corresponding Cartan subalgebra. Let Δ be the root system of $\mathfrak{g}$ and $\Delta_+ \subset \Delta$ be subset of positive roots. Denote by $\alpha_1, \ldots, \alpha_r \in \Delta_+$ enumerated fundamental roots. Here $r = \mathrm{rank}(\mathfrak{g}) = \dim(\mathfrak{h})$ is the rank of the Lie algebra $\mathfrak{g}$.

The Killing form on $\mathfrak{g}$ induces the nondegenerate bilinear form on its Cartan subalgebra. If $x = \sum_{a=1}^r x_a \alpha_a$ where α_a is the basis of simple roots, then $(x, y) = \sum_{a,b=1}^r x_a B_{ab} y_b$ where $B_{ab} = (\alpha_a, \alpha_b) = d_a C_{ab}$ is the symmetrized Cartain matrix. Here C_{ab} is the Cartan matrix and d_a is the length of root α_a. From now on we will identify the Cartain subalgebra $\mathfrak{h}$ and it dual vector space $\mathfrak{h}^*$ using the Killing form. Define the principal Weyl chamber $\mathfrak{h}_+ \subset \mathfrak{h}$ as the set of elements $x \in \mathfrak{h}$ with $(x, \alpha) \geqslant 0$ for all $\alpha \in \Delta_+$. In the interior of $\mathfrak{h}_+$, $(x, \alpha) > 0$ for all positive α on the walls some of $(x, \alpha) = 0$.

Fix an element $t \in \mathfrak{h}_+$ and let $\mathfrak{g}_0$ be a Lie subalgebra with the root system Δ_0 which is the subset of roots of $\mathfrak{g}$ that are orthogonal to t with respect to the Killing form. If t is in the interior of $\mathfrak{h}_+$, $\mathfrak{g}_0 = \{0\}$, if $t = 0$, then $\mathfrak{g}_0 = \mathfrak{g}$. Positive roots of $\mathfrak{g}$ decompose into the disjoint union $\Delta^+ = \Delta_0^+ \sqcup \Delta_1^+$ where $(\alpha, t) = 0$ when $\alpha \in \Delta_0^+$ and $(\alpha, t) > 0$ when $\alpha \in \Delta_1^+$. The Lie algebra $\mathfrak{g}$ decomposes as a vector space as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_1$ is direct sum of root subspaces for roots from Δ_1^+, $-\Delta_1^+$ and of the corresponding subspace of the Cartan subalgebra of $\mathfrak{g}$. Let r_0 be the rank of $\mathfrak{g}_0$. We will use notation $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$ for the half of the sum of positive roots of $\mathfrak{g}_0$ and $\rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha$.

1.1.2. *The character measure.* Let V_λ be the finite dimensional irreducible representations of simple Lie algebra $\mathfrak{g}$ with the highest weight λ and N be a positive integer. Any finite dimensional representation of a simple Lie algebra is completely reducible and therefore:

$$V_\nu^{\otimes N} \cong \bigoplus_\lambda W_\lambda(\nu, N) \otimes V_\lambda, \tag{1}$$

where the sum is taken over irreducible components of the tensor product, V_λ is the irreducible finite dimensional $\mathfrak{g}$-module with the highest weight λ and $W_\lambda(\nu, N)$ is the "space of multiplicities":

$$W_\lambda(\nu, N) \simeq \mathrm{Hom}_{\mathfrak{g}}(V_\nu^{\otimes N}, V_\lambda).$$

Its dimension $m_\lambda(\nu, N)$ is the multiplicity of V_λ in the tensor product and we have isomorphism of vector spaces. Let $D(\mu) \in \mathfrak{h}^*$ be the convex hull of the orbit of weight μ under the action of the Weyl group W. Highest weights of irreducible components in $V_\nu^{\otimes N}$ lie in $D(N\nu)$ and have the form $N\nu - \sum_{a=1}^r n_a\alpha_a$ where n_a are integers.

We will use notations $\chi_\lambda(g) = \mathrm{tr}_{V_\lambda(g)}$ for characters of irreducible representations.

Let $\mathfrak{g}_\mathbb{R}$ be the split real form of $\mathfrak{g}$ and $\mathfrak{h}_\mathbb{R}$ be its Cartan subalgebra. When $g = e^t$, $t \in \mathfrak{h}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ the characters $\chi_\nu(e^t)$ and $\chi_\lambda(e^t)$ are positive and as a consequence of the tensor product decomposition we have the identity

$$\chi_\nu(e^t)^N = \sum_\lambda m_\lambda(\nu, N)\chi_\lambda(e^t).$$

Therefore

$$p_\lambda^N(t) = \frac{m_\lambda(\nu, N)\chi_\lambda(e^t)}{\chi_\nu(e^t)^N} \tag{2}$$

is a natural probability measure on irreducible components of tensor product: $p_\lambda^{(N)}(t) \geqslant 0$ and $\sum_\lambda p_\lambda^{(N)}(t) = 1$. We will call it *the character measure.*

We will also assume that components of t in the basis $H_a, a = 1, \dots r$ in the Cartan subalgebra corresponding to simple roots are nonnegative, i.e., $t \geqslant 0$. Our goal is to describe this measure in the limit when $N \to \infty$ and ν and t remain finite.

In this limit the lattice region $D(N\nu)$ in the weight lattice rescaled by N becomes region $D(\nu)$ in the Euclidean space $\subset \mathbb{R}^r$. When t is strictly inside the principal Weyl chamber[1], i.e., when $t_a > 0$ for all $a = 1, \dots, r$ the limit of the probability measure was computed in [10]. Here we describe the limit of the probability measure (2) when t is on a wall of the main Weyl chamber, i.e., when for some a, $t_a = 0$.

[1]The principal Weyl chamber is in the dual space of the Cartan subalgebra, but we will apply the same term to the Cartan subalgebra using the Killing form

1.2. The main result. Consider an irreducible highest weight $\mathfrak{g}$-module V_ν as a $\mathfrak{g}_0$-module. It decomposes into the direct sum of $\mathfrak{g}_0$-irreducible modules $V_\mu^{\mathfrak{g}_0}$:

$$V_\nu|_{\mathfrak{g}_0} \simeq \oplus_\mu W_\mu^\nu \otimes V_\mu^{\mathfrak{g}_0}.$$

Here the $\dim(W_\mu^\nu)$ is the multiplicity of $V_\mu^{\mathfrak{g}_0}$ in V_ν. Define

$$\kappa = \frac{\sum_\mu \mathrm{tr}_{W_\mu^\nu}(e^t) c_2^{\mathfrak{g}_0}(\mu) \dim(V_\mu^{\mathfrak{g}_0})}{\dim(\mathfrak{g}^t) \sum_\mu \mathrm{tr}_{W_\mu}(e^t) \dim(V_\mu^{\mathfrak{g}_0})}, \tag{3}$$

where $c_2^{\mathfrak{g}_0}(\mu)$ is the value of the Casimir element for $\mathfrak{g}_0$ on $V_\mu^{\mathfrak{g}_0}$

As we explain in Section 3.4, the leading contribution to the character measure in the limit when $N \to \infty$ with t and ν being fixed and t is on a wall of the principal Weyl chamber comes from the region where

$$\lambda = N\eta + \sqrt{N\kappa}a + \sqrt{N}b.$$

Here $\eta = \eta(t, \nu)$ is defined in Section 3.4, b lies on the wall of the principal Weyl chamber where t is, i.e., $(\alpha, b) = 0$ for all $\alpha \in \Delta_1$, and a is strictly inside the principal Weyl chamber, i.e., $(\alpha, a) > 0$ for all $\alpha \in \Delta_0$.

Theorem 1. *As $N \to \infty$ the character measure weakly converges to the probability measure on $\mathbb{R}_{\geqslant 0}^{r_0} \times \mathbb{R}^{r-r_0}$ with density function*

$$\mathcal{P}(a, b, t) = \frac{\sqrt{\det K_1}}{(2\pi)^{\frac{r-r_0}{2}}} e^{-\frac{1}{2}(b, K_1(\eta) b)} \frac{\sqrt{\det B_0}}{(2\pi)^{\frac{r_0}{2}}} \prod_{\alpha \in \Delta_0^+} \frac{(a, \alpha)^2}{(\rho_0, \alpha)} e^{-\frac{1}{2}(a,a)_0}.$$

The matrix K_1 is described in Section 3.3, B_0 is the symmetrized Cartan matrix of $\mathfrak{g}_0$ and $(a, b)_0$ is the Killing form on $\mathfrak{g}_0$, restricted to $\mathfrak{h}_0$.

The key element in proving this theorem is the asymptotical behavior of the multiplicity $m_\lambda(\nu, N)$ in the limit when $N \to \infty$, ν is fixed and λ/N is finite. It was found in [9], see also [11].

The weak convergence of measures in the context of Theorem 1 means the following. For any bounded continuous function h on $\mathbb{R}_{\geqslant 0}^{r_0} \times \mathbb{R}^{r-r_0}$, and an increasing sequence of natural numbers N_n we have

$$\sum_{\lambda \in D} h(\sqrt{\kappa/N_n}\lambda, \sqrt{1/N_n}(\lambda - N_n\eta)) p_\lambda^{(N_n)}(t)$$
$$\to \int_{\mathbb{R}_{\geqslant 0}^{r_0}} \int_{\mathbb{R}^{r-r_0}} h(a, b) \mathcal{P}(a, b, t) da_1 \dots da_{r_0} db_1 \dots db_{r-r_0}. \tag{4}$$

Here we assume that ν and t are constant as $n \to \infty$.

1.3. The paper is organized as follows. This paper is a research announcement. The details of proofs are left to a follow up article.

§2. The multiplicity $m_\lambda(\nu, N)$ for large N

Here we give an overview of [9–11].

2.1. Character estimates. Consider the character of the irreducible $\mathfrak{g}$-module V_ν:

$$\chi_\nu(g) = \chi_\nu(e^t) = \sum_{\beta \in Q(\nu)} c(\nu;\beta) e^{(\beta,t)},$$

where $t \in \mathfrak{h}$ and $c(\nu;\beta)$ are the multiplicities of the weights $\beta \in \mathfrak{h}^*$ of V_ν. $Q(\nu) \subset \mathfrak{h}^*$ is the convex hull of the Weyl orbit of ν. The following lemma was proven in [9], see also [11]:

Lemma 1. *For any fixed $x \in X$ the following inequalities hold:*

$$|\chi_\nu(e^{x+i\theta})| \leqslant |\chi_\nu(e^x)|.$$

The number of points for which both expressions are equal is $\det C_{ab}$.

2.2. Integral formula for multiplicities. The character of tensor product of modules can be expressed as a sum of characters of irreducibles as

$$\chi_\nu(e^t)^N = \sum_\lambda m_\lambda(\nu, N)\chi_\lambda(e^t),$$

and due to the orthogonality of characters and the Weyl integration formula the multiplicities $m_\lambda(\nu, N)$ could be computed as the following integral:

$$\begin{aligned} m_\lambda(\nu, N) &= \int_G \chi_\nu(g)^N \overline{\chi_\lambda(g)} dg \\ &= \frac{1}{|W|} \int_T \chi_\nu(e^{i\theta})^N \chi_\lambda(e^{i\theta}) |\Delta(e^{i\theta})|^2 d\theta_1 \dots d\theta_r, \end{aligned}$$

where

$$\Delta(e^{i\theta}) = e^{i(\theta,\rho)} \prod_{\alpha \in \Delta^+} (1 - e^{-i(\theta,\alpha)})$$

is the denominator of the Weyl character formula

$$\chi_\lambda(e^{i\theta}) = \frac{\overline{\Phi_\lambda(e^{i\theta})}}{\overline{\Delta(e^{i\theta})}},$$

and

$$\Phi_\lambda(e^{i\theta}) = \sum_w \sigma(w) e^{i(\theta, w(\lambda+\rho))},$$

ρ is a half sum of positive roots, $|W|$ is the order of the Weyl group.

Remark 1. The characters are orthogonal on T:

$$\int_T \chi_\nu(e^{i\theta})\overline{\chi_\lambda(e^{i\theta})}|\Delta(e^{i\theta})|^2 d\theta_1 \dots d\theta_r = \int_T \Phi_\nu(e^{i\theta})\overline{\Phi_\lambda(e^{i\theta})} d\theta_1 \dots d\theta_r$$
$$= \int_T \sum_w \sigma(w) e^{i(\theta, w(\nu+\rho))} \overline{\sum_u \sigma(u) e^{i(\theta, u(\lambda+\rho))}} d\theta_1 \dots d\theta_r$$
$$= \int_T \sum_{u,w} \sigma(w)\sigma(u) e^{i(\theta, w(\nu+\rho)) - (\theta, u(\lambda+\rho))} d\theta_1 \dots d\theta_r = \delta_{\nu\lambda}|T||W|.$$

since the only nontrivial contribution to this sum is when $u = w$ and $\nu = \lambda$.

Thus, for the multiplicity we have the following integral formula:

$$m_\lambda(\nu, N) = \frac{1}{|T|}\int_T \left(\chi_\nu(e^{i\theta})\right)^N e^{-i(\theta, \lambda+\rho)} \Delta(e^{i\theta}) d\theta_1 \dots d\theta_r.$$

The function $\chi(z) = \chi(e^{x+i\theta})$ is a holomorphic function on z, therefore we can deform the contour of the integral from the unit circle $z = e^{i\theta}$ to $e^{x+i\theta}$:

$$m_\lambda(\nu, N) = \frac{1}{|T|}\int_T \left(\chi_\nu(e^{x+i\theta})\right)^N e^{-(x+i\theta, \lambda+\rho)} \Delta(e^{x+i\theta}) d\theta$$
$$= \frac{1}{|T|}\int_T e^{N\left(\ln\left(\chi_\nu(e^{x+i\theta})\right) - (x+i\theta, \xi)\right)} e^{-(x+i\theta, \rho)} \Delta(e^{i\theta}) d\theta, \qquad (5)$$

where $d\theta = d\theta_1 \dots d\theta_r$ and $\xi = \lambda/N$.

2.3. The steepest descent computation of the asymptotic. Now by a suitable deformation of the integration contour we can evaluate the integral (5) in the limit $N \to \infty$ when $\xi = \lambda/N$ and ν are fixed. For this we need critical points of

$$\ln(\chi_\nu(e^z)) - (z, \xi). \qquad (6)$$

As it is shown in [11] Lemma 1 implies that the leading contribution comes from real critical points. Because the character $\chi_\nu(e^x)$ is a strictly convex

, there exists unique such critical point x_{cr} in the principal Weyl chamber $\mathfrak{h}_+$. It is the unique solution in $\mathfrak{h}_+$ to the equations

$$\frac{\partial \ln(\chi_\nu(e^x))}{\partial x_a} = \sum_b B_{ab}\xi_b.$$

In the vicinity of this point

$$\ln(\chi_\nu(e^x)) - (x,\xi) = S(\xi) + \frac{1}{2}\sum_{a,b=1}^{r} D_{ab}(x_{cr})y_a y_b + O(y^3),$$

where $y_a = (x - x_{cr})_a$, $S(\xi) = \ln(\chi_\nu(e^{x_{cr}})) - (x_{cr},\xi)$ is the critical value of function (6), and

$$D_{ab}(x) = \frac{\partial^2}{\partial x_a \partial x_b}\ln(\chi_\nu(e^x)).$$

Note that because $\chi_\nu(e^x)$ is a convex function of x, $S(\xi)$ is also the Legendre transform of $\chi_\nu(e^x)$:

$$S(\xi) = \min_x \left(\ln(\chi_\nu(e^x)) - (x,\xi)\right) = \ln(\chi_\nu(e^{x_{cr}})) - (x_{cr},\xi).$$

After the steepest descent analysis we obtain the asymptotical behavior of multiplicities for generic values of ξ [9, 11]:

$$m_\lambda(\nu,N) = N^{-\frac{r}{2}}\frac{\Delta(x_{cr})}{(2\pi)^{\frac{r}{2}}\sqrt{\det D(x_{cr})}}e^{-(\rho,x_{cr})}e^{NS(\xi)}\Big(1 + O\Big(\frac{1}{N}\Big)\Big).$$

§3. The asymptotic of the character measure

3.1. The characters $\chi_\lambda(e^t)$. Let t be on the wall of the principal Weyl chamber, $\mathfrak{g}_0 \subset \mathfrak{g}$ is a Lie subalgebra with roots orthogonal to t. We will use notations of section 1.2.

Let Δ_0^+ be the subset of positive roots of $\mathfrak{g}_0$. Denote by $V_\lambda^{\mathfrak{g}_0}$ the irreducible finite dimensional $\mathfrak{g}_0$ module with highest weight λ. Let $\lambda \in \mathfrak{g}^*$ be a highest weight for the Lie algebra $\mathfrak{g}$. Denote by λ_0 its projection to $\mathfrak{g}_0^*$. Note that if t is an a wall of the principal Weyl chamber, $(\lambda,t) = (\lambda_0,t)$.

For $t \geqslant 0$[2] the leading term in $\chi_\lambda(e^t)$ when $\lambda \to \infty$ can be computed as follows:

$$\chi_\lambda(e^t) = e^{(\lambda_0,t)}\dim(V_{\lambda_0}^{\mathfrak{g}_0}) + \sum_{\substack{n_i \geqslant 0,\\ \alpha_i \in \Delta_1^+}} \dim(V_{\lambda_0}^{\mathfrak{g}_0})e^{\left(\lambda - \sum_i n_i\alpha_i, t\right)} + \dots$$

[2]This means $(\alpha,t) \geqslant 0$ for each $\alpha \in \Delta^+$.

$$= e^{(\lambda_0,t)} \frac{\dim(V_{\lambda_0}^{\mathfrak{g}_0})}{\prod\limits_{\alpha_i\in\Delta_1^+}(1-e^{-(\alpha,t)})}(1+o(1))$$

$$= e^{(\lambda_0,t)+(\rho_1,t)} \frac{\dim(V_{\lambda_0}^{\mathfrak{g}_0})}{\prod\limits_{\alpha_i\in\Delta_1^+}\left(e^{\frac{(\alpha,t)}{2}}-e^{-\frac{(\alpha,t)}{2}}\right)}(1+o(1))$$

$$= e^{(\lambda,t)+(\rho_1,t)} \prod_{\alpha_i\in\Delta_0^+}\frac{(\lambda+\rho_0,\alpha)}{(\rho_0,\alpha)}\frac{1}{\prod\limits_{\alpha_i\in\Delta_1^+}\left(e^{\frac{(\alpha,t)}{2}}-e^{-\frac{(\alpha,t)}{2}}\right)}(1+o(1)).$$

Thus, when $\lambda = N\xi$, $N\to\infty$, the leading part of the asymptotics is

$$\chi_{N\xi}(e^t) = e^{N(\xi,t)+(\rho_1,t)}N^{|\Delta_+^0|} \times \prod_{\alpha_i\in\Delta_0^+}\frac{(\xi,\alpha)}{(\rho_0,\alpha)}\frac{1}{\prod\limits_{\alpha_i\in\Delta_1^+}\left(e^{\frac{(\alpha,t)}{2}}-e^{-\frac{(\alpha,t)}{2}}\right)}(1+o(1)).$$

3.2. The matrix $D(y)$ when y is on a wall of the principal Weyl chamber. Denote by $f(y)=\ln\chi_\nu(e^y)$. Let us repeat some steps from [11] and compute the matrix of second derivatives of $f(y)$:

$$D_{ab}(y) = \frac{\partial^2 f}{\partial y_a\partial y_b} = \frac{\partial}{\partial y_a}\frac{\frac{\partial}{\partial y_b}\mathrm{tr}_{V_\nu}(e^y)}{\mathrm{tr}_{V_\nu}(e^y)}$$
$$= \frac{\frac{\partial^2}{\partial y_a\partial y_b}\left(\mathrm{tr}_{V_\nu}(e^y)\right)\mathrm{tr}_{V_\nu}(e^y) - \frac{\partial}{\partial y_a}\left(\mathrm{tr}_{V_\nu}(e^y)\right)\frac{\partial}{\partial y_b}\left(\mathrm{tr}_{V_\nu}(e^y)\right)}{\left(\mathrm{tr}_{V_\nu}(e^y)\right)^2}$$
$$= \frac{\mathrm{tr}_{V_\nu}(H_aH_be^y)\,\mathrm{tr}_{V_\nu}(e^y) - \mathrm{tr}_{V_\nu}(H_ae^y)\,\mathrm{tr}_{V_\nu}(H_be^y)}{\left(\mathrm{tr}_{V_\nu}(e^y)\right)^2}.$$

The irreducible $\mathfrak{g}$-module V_ν, being restricted to $\mathfrak{g}_0$ decomposes as

$$V_\nu \simeq \oplus_\mu W_\mu^\nu\otimes V_\mu^{\mathfrak{g}_0}$$

where $V_\mu^{\mathfrak{g}_0}$ are irreducible representations of $\mathfrak{g}_0$, where the lower index denotes the highest weight. Denote by α_a simple roots of $\mathfrak{g}_0$ and by α_i simple roots corresponding to simple root subspaces of $\mathfrak{g}_1$ and let us compute matrix elements of $D(y)$ when y belongs to the same wall of P_+ as t.

- When α_s, α_v are simple roots of $\mathfrak{g}_0$ we have $(\alpha_v, y) = 0$ and $(\alpha_s, y) = 0$ and we can simplify the factor

$$\mathrm{tr}_{V_\nu}\,(H_s H_v e^y) = \sum_\mu \mathrm{tr}_{W_\mu^\nu}(e^y)\mathrm{tr}_{V_\mu^{\mathfrak{g}_0}}(H_s H_v).$$

One can note that in $H_a, H_b \in \mathfrak{h}_0 \subset \mathfrak{g}_0$ and $V_\mu^{\mathfrak{g}_0}$ is a $\mathfrak{g}_0$-module, so we have:

$$\mathrm{tr}_{V_\mu^{\mathfrak{g}_0}}(H_s H_v) = \frac{c_2(\mathfrak{g}_0, \mu)\dim(V_\mu^{\mathfrak{g}_0})}{\dim(\mathfrak{g}_0)} B_{sv}^{(0)}.$$

Taking into account the identity

$$\mathrm{tr}_{V_\nu}\,(e^y) = \sum_\mu \mathrm{tr}_{W_\mu^\nu}(e^y)\dim(V_\mu^{\mathfrak{g}_0})$$

and

$$\mathrm{tr}_{V_\nu}\,(H_s e^y) = \sum_\mu \mathrm{tr}_{W_\mu^\nu}(e^y)\mathrm{tr}_{V_\mu^{\mathfrak{g}_0}}(H_s) = 0$$

we obtain

$$\frac{\partial^2 f(y)}{\partial y_s \partial y_v} = \kappa B_{sv}^{(0)},$$

where κ is given by (3). Note that when $y = 0$, i.e., $\mathfrak{g}_0 = \mathfrak{g}$ this expression coincides with the one obtained for Plancherel measure

$$\frac{\partial^2 f(y)}{\partial y_a \partial y_b}|_{y=0} = \frac{c_2(\nu)}{\dim(\mathfrak{g})} B_{ab}.$$

- Now assume that α_s is a simple root for $\mathfrak{g}_0$, i.e., $(\alpha_a, y) = 0$ and α_i is the root system of $\mathfrak{g}_1$, i.e., $(\alpha_i, y) > 0$. In this case we have:

$$\mathrm{tr}_{V_\nu}\,(H_s H_i e^y) = \sum_\mu \mathrm{tr}_{W_\mu^\nu}(H_i e^y)\mathrm{tr}_{V_\mu^{\mathfrak{g}_0}}(H_a) = 0$$

because

$$\mathrm{tr}_{V_\nu^{\mathfrak{g}_0}}(H_s) = 0.$$

Thus, we have

$$\frac{\partial^2 f(y)}{\partial y_a \partial y_i} = 0.$$

Thus the matrix of second derivatives of $f(y)$ at $y = t$ has the following block form

$$D(y) = \left[\begin{array}{c|c} \kappa B_0 & 0 \\ \hline 0 & D_1(y) \end{array}\right],$$

where κ is given by (3) and

$$D_1(y)_{ij} = \frac{\partial^2 f(y)}{\partial y_i \partial y_j}.$$

3.3. Pointwise asymptotic of the density of the character measure. Taking into account the asymptotic of the multiplicity and the asymptotic of the characters when $N \to \infty$ and $\xi = \lambda/N$, ν and t are fixed, we obtain the following pointwise asymptotic of the probability $p_\lambda^{(N)}(t)$ assuming ξ is generic:

$$p_\lambda^{(N)}(t) = N^{-\frac{r}{2}} \frac{\Delta(e^{x_{cr}})}{(2\pi)^{\frac{r}{2}} \sqrt{\det D(x_{cr})}} e^{-(\rho, x_{cr})} e^{N(S(\xi) - \ln \chi_\nu(e^t))} \tag{7}$$

$$\times\, e^{N(\xi,t)) + (\rho_1, t)} N^{|\Delta_0^+|} \prod_{\alpha \in \Delta_0^+} \frac{(\xi, \alpha)}{(\rho_0, \alpha)} \frac{1}{\prod\limits_{\alpha \in \Delta_1^+} \left(e^{\frac{(\alpha,t)}{2}} - e^{-\frac{(\alpha,t)}{2}}\right)} (1 + o(1))$$

$$= N^{-\frac{r}{2} + |\Delta_0^+|} \frac{\Delta(e^{x_{cr}})}{(2\pi)^{\frac{r}{2}} \sqrt{\det D(x_{cr})}} \prod_{\alpha_i \in \Delta_0^+} \frac{(\xi, \alpha)}{(\rho_0, \alpha)} \frac{1}{\prod\limits_{\alpha_i \in \Delta_1^+} \left(e^{\frac{(\alpha,t)}{2}} - e^{-\frac{(\alpha,t)}{2}}\right)}$$

$$\times\, e^{-(\rho, x_{cr}) + (\rho_1, t)} e^{N\widetilde{S}(\tau, \xi)} (1 + o(1)),$$

where

$$\widetilde{S}(\xi) = S(\xi) - \ln \chi_\nu(e^t) + (t, \xi).$$

From this we see that the character measure concentrates in a vicinity of the maximum of $\widetilde{S}(\xi)$.

3.4. The function $\widetilde{S}(\xi)$ near its relevant points. Because $-S(\xi)$ is the Legendre transform of a strictly convex function, it is strictly convex, so is $-\widetilde{S}(\xi)$, as it follows from its definition. Let us find its minimum. We have

$$\frac{\partial \widetilde{S}(\xi)}{\partial \xi_a} = \sum_b \frac{\partial (x_{cr})_b}{\partial \xi_a} \left(\frac{\partial \chi_\nu(e^x)}{\partial x_b}|_{x = x_{cr}} - \xi_b \right) + (-(x_{cr})_a + t_a).$$

The first term vanishes because x_{cr} is the critical point of (6). Thus, the critical point $\eta(t, \nu)$ is the unique globally defined (because of the convexity of (6)) solution of

$$x_{cr}(\xi, t, \nu) = t.$$

This point is also the global maximum of $\widetilde{S}(\xi)$.

Now let us consider the function $\widetilde{S}(\xi)$ near its critical point η:

$$\widetilde{S}(\xi) = \widetilde{S}(\eta) + \frac{1}{2}\sum_{a,b} \frac{\partial^2 \widetilde{S}(\xi)}{\partial \xi_a \partial \xi_b}(\xi-\eta)_a(\xi-\eta)_b + O((\xi-\eta)^3).$$

First, let us compute the critical value:

$$\begin{aligned}\widetilde{S}(\eta) &= S(\xi) - \ln \chi_\nu(e^t) + (t,\xi)\\ &= \chi_\nu(e^{x_{cr}}) - (x_{cr},\eta) - \ln \chi_\nu(e^t) + (t,\eta).\end{aligned}$$

This expression is zero because $x_{cr} = t$.

Define

$$K_{ab} = -\frac{\partial^2 S(\xi)}{\partial \xi_a \partial \xi_b}|_\eta.$$

It can be written as

$$K_{ac} = \sum_{b,d} B_{ad}(D(x_{cr})^{-1})_{db} B_{bc},$$

where $B_{ab} = d_a C_{ab}$ is the symmetrized Cartan matrix.

Thus, in the vicinity of $\xi = \eta$ we have:

$$\widetilde{S}(\xi) = -\frac{1}{2} K_{a,b}(\xi-\eta)_a(\xi-\eta)_b + O((\xi-\eta)^3)).$$

If t is on a wall of the principal Weyl chamber $D(x_{cr})$ has the block structure described in Section 3.2. As a consequence, in this case the matrix K has the block structure

$$K = \left[\begin{array}{c|c} \kappa^{-1}B_0 & 0 \\ \hline 0 & K_1 \end{array}\right].$$

3.5. The weak limit of the character measure. The function $\widetilde{S}(\xi)$ has maximum at η which is the Legendre image of t. If t is on a wall of the positive Weyl chamber, its Legendre image is on the same wall. Let us compute $\widetilde{S}(\xi)$ in a neighborhood of the critical point η.

Based the block diagonal structure of K we can treat the problem on the subspaces and note that on $\mathfrak{h}_0$-subspace the matrix of second derivatives is equal to xB_{ab} and this is equivalent for the one for Plancherel measure. On $\mathfrak{h}_1$-subspace the problem is equivalent to the general case when t is regular. Therefore we will admit the following rescale

$$\lambda = N\eta + \sqrt{\kappa N}a + \sqrt{N}b, \qquad (8)$$

where $a \in \mathfrak{h}_0$ and $b \in \mathfrak{h}_1$, i.e., $(\alpha, a) = 0$ for $\alpha \in \Delta_1$, $(\alpha, b) = 0$ for $\alpha \in \Delta_0$. Because η lies on the same wall as t, it belongs to the $\mathfrak{h}_1$-subspace. Therefore, in $\mathfrak{h}_0$-subspace the fluctuations are centered around 0, whereas in $\mathfrak{h}_1$-subspace they will be centered around η.

When λ is in the asymptotical region (8) we have:

$$p_\lambda^N(t) = N^{-\frac{r}{2}} \frac{\sqrt{\det K_1}}{(2\pi)^{\frac{r-r_0}{2}}} e^{-\frac{1}{2}(b,K_1 b)} \frac{\sqrt{detB_0}}{(2\pi)^{\frac{r_0}{2}}} \prod_{\alpha\in\Delta_0^+} \frac{(a,\alpha)^2}{(\rho_0,\alpha)} e^{-\frac{1}{2}(a,a)_0}.$$

Taking the expectation value of a test function, as in (4), we obtain the weak convergence of the character measure and thus, the proof of Theorem 1.

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Euler International
Mathematical Institute,
St. Petersburg Department
of Steklov Mathematical Institute,
Fontanka 27, 191023 St. Petersburg, Russia
E-mail: postnova.olga@gmail.com

YMSC, Tsinghua University,
Beijing,& BIMSA, Beijing& Department of Mathematics
and Computer Sciences,
St. Petersburg University, Russia
E-mail: reshetik@math.berkeley.edu

Department of Mathematics,
University of California,
Berkeley, CA 94720, USA
E-mail: serganov@math.berkeley.edu