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FACTORIZATION OF THE SL(2|1) INVARIANT R-MATRIX

Abstract. This work presents a new approach to the construction of general solution of the Yang–Baxter equation with supersymmetry algebra. The factorizing R-operator acts in a tensor product of two arbitrary Verma module representations of superalgebra $\mathfrak{sl}(2|1)$. The main building blocks in construction are even and odd intertwining operators.

Dedicated to N. M. Bogoliubov on the occasion of his 75th birthday

§1. Introduction

This paper is devoted to the construction of the general solution of the graded Yang–Baxter relation [1]. To provide context, we recall that in supersymmetric Yang–Mills (SYM) theory, integrable spin chains with $\mathfrak{sl}(2|1)$ symmetry arise in the analysis of the anomalous dimension spectrum of composite operators [2,3]. The dilatation operator in $\mathcal{N} = 1$ SYM can be expressed as the logarithmic derivative of R-operator which is the general solution of the Yang–Baxter equation. The R-operator acting in the tensor product of two generic Verma module representations of superalgebra $\mathfrak{sl}(2|1)$ is obtained in the form of an integral operator in [4,5]. In [6–8] an equivalent factorizing representation for the R-operator is given.

A general construction of the factorized R-operators with the symmetry group $\mathrm{SL}(n, \mathbb{C})$ is worked out in [9]. The main building blocks in this construction are Knapp–Stein elementary intertwining operators [10,11] between equivalent representations of the group $\mathrm{SL}(n, \mathbb{C})$ [12]. Each elementary intertwining operator corresponds to the elementary reflection (defined by a simple root) in the Weyl group. For $\mathfrak{sl}(n)$ the Weyl group

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coincides with the symmetric group S_n, and an elementary reflection corresponds to an elementary transposition in the symmetric group.

In the case of superalgebra $\mathfrak{sl}(n|m)$, the root system contains even and odd roots [13, 14]. The appropriate analog of the Weyl group is Weyl groupoid [15, 16] and there appears two type of intertwining operators. Even intertwining operators correspond to reflections in even roots and they are analogues of the corresponding operators present in the case of $\mathfrak{sl}(n)$. The new phenomenon is an appearance of odd intertwining operators corresponding to the odd reflections [17]. In the present paper, we construct these intertwining operators in the simplest nontrivial case of the superalgebra $\mathfrak{sl}(2|1)$ and work out the construction of $\mathfrak{sl}(2|1)$-invariant factorized R-matrix in analogy with $\mathfrak{sl}(n)$.

In this way, we obtain the representation for the R-operator which is equivalent to the ones obtained in [6–8], and it is a strong crosscheck for the whole construction.

Beyond its computational utility, this approach elucidates key algebraic structures essential for the integrability. For example, in the case of $\mathrm{SL}(n, \mathbb{C})$, the fundamental relation for the intertwining operators is equivalent to the well-known star-triangle relation [9]. The natural analog of the star-triangle relation for the case of $\mathfrak{sl}(2|1)$ is given in the present work.

Recent developments in the study of integrability in quantum field theories, particularly within the context of dynamical fishnet models [18, 19], have been fundamentally based on a new class of relations known as generalized, or supersymmetric, star-triangle relations (SSTR) [20]. While the standard bosonic star-triangle relation is well-known to underpin the integrability, the authors of [18, 19] noted that an integrable spin chain whose integrability is governed solely by the SSTR still has not been identified. We find that the integrability of general supersymmetric spin chains is guaranteed by the combination of the SSTR and certain local fermionic relations, the latter having first appeared in [17]. For clarity and simplicity in presenting the underlying machinery, we focus our explicit constructions here on the case $\mathfrak{sl}(2|1)$.

The paper is organized as follows. In Section 2, we develop the theory of induced representations for the supergroup SL(2|1). A key distinction from the $\mathrm{SL}(n, \mathbb{C})$ case is that SL(2|1) possesses non-conjugate Borel subgroups [13,15]. We construct representations for a representative of each conjugacy class and provide a complete classification of their intertwining operators in the next section. A subsequent analysis of the even intertwiner, which corresponds to the Weyl reflection in an even root, leads directly to the SSTR.

In Section 3, we construct all needed intertwining operators.

In Section 4, we demonstrate a novel method for factorizing the $\mathfrak{sl}(2|1)$-invariant R-operator within the context of induced representations. We study the structure of intertwining operators identifying two additional intertwiners in the tensor product that permute odd and even elements between the tensor factors. From these elementary building blocks, we construct the operators $\mathcal{R}_i$. Direct computation verifies that our factorization formulas reproduce known results [7, 8].

§2. Representations of SL(2|1)

2.1. Lie superalgebras $\mathfrak{sl}(2|1)$ and $\mathfrak{gl}(2|1)$. We fix the grading of the superspace $\mathbb{C}^{2|1}$ as follows: $[1] = [3] = 0$ and $[2] = 1$. The Lie superalgebra $\mathfrak{gl}(2|1)$ is defined as a real Lie superalgebra by the commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-1)^{([i]+[j])([k]+[l])} \delta_{li} e_{kj}, \tag{2.1}$$

where the graded commutator is defined as

$$[e_{ij}, e_{kl}] = e_{ij}\, e_{kl} - (-1)^{([i]+[j])([k]+[l])}\, e_{kl}\, e_{ij}.$$

In the simplest three-dimensional representation e_{ij} are given by the standard matrix units

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl}. \tag{2.2}$$

The superalgebra $\mathfrak{sl}(2|1)$ is the subalgebra of $\mathfrak{gl}(2|1)$ consisting of elements with vanishing supertrace,

$$\mathfrak{sl}(2|1) = \{a \in \mathfrak{gl}(2|1) \mid \operatorname{str}(a) = a_{11} - a_{22} + a_{33} = 0\}.$$

In the simplest three-dimensional representation, it means the corresponding redefinition of generators e_{ij},

$$e_{ij} \longmapsto e_{ij} - (-1)^{[j]} \delta_{ij} I, \tag{2.3}$$

where I is the identity matrix. This redefinition changes the diagonal generators e_{ii} only. To avoid additional complications with notations, we will denote diagonal generators by the same symbols e_{ii} for both superalgebras $\mathfrak{gl}(2|1)$ and $\mathfrak{sl}(2|1)$, assuming substitution (2.3) in the case of $\mathfrak{sl}(2|1)$.

Both superalgebras admit a standard root decomposition,

$$\mathfrak{h} = \langle e_{ii} \rangle_{i=1}^{3}, \quad \mathfrak{n}_+ = \langle e_{ij} \rangle_{i<j}, \quad \mathfrak{n}_- = \langle e_{ij} \rangle_{i>j}, \tag{2.4}$$

where $\mathfrak{h}$ denotes the Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(2|1)$, or $\mathfrak{sl}(2|1)$. The standard Borel subalgebra is given by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$.

Lie superalgebras typically possess non-conjugate root systems. In what follows, we restrict our attention to the Borel subalgebras where the even part coincides with the standard even Borel subalgebra of the upper triangular matrices. According to the general theory [13, 15, 16], there exist precisely three non-conjugate Borel subalgebras sharing the same even part. These subalgebras are given explicitly as follows:

(1) The standard Borel subalgebra of upper-triangular matrices

$$\mathfrak{b} = \left\{ \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ 0 & h_{22} & h_{23} \\ 0 & 0 & h_{33} \end{pmatrix} \middle| \, h_{ij} \in \mathbb{C} \right\}$$

corresponds to the standard root system (2.4)

$$\mathfrak{h} = \langle e_{11}\,, e_{22}\,, e_{33} \rangle, \quad \mathfrak{n}_+ = \langle e_{12}\,, e_{13}\,, e_{23} \rangle, \quad \mathfrak{n}_- = \langle e_{21}\,, e_{31}\,, e_{32} \rangle.$$

(2) The Borel subalgebra

$$\mathfrak{b}^- = \left\{ \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ 0 & h_{22} & 0 \\ 0 & h_{32} & h_{33} \end{pmatrix} \middle| \, h_{ij} \in \mathbb{C} \right\} \tag{2.5}$$

corresponds to the root system

$$\mathfrak{h} = \langle e_{11}\,, e_{22}\,, e_{33} \rangle, \quad \mathfrak{n}_+ = \langle e_{12}\,, e_{13}\,, e_{32} \rangle, \quad \mathfrak{n}_- = \langle e_{21}\,, e_{31}\,, e_{23} \rangle. \tag{2.6}$$

(3) The Borel subalgebra

$$\mathfrak{b}^+ = \left\{ \begin{pmatrix} h_{11} & 0 & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & 0 & h_{33} \end{pmatrix} \middle| \, h_{ij} \in \mathbb{C} \right\} \tag{2.7}$$

corresponds to the root system

$$\mathfrak{h} = \langle e_{11}\,, e_{22}\,, e_{33} \rangle, \quad \mathfrak{n}_+ = \langle e_{21}\,, e_{13}\,, e_{23} \rangle, \quad \mathfrak{n}_- = \langle e_{12}\,, e_{31}\,, e_{32} \rangle. \tag{2.8}$$

2.2. Induced representations. We now recall the construction of induced representations [10, 21]. Let Z and B be the supergroups associated with the Lie superalgebras $\mathfrak{n}_-$ and $\mathfrak{b}$, respectively. For a generic $g \in \mathrm{GL}(2|1)$ and $z \in Z$, there exists an unique decomposition

$$g^{-1}\, z = z'\, h(z, g), \tag{2.9}$$

with $z' \in Z$ and $h \in B$. The induced representation is then defined by the action

$$[T^{\boldsymbol{\sigma}}(g)\Phi](z) = h(z, g; \boldsymbol{\sigma})\, \Phi(z'), \tag{2.10}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and in the detailed notation

$$h(z, g; \boldsymbol{\sigma}) = (h_{11}(z, g))^{\sigma_1}\ (h_{22}(z, g))^{\sigma_2}\ (h_{33}(z, g))^{\sigma_3}\,. \tag{2.11}$$

The matrix $z \in Z$ possesses a single non-trivial even element and two odd elements. General principles imply that the representation can be realized on functions defined on a superspace $\mathbb{C}^{2|1}$. We will consider Verma modules so that the representation is realized on polynomials which depend on one even and two odd variables.

For the case of SL(2|1), the representation depends on two complex parameters, instead of three, since

$$\mathrm{sdet}(h) = h_{11}\, h_{22}^{-1}\, h_{33} = 1.$$

We can fix the remaining freedom by imposing the linear constraint

$$\sigma_1 + \sigma_2 + \sigma_3 = 0, \tag{2.12}$$

or, explicitly,

$$\sigma_1 = -\ell - b, \quad \sigma_2 = 2b, \quad \sigma_3 = \ell - b.$$

We will denote these representations by $T^{\ell,b}$,

$$[T^{\ell,b}(g)\Phi](z) = h_{11}^{-\ell-b}\, h_{22}^{2b}\, h_{33}^{\ell-b}\, \Phi(z'). \tag{2.13}$$

The generators in representation $T^{\ell,b}$ are defined by the following formula:

$$[T^{\ell,b}(g)\Phi](z) = \Phi(z) + \varepsilon(-1)^{[i]([i]+[j])}\, [E_{ij}\Phi](z) + O(\varepsilon^2), \quad g = 1 + \varepsilon e_{ij}, \tag{2.14}$$

where the sign factor is introduced to ensure that the resulting generators satisfy the commutation relations (2.1).

Using the generators e_{ij} in three-dimensional representation (2.2) and the generators E_{ij} in the representation $T^{\ell,b}$, we construct the L-operator [1],

$$L(u) = u + \sum_{i,j=1}^{3} (-1)^{[j]} e_{ij} \otimes E_{ji}. \tag{2.15}$$

The nontrivial part of the L-operator essentially coincides with the quadratic Casimir operator for the tensor product of the three-dimensional representation and the representation $T^{\ell,b}$.

Remark 2.1. We should note that there is no difference in using the standard matrix units or the transformed ones (2.3) in the last formula. It is easy to check that the extra terms cancel,

$$L(u) = u + \sum_{i,j=1}^{3} (-1)^{[j]} \Big(e_{ij} - (-1)^{[j]} \delta_{ij} I \Big) \otimes E_{ji}$$
$$= u + \sum_{i,j=1}^{3} (-1)^{[j]} e_{ij} \otimes E_{ji} - I \otimes \sum_{i=1}^{3} E_{ii} = u + \sum_{i,j=1}^{3} (-1)^{[j]} e_{ij} \otimes E_{ji},$$

since $E_{11} + E_{22} + E_{33} = 0$ in the representation $T^{\ell,b}$.

Let us represent the operator under the consideration in the matrix form in the standard basis, with the grading $[1] = [3] = 0\,, [2] = 1$,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To calculate the matrix of an L-operator, we use the following definition of the matrix of an operator:

$$F e_i = \sum_{j=1}^{3} e_j F_{ji}.$$

We have

$$(e_{ij} E_{ji}) e_k = (-)^{([i]+[j])[k]}\, e_{ij}\, e_k\, E_{ji} = (-)^{([i]+[j])[k]} \sum_{l=1}^{3} e_l\, (e_{ij})_{lk}\, E_{ji},$$

so that

$$(e_{ij} E_{ji})_{lk} = (-)^{([i]+[j])[k]}\, (e_{ij})_{lk}\, E_{ji} = (-)^{([i]+[j])[k]}\, \delta_{il} \delta_{jk}\, E_{ji},$$

and finally

$$(L(u))_{lk} = u \delta_{lk} + (-)^{[l][k]}\, E_{kl}.$$

Or, in an explicit matrix form,

$$L(u) = u + \sum_{i,j=1}^{3} (-1)^{[j]} e_{ij} \otimes E_{ji} = \begin{pmatrix} u + E_{11} & E_{21} & E_{31} \\ E_{12} & u - E_{22} & E_{32} \\ E_{13} & E_{23} & u + E_{33} \end{pmatrix}. \quad (2.16)$$

Remark 2.2. To avoid any misunderstanding, we should note that in the construction of the L-operator, each term $e_{ij} \otimes E_{ji}$ is even or the bosonic operator by the definition. The element e_{ij} has grading $[i] + [j]$ and the generator E_{ji} has the same grading $[j] + [i]$, so that finally the grading of the element $e_{ij} \otimes E_{ji}$ is zero.

2.3. Explicit formulas for the superconformal transformations. Existence of non-conjugate Borel subalgebras make it natural to consider representations induced from these distinct Borel subgroups. The matrix $z \in Z$ possesses a single non-trivial even element and two odd elements. The general principles imply that the representations can be realized on functions defined on a superspace $\mathbb{C}^{2|1}$. We will consider Verma modules so that the representation is realised on polynomials depending on one even and two odd variables.

We now present explicit formulas for the superconformal transformations and corresponding L-operators for different choices of the Borel subalgebra. We will start from the representation $T^{\ell,b}$ induced from the subgroup of upper-triangular matrices. We parameterize the group element from Z in the following way:

$$z = \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & 0 \\ z + \frac{1}{2}\theta_1\theta_2 & \theta_1 & 1 \end{pmatrix}.$$

In these coordinates one obtains the formulae for group-like elements constructed for the Cartan subalgebra generators,

$$\begin{aligned} \lambda^{E_{11}}\,\Phi(z,\theta_1,\theta_2) &= \lambda^{\ell+b}\,\Phi(\lambda z,\theta_1,\lambda\theta_2), \\ \lambda^{E_{22}}\,\Phi(z,\theta_1,\theta_2) &= \lambda^{-2b}\,\Phi(z,\lambda\theta_1,\lambda^{-1}\theta_2), \\ \lambda^{E_{33}}\,\Phi(z,\theta_1,\theta_2) &= \lambda^{-\ell+b}\,\Phi\left(\lambda^{-1}z,\lambda^{-1}\theta_1,\theta_2\right), \end{aligned} \tag{2.17}$$

for the lowering $\mathfrak{n}_-$-generators,

$$\begin{aligned} e^{\lambda E_{31}}\Phi(z,\theta_1,\theta_2) &= \Phi(z-\lambda,\theta_1,\theta_2), \\ e^{\varepsilon E_{32}}\Phi(z,\theta_1,\theta_2) &= \Phi\left(z - \frac{\varepsilon\theta_2}{2},\theta_1-\varepsilon,\theta_2\right), \\ e^{-\varepsilon E_{21}}\Phi(z,\theta_1,\theta_2) &= \Phi\left(z + \frac{\varepsilon\theta_1}{2},\theta_1,\theta_2+\varepsilon\right), \end{aligned} \tag{2.18}$$

and for the raising $\mathfrak{n}_+$-generators,

$$e^{\varepsilon E_{12}}\Phi(z,\theta_1,\theta_2) = \frac{1}{[1-\varepsilon\theta_2]^{\ell-b}}\Phi\left(\frac{z}{1-\varepsilon\theta_2/2}, \frac{\theta_1+\varepsilon z}{1-\varepsilon\theta_2/2}, \theta_2\right),$$
$$e^{\lambda E_{13}}\Phi(z,\theta_1,\theta_2) = \frac{1}{[1-\lambda z]^{2\ell}}\left[1+\frac{\lambda\theta_1\theta_2}{1-\lambda z}\right]^{-b}\Phi\left(\frac{z}{1-\lambda z}, \frac{\theta_1}{1-\lambda z}, \frac{\theta_2}{1-\lambda z}\right),$$
$$e^{-\varepsilon E_{23}}\Phi(z,\theta_1,\theta_2) = \frac{1}{[1+\varepsilon\theta_1]^{\ell+b}}\Phi\left(\frac{z}{1+\varepsilon\theta_1/2}, \theta_1, \frac{\theta_2-\varepsilon z}{1+\varepsilon\theta_1/2}\right). \tag{2.19}$$

The formulas above reproduce the known results from [4, 5, 7]. After extraction of the explicit expressions for the generators and substitution inside $L(u)$ (2.16), one obtains the following factorized representation for the L-operator:

$$L(u) = L(u_1,u_2,u_3)$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & 0 \\ z+\frac{\theta_1\theta_2}{2} & \theta_1 & 1 \end{pmatrix} \begin{pmatrix} u_1 & \partial_{\theta_2}-\frac{1}{2}\theta_1\partial_z & -\partial_z \\ 0 & u_2-1 & -\partial_{\theta_1}+\frac{1}{2}\theta_2\partial_z \\ 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & 0 \\ z+\frac{\theta_1\theta_2}{2} & \theta_1 & 1 \end{pmatrix}^{-1},$$

where we have shown explicitly in the matrix $L(u)$ the dependence on all the parameters, and have used the following compact notation for the linear combinations of the spectral parameter u and the representation ℓ and b parameters:

$$u_1 = u+b+\ell, \quad u_2 = u+2b, \quad u_3 = u+b-\ell. \tag{2.20}$$

Remark 2.3. Note that, we consider the Verma module representations of the superalgebras $\mathfrak{sl}(2|1)$ and $\mathfrak{gl}(2|1)$ [4, 5, 7]. It means that all formulae like (2.17), (2.18), and (2.19) we consider as formulae for the group-like elements — the generating functions, which collect the action of arbitrary powers of the corresponding generators on the polynomials $\Phi(z,\theta_1,\theta_2)$.

We now provide explicit formulas for the action in the representation $T_-^{\ell,b}$, induced from a Borel subgroup B^-. We parameterize the group element as

$$z_- = \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & \theta_1 \\ z & 0 & 1 \end{pmatrix}.$$

The L-operator in this realization is given by

$$L^-(u_1,u_2,u_3) = \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & \theta_1 \\ z & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 & \partial_{\theta_2} & -\partial_z-\theta_1\partial_{\theta_2} \\ 0 & u_2 & 0 \\ 0 & \partial_{\theta_1} & u_3-1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & \theta_1 \\ z & 0 & 1 \end{pmatrix}^{-1}.$$

A global form of the lowering operators is:

$$\begin{aligned} e^{\lambda E_{31}}\Phi &= \Phi(z-\lambda,\theta_1,\theta_2), \\ e^{-\varepsilon E_{23}}\Phi &= \Phi(z,\theta_1+\varepsilon,\theta_2+\varepsilon z), \\ e^{-\varepsilon E_{21}}\Phi &= \Phi(z,\theta_1,\theta_2+\varepsilon). \end{aligned} \tag{2.21}$$

and there are corresponding formulae for the Cartan subalgebra operators,

$$\begin{aligned} \lambda^{E_{11}}\,\Phi &= \lambda^{l+b}\,\Phi(\lambda z,\theta_1,\lambda\theta_2), \\ \lambda^{E_{22}}\,\Phi &= \lambda^{-2b}\,\Phi(\lambda^{-1}z,\lambda\theta_1,\theta_2), \\ \lambda^{E_{33}}\,\Phi &= \lambda^{-l+b}\,\Phi(z,\lambda^{-1}\theta_1,\lambda^{-1}\theta_2), \end{aligned}$$

and for the raising operators,

$$\begin{aligned} e^{\lambda E_{13}}\Phi &= \frac{1}{[1-\lambda z]^{2\ell}}\,\Phi\left(\frac{z}{1-\lambda z},(1-\lambda z)\theta_1,\frac{\theta_2}{1-\lambda z}\right), \\ e^{\varepsilon E_{12}}\Phi &= \frac{1}{[1-\varepsilon\theta_2]^{\ell-b}[1+\varepsilon z\theta_1]^{l+b}}\,\Phi\left(\frac{z}{1-\varepsilon\theta_2},\frac{\theta_1}{1+\varepsilon\theta_2},\theta_2\right), \\ e^{\varepsilon E_{32}}\Phi &= \frac{1}{[1-\varepsilon\theta_1]^{\ell+b}}\,\Phi(z-\varepsilon\theta_2,\theta_1,\theta_2). \end{aligned} \tag{2.22}$$

Finally, we consider the representation $T_+^{\ell,b}$ induced from a Borel subgroup B^+. We parameterize the group element as

$$z_+ = \begin{pmatrix} 1 & \theta_2 & 0 \\ 0 & 1 & 0 \\ z & \theta_1 & 1 \end{pmatrix}.$$

The L-operator in this realization is given by

$$L^+(u_1,u_2,u_3) = \begin{pmatrix} 1 & \theta_2 & 0 \\ 0 & 1 & 0 \\ z & \theta_1 & 1 \end{pmatrix} \begin{pmatrix} u_1-1 & 0 & -\partial_z \\ -\partial_{\theta_2}-z\partial_{\theta_1} & u_2-2 & -\partial_{\theta_1} \\ 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} 1 & \theta_2 & 0 \\ 0 & 1 & 0 \\ z & \theta_1 & 1 \end{pmatrix}^{-1}.$$

The global form of the lowering operators is

$$\begin{aligned} e^{\varepsilon E_{12}}\Phi &= \Phi(z,\theta_1,\theta_2-\varepsilon), \\ e^{\lambda E_{31}}\Phi &= \Phi(z-\lambda,\theta_1-\lambda\theta_2,\theta_2), \\ e^{\varepsilon E_{32}}\Phi &= \Phi(z,\theta_1-\varepsilon,\theta_2), \end{aligned} \tag{2.23}$$

and there are corresponding formulae for the Cartan subalgebra operators,

$$\begin{aligned}\lambda^{E_{11}}\,\Phi &= \lambda^{\ell+b}\,\Phi(z,\lambda\theta_1,\lambda\theta_2),\\ \lambda^{E_{22}}\,\Phi &= \lambda^{-2b}\,\Phi(\lambda z,\theta_1,\lambda^{-1}\theta_2),\\ \lambda^{E_{33}}\,\Phi &= \lambda^{-\ell+b}\,\Phi(\lambda^{-1}z,\lambda^{-1}\theta_1,\theta_2),\end{aligned}$$

and for the raising operators,

$$\begin{aligned}e^{-\varepsilon E_{21}}\Phi &= \frac{1}{[1-\varepsilon\theta_2]^{\ell-b}}\,\Phi\left(\frac{z+\varepsilon\theta_1}{1+\varepsilon\theta_2},\frac{\theta_1}{1+\varepsilon\theta_2},\theta_2\right),\\ e^{-\varepsilon E_{23}}\Phi &= \frac{1}{[1-\varepsilon\theta_1]^{\ell+b}[1+\varepsilon z\theta_2]^{2\ell}}\,\Phi\left(\frac{z}{1+\varepsilon(\theta_1-z\theta_2)},\theta_1,\frac{\theta_2}{1+\varepsilon\theta_1}\right),\\ e^{\lambda E_{13}}\Phi &= \frac{1}{[1-\lambda z]^{2\ell}}\,\Phi\left(\frac{z}{1-\lambda z},\theta_1,\theta_2-\lambda\theta_1\right).\end{aligned} \tag{2.24}$$

§3. Intertwining operators

In this section, we consider the intertwining operators for the case under consideration. One of these operators generalizes the $\mathfrak{gl}(2)$ intertwiner, corresponding to the action of the Weyl group on the weights, while the others represent a new type of intertwiner corresponding to odd reflections. We refer to the latter as *odd intertwining operators* and begin our discussion focusing on them.

3.1. Local intertwining operators. First of all, introduce two operators which depend on the parameter a:

$$\mathcal{D}_{23}(a)=(\partial_{\theta_1}+a\theta_1)\,e^{-\frac{\theta_1\theta_2}{2}\partial_z};\quad \mathcal{D}_{12}(a)=e^{-z\theta_2\partial_{\theta_1}}\,(\partial_{\theta_2}+a\theta_2)\,e^{\frac{\theta_1\theta_2}{2}\partial_z}. \tag{3.1}$$

The inverse operators have the form:

$$\mathcal{D}_{23}^{-1}(a)=a^{-1}e^{\frac{\theta_1\theta_2}{2}\partial_z}\,(\partial_{\theta_1}+a\theta_1)\,;\quad \mathcal{D}_{12}^{-1}(a)=a^{-1}e^{-\frac{\theta_1\theta_2}{2}\partial_z}\,(\partial_{\theta_2}+a\theta_2)\,e^{z\theta_2\partial_{\theta_1}}. \tag{3.2}$$

Proposition 1. *For $a=-(\ell+b)=u_3-u_2$, the operator $\mathcal{D}_{23}(a)$ is an intertwining operator,*

$$\mathcal{D}_{23}:T^{\ell,b}\longrightarrow T_{-}^{\ell+1/2,b-1/2}.$$

The set of the intertwining relations can be expressed in a compact matrix form using the corresponding L-operators,

$$\mathcal{D}_{23}(u_3-u_2)\,L(u_1,u_2,u_3)=\,L^{-}(u_1,u_2-1,u_3+1)\,\mathcal{D}_{23}(u_3-u_2). \tag{3.3}$$

The relation for the inverse operator

$$\mathcal{D}_{23}^{-1} : T_{-}^{\ell,b} \longrightarrow T^{\ell-1/2,b+1/2}$$

has the following form:

$$\mathcal{D}_{23}^{-1}(u_3 - u_2 - 2)\, L^{-}(u_1, u_2, u_3) = \, L(u_1, u_2 + 1, u_3 - 1)\, \mathcal{D}_{23}^{-1}(u_3 - u_2 - 2).$$

Proof. The derivation of all the relations is based on simple formulae

$$(\partial_\theta + a\,\theta)(\partial_\theta + a\,\theta) = a\,,$$
$$(\partial_\theta + a\,\theta)\,\theta\,(\partial_\theta + a\,\theta) = \partial_\theta\,,$$
$$(\partial_\theta + a\,\theta)\,\partial_\theta\,(\partial_\theta + a\,\theta) = a^2\,\theta\,,$$

where θ is a grassmanian variable and $a \in \mathbb{C}$. Using these formulae, we obtain transformation rules for the building blocks of the matrix $L(u_1, u_2, u_3)$,

$$\mathcal{D}_{23}(a) \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & 0 \\ z + \frac{1}{2}\theta_1\theta_2 & \theta_1 & 1 \end{pmatrix} \mathcal{D}_{23}^{-1}(a) = \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & \theta_1 \\ z & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + a^{-1}\theta_1\partial_{\theta_1} & -\theta_1 \\ 0 & -a^{-1}\partial_{\theta_1} & 1 \end{pmatrix},$$

$$\mathcal{D}_{23}(a) \begin{pmatrix} u_1 & \partial_{\theta_2} - \frac{1}{2}\theta_1\partial_z & -\partial_z \\ 0 & u_2 - 1 & -\partial_{\theta_1} + \frac{1}{2}\theta_2\partial_z \\ 0 & 0 & u_3 \end{pmatrix} \mathcal{D}_{23}^{-1}(a) = \begin{pmatrix} u_1 & \partial_{\theta_2} + a^{-1}\partial_{\theta_1}\partial_z & -\partial_z \\ 0 & u_2 - 1 & a\,\theta_1 \\ 0 & 0 & u_3 \end{pmatrix},$$

for an arbitrary parameter a. After the substitution $a = -(\ell + b)$ and multiplication of all the corresponding matrices, one obtains

$$\mathcal{D}_{23}\, L(u_1, u_2, u_3)\, \mathcal{D}_{23}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & \theta_1 \\ z & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 & \partial_{\theta_2} & -\partial_z - \theta_1\partial_{\theta_2} \\ 0 & u_2 - 1 & 0 \\ 0 & \partial_{\theta_1} & u_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & \theta_1 \\ z & 0 & 1 \end{pmatrix}^{-1},$$

so that one can recognize the matrix $L^{-}(u_1, u_2 - 1, u_3 + 1)$. $\square$

Remark 3.1. To avoid any misunderstanding and to explain the absence of some sign factors, we illustrate our calculations by the simple example (see Remark 2.2)

$$\mathcal{D}_{23}(a)\begin{pmatrix}0&0&0\\ \theta_2&0&0\\ 0&0&0\end{pmatrix}\mathcal{D}_{23}^{-1}(a)=\mathcal{D}_{23}(a)\,e_{21}\otimes\theta_2\,\mathcal{D}_{23}^{-1}(a)$$

$$=-e_{21}\otimes\mathcal{D}_{23}(a)\,\theta_2\,\mathcal{D}_{23}^{-1}(a)=-e_{21}\otimes(-\theta_2)=e_{21}\otimes\theta_2=\begin{pmatrix}0&0&0\\ \theta_2&0&0\\ 0&0&0\end{pmatrix}.$$

Proposition 2. *For $a=\ell-b=u_1-u_2$ the operator $\mathcal{D}_{12}(a)$ is the intertwining operator,*

$$\mathcal{D}_{12}:T^{\ell,b}\longrightarrow T_{+}^{\ell-1/2,b+1/2}.$$

The set of the intertwining relations can be expressed in a compact matrix form using the corresponding L-operators,

$$\mathcal{D}_{12}(u_1-u_2)\,L(u_1,u_2,u_3)=\,L^{+}(u_1-1,u_2+1,u_3)\;\mathcal{D}_{12}(u_1-u_2).$$

The corresponding relation for the inverse operator

$$\mathcal{D}_{12}^{-1}:T_{+}^{\ell,b}\longrightarrow T^{\ell+1/2,b-1/2}\tag{3.4}$$

has the following form:

$$\mathcal{D}_{12}^{-1}(u_1-u_2+2)L^{+}(u_1,u_2,u_3)=L(u_1+1,u_2-1,u_3)\mathcal{D}_{12}^{-1}(u_1-u_2+2).$$

Proof. All calculations are based on the same simple basic formulae and after calculation of the product of the transformed matrices, one obtains

$$\mathcal{D}_{12}\,L(u_1,u_2,u_3)\,\mathcal{D}_{12}^{-1}$$
$$=\begin{pmatrix}1&\theta_2&0\\ 0&1&0\\ z&\theta_1&1\end{pmatrix}\begin{pmatrix}u_1&0&-\partial_z\\ -\partial_{\theta_2}-z\partial_{\theta_1}&u_2-1&-\partial_{\theta_1}\\ 0&0&u_3\end{pmatrix}\begin{pmatrix}1&\theta_2&0\\ 0&1&0\\ z&\theta_1&1\end{pmatrix}^{-1},$$

so that one can recognize the matrix $L^{+}(u_1-1,u_2+1,u_3)$. □

A key ingredient for an understanding the structure of the standard intertwining operators — which we refer to as *even*, or *bosonic*, intertwiners — is the generalized differentiation operator. In the case of Verma modules, we define this operator as follows:

$$\partial^{\alpha}\equiv\frac{1}{z^{\alpha}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha)}.\tag{3.5}$$

For α from the discrete set $1, 2, \ldots$, one obtains an usual multiple derivative and in the generic situation all expected properties:

- *commutativity and group property,* $\partial^{\alpha}\,\partial^{\beta} = \partial^{\beta}\,\partial^{\alpha} = \partial^{\alpha+\beta}$ *and* $\partial^0 = 1$*;*
- *Euler differentiation rule,* $\partial^{\alpha}\, z = z\,\partial^{\alpha} + \alpha\partial^{\alpha-1}$*;*
- *star-triangle relation in operator form,* $\partial^{\alpha}\, z^{\alpha+\beta}\partial^{\beta} = z^{\beta}\partial^{\alpha+\beta}\, z^{\alpha}$*;*
- *transition to* Γ*-functions,*

$$z^{\beta}\partial^{\alpha+\beta}\, z^{\alpha} = \frac{\Gamma(z\partial+1+\alpha)}{\Gamma(z\partial+1-\beta)}.$$

All these formulae can be proven in a direct way:

- commutativity,

$$\begin{aligned}\partial^{\alpha}\,\partial^{\beta} &= \frac{1}{z^{\alpha}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha)}\frac{1}{z^{\beta}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\beta)},\\ &= \frac{1}{z^{\alpha+\beta}}\frac{\Gamma(z\partial+1-\beta)}{\Gamma(z\partial+1-\alpha-\beta)}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\beta)}\\ &= \frac{1}{z^{\alpha+\beta}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha-\beta)} = \partial^{\alpha+\beta}\,,\end{aligned}$$

- Euler rule,

$$\begin{aligned}\partial^{\alpha} z &= \frac{1}{z^{\alpha}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha)} z = \frac{z}{z^{\alpha}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha)}\frac{z\partial+1}{z\partial+1-\alpha}\\ &= \frac{z}{z^{\alpha}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha)} + \frac{\alpha}{z^{\alpha-1}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+2-\alpha)} = z\,\partial^{\alpha} + \alpha\partial^{\alpha-1}\,,\end{aligned}$$

- the star-triangle relation,

$$\begin{aligned}\partial^{\alpha}\, z^{\alpha+\beta}\,\partial^{\beta} &= \frac{1}{z^{\alpha}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha)} z^{\alpha+\beta}\frac{1}{z^{\beta}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\beta)}\\ &= \frac{\Gamma(z\partial+1+\alpha)}{\Gamma(z\partial+1)}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\beta)} = \frac{\Gamma(z\partial+1+\alpha)}{\Gamma(z\partial+1-\beta)}\\ &= \frac{1}{z^{\alpha}}\frac{\Gamma(z\partial+1)}{\Gamma(z\partial+1-\alpha-\beta)} z^{\alpha} = z^{\beta}\partial^{\alpha+\beta}\, z^{\alpha}\,.\end{aligned}$$

Let us define the operator $\mathcal{D}^{-}_{13}(a)$ by the formula

$$\mathcal{D}^{-}_{13}(a) = (\partial_z + \theta_1\partial_{\theta_2})^{a}. \tag{3.6}$$

Proposition 3. *For $a = -2\ell - 1 = u_3 - u_1 - 1$ the operator $S^{\emptyset}_{13}(a)$ is an intertwining operator,*

$$\mathcal{D}^-_{13} : T^{\ell,b}_- \to T^{-\ell-1,b}_-. \tag{3.7}$$

The set of the intertwining relations can be expressed in a compact matrix form using corresponding L-operators

$$\mathcal{D}^-_{13}(u_3-u_1-1)L^-(u_1,u_2,u_3) = L^-(u_3-1,u_2,u_1+1)\mathcal{D}^-_{13}(u_3-u_1-1). \tag{3.8}$$

Proof. We have to check the following explicit relation:

$$\mathcal{D}^-_{13}(u_3-u_1-1)\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}\begin{pmatrix}u_1 & \partial_{\theta_2} & -\partial_z-\theta_1\partial_{\theta_2}\\ 0 & u_2 & 0\\ 0 & \partial_{\theta_1} & u_3-1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}\begin{pmatrix}u_3-1 & \partial_{\theta_2} & -\partial_z-\theta_1\partial_{\theta_2}\\ 0 & u_2 & 0\\ 0 & \partial_{\theta_1} & u_1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}^{-1}\mathcal{D}^-_{13}(u_3-u_1-1).$$

Everything is based on simple formulas that are valid for an arbitrary parameter a,

$$\mathcal{D}^-_{13}(a)\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a(\partial_z+\theta_1\partial_{\theta_2})^{-1} & 0 & 1\end{pmatrix}\mathcal{D}^-_{13}(a),$$

$$\mathcal{D}^-_{13}(a)\begin{pmatrix}u_1 & \partial_{\theta_2} & -\partial_z-\theta_1\partial_{\theta_2}\\ 0 & u_2 & 0\\ 0 & \partial_{\theta_1} & u_3-1\end{pmatrix}$$
$$=\begin{pmatrix}u_1 & \partial_{\theta_2} & -\partial_z-\theta_1\partial_{\theta_2}\\ 0 & u_2 & 0\\ 0 & \partial_{\theta_1}-a(\partial_z+\theta_1\partial_{\theta_2})^{-1} & u_3-1\end{pmatrix}\mathcal{D}^-_{13}(a).$$

After multiplication of the transformed matrices, one obtains

$$\mathcal{D}^-_{13}(a)\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}\begin{pmatrix}u_1 & \partial_{\theta_2} & -\partial_z-\theta_1\partial_{\theta_2}\\ 0 & u_2 & 0\\ 0 & \partial_{\theta_1} & u_3-1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}\begin{pmatrix}u_1+a & \partial_{\theta_2} & -\partial_z-\theta_1\partial_{\theta_2}\\ 0 & u_2 & 0\\ \frac{u_1-u_3+1+a}{\partial_z+\theta_1\partial_{\theta_2}} & \partial_{\theta_1} & u_3-1-a\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ \theta_2 & 1 & \theta_1\\ z & 0 & 1\end{pmatrix}^{-1}\mathcal{D}^-_{13}(a),$$

so that for $a = u_3 - u_1 - 1$, we obtain the expected result. □

By combining Propositions 1 and 3, we obtain the following formula for an even intertwiner acting on $T^{\ell,b}$.

Proposition 4. *There exists an intertwining operator*

$$\mathcal{D}_{13} : T^{\ell,b} \to T^{-\ell,b}, \tag{3.9}$$

defined by the formula

$$\mathcal{D}_{13}(u_1,u_2,u_3) = \mathcal{D}_{23}^{-1}(u_1-u_2)\mathcal{D}_{13}^{-}(u_3-u_1)\mathcal{D}_{23}(u_3-u_2) \tag{3.10}$$

$$= \partial_z^{u_3-u_1-1}\left(\partial_z + \frac{u_1-u_3}{u_1-u_2}(\partial_{\theta_1} - \frac{1}{2}\theta_2\partial_z)(\partial_{\theta_2} - \frac{1}{2}\theta_1\partial_z)\right) \tag{3.11}$$

$$= \partial_z^{-2\ell-1}\left(\partial_z + \frac{2\ell}{\ell-b}(\partial_{\theta_1} - \frac{1}{2}\theta_2\partial_z)(\partial_{\theta_2} - \frac{1}{2}\theta_1\partial_z)\right). \tag{3.12}$$

Proof. The construction (3.10) of the intertwining operator S_{13} is defined by the following diagram:

$$\begin{aligned} L(u_1,u_2,u_3) &\xrightarrow{\mathcal{D}_{23}(u_3-u_2)} L^-(u_1,u_2-1,u_3+1) \\ &\xrightarrow{\mathcal{D}_{13}^-(u_3-u_1)} L^-(u_3,u_2-1,u_1+1) \\ &\xrightarrow{\mathcal{D}_{23}^{-1}(u_1-u_2)} L(u_3,u_2,u_1). \end{aligned}$$

This diagram shows the needed commutation relation for intertwining operators $\mathcal{D}_{23}$ and $\mathcal{D}_{13}^-$. Substituting the explicit expressions for the operators, yields

$$\begin{aligned} \mathcal{D}_{13} &= \frac{1}{u_1-u_2} e^{\frac{\theta_1\theta_2}{2}\partial_z}\left(\partial_{\theta_1} + (u_1-u_2)\theta_1\right) \\ &\quad \times (\partial_z + \theta_1\partial_{\theta_2})^{u_3-u_1}\left(\partial_{\theta_1} + (u_3-u_1)\theta_1\right) e^{-\frac{\theta_1\theta_2}{2}\partial_z} \\ &= \frac{1}{u_1-u_2}\partial_z^{u_3-u_1-1} e^{\frac{\theta_1\theta_2}{2}\partial_z}((u_1-u_2)\partial_z \\ &\quad + (u_1-u_3)\,\partial_{\theta_1}(\partial_{\theta_2} - \theta_1\partial_z))e^{-\frac{\theta_1\theta_2}{2}\partial_z}, \end{aligned}$$

where we have used the transformations

$$\begin{aligned} (\partial_z + \theta_1\partial_{\theta_2})^{u_3-u_1} &= \partial_z^{u_3-u_1} + (u_3-u_1)\,\partial_z^{u_3-u_1-1}\theta_1\partial_{\theta_2} \\ &= \partial_z^{u_3-u_1-1}(\partial_z + (u_3-u_1)\,\theta_1\partial_{\theta_2}) \end{aligned}$$

and

$$\begin{aligned} (\partial_{\theta_1} + (u_1-u_2)\theta_1)&(\partial_z + (u_3-u_1)\,\theta_1\partial_{\theta_2})\,(\partial_{\theta_1} + (u_3-u_2)\theta_1) \\ &= (u_1-u_2)\partial_z + (u_1-u_3)\,\partial_{\theta_1}(\partial_{\theta_2} - \theta_1\partial_z). \end{aligned}$$

Noting that

$$e^{\frac{\theta_1\theta_2}{2}\partial_z}\partial_{\theta_1}e^{-\frac{\theta_1\theta_2}{2}\partial_z} = \partial_{\theta_1} - \frac{1}{2}\theta_2\partial_z,$$
$$e^{\frac{\theta_1\theta_2}{2}\partial_z}(\partial_{\theta_2} - \theta_1\partial_z)e^{-\frac{\theta_1\theta_2}{2}\partial_z} = \partial_{\theta_2} - \frac{1}{2}\theta_1\partial_z,$$

we arrive to the expression given in (3.11). □

The operators $\mathcal{D}_{12}$ and $\mathcal{D}_{23}$ become non-invertible in the special cases, where $\ell + b = 0$, or $\ell - b = 0$. In these degenerate situations, the kernels of these intertwining operators form invariant subspaces. It can be directly verified, that these invariant subspaces correspond to the states of a fixed chirality.

Additionally, the finite-dimensional invariant subspaces arise when $2\ell \in \mathbb{N}_0$, which are defined as the kernel of the operator $\mathcal{D}_{13}$.

3.2. Intertwining operators for the tensor products. Consider the tensor product of two representations $T_\lambda^{\ell,b}\otimes T_{\lambda'}^{\ell',b'}$. We identify the representation space with polynomial functions depending on two sets of variables

$$\Phi(z, \theta_1, \theta_2\,, w, \tilde{\theta}_1, \tilde{\theta}_2) \in T_\lambda^{\ell,b} \otimes T_{\lambda'}^{\ell',b'}\,.$$

We denote $L_1^\lambda(u_1, u_2, u_3)$ the L-operator with the spectral parameter u acting in $\mathbb{C}^{2|1} \otimes T_\lambda^{\ell,b}$, so that its matrix elements are generators $T_\lambda^{\ell,b}$ acting on polynomials of the variables z, θ_1, θ_2. In a similar way, $L_2^\lambda(v_1, v_2, v_3)$ is an L-operator with the spectral parameter v acting in $\mathbb{C}^{2|1} \otimes T_{\lambda'}^{\ell',b'}$ and its matrix elements are the generators of $T_{\lambda'}^{\ell',b'}$ acting on polynomials of the variables $w, \tilde{\theta}_1, \tilde{\theta}_2$. The expressions for the parameters have the form

$$u_1 = u + \ell + b, \quad u_2 = u + 2b, \quad u_3 = u + b - \ell, \tag{3.13}$$
$$v_1 = v + \ell' + b', \quad v_2 = v + 2b', \quad v_3 = v + b' - \ell'. \tag{3.14}$$

We will use the compact notation $\mathcal{D}_{\tilde{1}\tilde{2}}, \mathcal{D}_{\tilde{1}\tilde{3}}, \mathcal{D}_{\tilde{2}\tilde{3}}$ for the analogs of intertwining operators $\mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{23}$ acting on polynomials of the variables $w, \tilde{\theta}_1, \tilde{\theta}_2$. For example,

$$\mathcal{D}_{\tilde{2}\tilde{3}}(a) = \left(\partial_{\tilde{\theta}_1} + a\tilde{\theta}_1\right)\, e^{-\frac{\tilde{\theta}_1\tilde{\theta}_2}{2}\partial_w}, \quad \mathcal{D}_{\tilde{1}\tilde{2}}(a) = e^{-w\tilde{\theta}_2\partial_{\tilde{\theta}_1}}\left(\partial_{\tilde{\theta}_2} + a\tilde{\theta}_2\right) e^{\frac{\tilde{\theta}_1\tilde{\theta}_2}{2}\partial_w}.$$

In addition to the permutations within the tensor factors, we have additional operators that permute parameters between the tensor factors.

Proposition 5. *For any $\alpha \in \mathbb{C}$, there exists an intertwining operator for the tensor product of two representations of $g\ell(2|1)$,*

$$\mathcal{S}_{1\tilde{3}}(\alpha) : T^{\boldsymbol{\sigma}} \otimes T^{\boldsymbol{\sigma}'} \longrightarrow T^{(\sigma_1+\alpha,\sigma_2,\sigma_3)} \otimes T^{(\sigma'_1,\sigma'_2,\sigma'_3-\alpha)},$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\boldsymbol{\sigma}' = (\sigma'_1, \sigma'_2, \sigma'_3)$, $\boldsymbol{\sigma}_\alpha = (\sigma_1 + \alpha, \sigma_2, \sigma_3)$, and $\boldsymbol{\sigma}'_\alpha = (\sigma'_1, \sigma'_2, \sigma'_3 - \alpha)$. In an equivalent way, this relation can be expressed as a relation for the L-operators

$$\begin{aligned} &\mathcal{S}_{1\tilde{3}}(\alpha)\ (L_1(u_1, u_2, u_3) + L_2(v_1, v_2, v_3)) \\ &\qquad = (L_1(u_1 - \alpha, u_2, u_3) + L_2(v_1, v_2, v_3 + \alpha))\ \mathcal{S}_{1\tilde{3}}(\alpha). \end{aligned} \tag{3.15}$$

The operator $\mathcal{S}_{1\tilde{3}}(\alpha)$ is defined by the formula

$$\Phi \longmapsto \left(z - w + \frac{\theta_2\tilde{\theta}_1 + \theta_1\tilde{\theta}_2}{2} + \frac{(\theta_1 - \tilde{\theta}_1)(\theta_2 - \tilde{\theta}_2)}{2} \right)^\alpha \Phi. \tag{3.16}$$

For the special value $\alpha = u_1 - v_3$, it satisfies the Yangian intertwining relation,

$$\begin{aligned} &\mathcal{S}_{1\tilde{3}}(u_1 - v_3)\, L_1(u_1, u_2, u_3)\, L_2(v_1, v_2, v_3) \\ &\qquad = L_1(v_3, u_2, u_3)\, L_2(v_1, v_2, u_1)\, \mathcal{S}_{1\tilde{3}}(u_1 - v_3). \end{aligned} \tag{3.17}$$

Proof. Denote

$$S = z - w + \frac{\theta_2\tilde{\theta}_1 + \theta_1\tilde{\theta}_2}{2} + \frac{(\theta_1 - \tilde{\theta}_1)(\theta_2 - \tilde{\theta}_2)}{2}.$$

We demonstrate that the operator (3.16) is intertwining under the stated conditions. Observe that S is a matrix element of the product

$$\begin{aligned} w^{-1}\, z &= \begin{pmatrix} 1 & 0 & 0 \\ \tilde{\theta}_2 & 1 & 0 \\ w + \frac{\tilde{\theta}_1\tilde{\theta}_2}{2} & \tilde{\theta}_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & 0 \\ z + \frac{\theta_1\theta_2}{2} & \theta_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 - \tilde{\theta}_2 & 1 & 0 \\ S & \theta_1 - \tilde{\theta}_1 & 1 \end{pmatrix}. \end{aligned}$$

Under a group action

$$z \to g^{-1}z = z'h(z, g), \qquad w \to g^{-1}w = w'h(w, g), \tag{3.18}$$

the matrix $w^{-1}\, z$ transforms as

$$w^{-1}\, z \longmapsto w'^{-1}\, z' = h(w, g)w^{-1}gg^{-1}zh^{-1}(z, g) = h(w, g)w^{-1}zh^{-1}(z, g),$$

whence S transforms according to

$$S(z,w) \longmapsto S(z',w') = h_{33}(w,g)h_{11}^{-1}(z,g)S(z,w). \tag{3.19}$$

Consequently, multiplication by S^α implements the intertwining as claimed. Indeed, we have (2.10),

$$\begin{aligned}
&S^\alpha(z,w)\,[T^{\boldsymbol{\sigma}}(g) \otimes T^{\boldsymbol{\sigma}'}(g)\,\Phi](z,w) \\
&= S^\alpha(z,w)\,h(z,g;\boldsymbol{\sigma})\,h(w,g;\boldsymbol{\sigma}')\Phi(z',w') \\
&= h_{11}^\alpha(z,g)\,h(z,g;\boldsymbol{\sigma})\,h(w,g;\boldsymbol{\sigma}')\,h_{33}^{-\alpha}(w,g)\,S^\alpha(z',w')\,\Phi(z',w') \\
&= [T^{(\sigma_1+\alpha,\sigma_2,\sigma_3)}(g) \otimes T^{(\sigma_1,\sigma_2,\sigma_3-\alpha)}(g)\,S^\alpha\,\Phi](z,w).
\end{aligned}$$

We now verify relation (3.17) by a direct computation. First, consider the transformation of the central matrix,

$$\begin{aligned}
&S^\alpha \begin{pmatrix} u_1 & \partial_{\theta_2} - \frac{1}{2}\theta_1\partial_z & -\partial_z \\ 0 & u_2 - 1 & -\partial_{\theta_1} + \frac{1}{2}\theta_2\partial_z \\ 0 & 0 & u_3 - 1 \end{pmatrix} S^{-\alpha} \\
&= \begin{pmatrix} u_1 - \alpha & \partial_{\theta_2} - \frac{1}{2}\theta_1\partial_z & -\partial_z \\ 0 & u_2 - 1 & -\partial_{\theta_1} + \frac{1}{2}\theta_2\partial_z \\ 0 & 0 & u_3 \end{pmatrix} + \frac{\alpha}{S} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} S & \theta_1 - \tilde\theta_1 & 1 \end{pmatrix}.
\end{aligned}$$

Note, that there appears a row vector from the product $w^{-1}\,z$. Similarly, one finds

$$\begin{aligned}
&S^\alpha \begin{pmatrix} v_1 & \partial_{\tilde\theta_2} - \frac{1}{2}\tilde\theta_1\partial_w & -\partial_w \\ 0 & v_2 - 1 & -\partial_{\tilde\theta_1} + \frac{1}{2}\tilde\theta_2\partial_w \\ 0 & 0 & v_3 \end{pmatrix} S^{-\alpha} \\
&= \begin{pmatrix} v_1 & \partial_{\tilde\theta_2} - \frac{1}{2}\tilde\theta_1\partial_w & -\partial_w \\ 0 & v_2 - 1 & -\partial_{\tilde\theta_1} + \frac{1}{2}\tilde\theta_2\partial_w \\ 0 & 0 & v_3 + \alpha \end{pmatrix} - \frac{\alpha}{S} \begin{pmatrix} 1 \\ \theta_2 - \tilde\theta_2 \\ S \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

so that there appears a column vector from $w^{-1}\,z$. For the whole L-operators, we obtain

$$\begin{aligned}
S^\alpha L_1(u_1,u_2,u_3)S^{-\alpha} &= L_1(u_1-\alpha,u_2,u_3) \\
&\quad + \frac{\alpha}{S} \begin{pmatrix} 1 \\ \theta_2 \\ z + \frac{\theta_1\theta_2}{2} \end{pmatrix} \begin{pmatrix} -w + \frac{\tilde\theta_1\tilde\theta_2}{2} & -\tilde\theta_1 & 1 \end{pmatrix}, \\
S^\alpha L_2(v_1,v_2,v_3)S^{-\alpha} &= L_2(v_1,v_2,v_3+\alpha)
\end{aligned}$$

$$- \frac{\alpha}{S} \begin{pmatrix} 1 \\ \theta_2 \\ z + \frac{\theta_1\theta_2}{2} \end{pmatrix} \begin{pmatrix} -w + \frac{\tilde\theta_1\tilde\theta_2}{2} & -\tilde\theta_1 & 1 \end{pmatrix}.$$

The relation (3.15) is a direct consequence of these two relations. Next, using the formulae

$$\begin{pmatrix} -w + \frac{\tilde\theta_1\tilde\theta_2}{2} & -\tilde\theta_1 & 1 \end{pmatrix} L_2(v_1, v_2, v_3 + \alpha) = (v_3+\alpha) \begin{pmatrix} -w + \frac{\tilde\theta_1\tilde\theta_2}{2} & -\tilde\theta_1 & 1 \end{pmatrix},$$

$$L_1(u_1 - \alpha, u_2, u_3) \begin{pmatrix} 1 \\ \theta_2 \\ z + \frac{\theta_1\theta_2}{2} \end{pmatrix} = (u_1 - \alpha) \begin{pmatrix} 1 \\ \theta_2 \\ z + \frac{\theta_1\theta_2}{2} \end{pmatrix},$$

we arrive to the relation

$$S^{\alpha} L_1(u_1, u_2, u_3)\, L_2(v_1, v_2, v_3)\, S^{-\alpha} = L_1(u_1 - \alpha, u_2, u_3)\, L_2(v_1, v_2, v_3 + \alpha) + \frac{\alpha(v_3 - u_1 + \alpha)}{S} \begin{pmatrix} 1 \\ \theta_2 \\ z + \frac{\theta_1\theta_2}{2} \end{pmatrix} \begin{pmatrix} -w + \frac{\tilde\theta_1\tilde\theta_2}{2} & -\tilde\theta_1 & 1 \end{pmatrix}.$$

Thus, the intertwining operator (3.16) satisfies (3.17) for $\alpha = u_1 - v_3$. □

For other tensor products with $\lambda = \lambda' = +$, or $\lambda = \lambda' = -$, there no intertwining operator exists, that permutes parameters between the tensor factors and acts by multiplication. The suitable matrix element satisfying the relations of type (3.19) is odd, precluding raising it to an arbitrary power.

However, such an operator does arise in the tensor product of representations with $\lambda = +$ and $\lambda' = -$. A mnemonic rule here, is that the non-trivial intertwining operators acting by multiplication on functions arise only when the number of pluses minus the number of minuses, associated with both the L-operators, are equal to zero.

Proposition 6. *For any $\alpha \in \mathbb{C}$, there exists an intertwining operator,*

$$\mathcal{S}_{2\tilde 2}(a) : T_+^{\boldsymbol\sigma} \otimes T_-^{\boldsymbol\sigma'} \longrightarrow T_+^{(\sigma_1, \sigma_2 + \alpha\,\sigma_3)} \otimes T_-^{(\sigma_1', \sigma_2' - \alpha\,\sigma_3')},$$

defined by

$$\Phi \longmapsto [1 + \theta_2(\tilde\theta_2 - \tilde\theta_1 w) + \theta_1\tilde\theta_1]^{\alpha}\Phi. \tag{3.20}$$

The intertwining relation is equivalent to the relation for L-operators

$$\mathcal{S}_{2\tilde 2}(\alpha) \left(L_1^+(u_1, u_2, u_3) + L_2^-(v_1, v_2, v_3) \right) = \left(L_1^+(u_1, u_2 + \alpha, u_3) + L_2^-(v_1, v_2 - \alpha, v_3) \right) \mathcal{S}_{2\tilde 2}(\alpha). \tag{3.21}$$

For the special value $\alpha = v_2 - u_2 + 2$, the operator $\mathcal{S}_{2\tilde{2}}$ satisfies the Yangian intertwining relation

$$\begin{aligned}&\mathcal{S}_{2\tilde{2}}(v_2 - u_2 + 2)L_1^+(u_1, u_2, u_3)L_2^-(v_1, v_2, v_3)\\ &\qquad = L_1^+(u_1, u_2 + \alpha, u_3)L_2^-(v_1, v_2 - \alpha, v_3)\mathcal{S}_{2\tilde{2}}(v_2 - u_2 + 2).\end{aligned}$$

Proof. The proof closely follows that of Proposition 5. Indeed, the function

$$S = 1 + \theta_2(\tilde{\theta}_2 - \tilde{\theta}_1 w) + \theta_1\tilde{\theta}_1 \tag{3.22}$$

is a matrix element of the corresponding product,

$$\begin{aligned}w_-^{-1}z_+ &= \begin{pmatrix}1 & 0 & 0\\ \tilde{\theta}_2 & 1 & \tilde{\theta}_1\\ w & 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & \theta_2 & 0\\ 0 & 1 & 0\\ z & \theta_1 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & \theta_2 & 0\\ -\tilde{\theta}_2 - \tilde{\theta}_1(z-w) & S & -\tilde{\theta}_1\\ z - w & \theta_1 - w\theta_2 & 1\end{pmatrix}.\end{aligned}$$

The matrix $w_-^{-1}\, z_+$ transforms as

$$w_-'^{-1}\, z_+' = h(w_-, g)w_-^{-1}gg^{-1}z_+h^{-1}(z_+, g) = h(w_-, g)w_-^{-1}z_+h^{-1}(z_+, g),$$

whence $S(z_+, w_-)$ transforms according to

$$S(z_+', w_-') = h_{22}(w_-, g)h_{22}^{-1}(z_+, g)S(z_+, w_-).$$

Consequently, multiplication by S^α implements the intertwining as claimed,

$$\begin{aligned}&S^\alpha(z_+, w_-)\,[T_+^{\boldsymbol{\sigma}}(g) \otimes T_-^{\boldsymbol{\sigma}'}(g)\,\Phi](z_+, w_-)\\ &= S^\alpha(z_+, w_-)\,h(z_+, g; \boldsymbol{\sigma})\,h(w_-, g; \boldsymbol{\sigma}')\Phi(z_+', w_-')\\ &= h_{11}^\alpha(z_+, g)\,h(z_+, g; \boldsymbol{\sigma})\,h(w_-, g; \boldsymbol{\sigma}')\,h_{22}^{-\alpha}(w_-, g)\,S^\alpha(z_+', w_-')\,\Phi(z_+', w_-')\\ &= [T_+^{(\sigma_1, \sigma_2+\alpha, \sigma_3)}(g) \otimes T_-^{(\sigma_1, \sigma_2-\alpha, \sigma_3)}(g)\,S^\alpha\,\Phi](z_+, w_-).\end{aligned}$$

Let us now turn to the relations with L-operators. By analogy, we compute,

$$S^\alpha L_1^+(u_1, u_2, u_3)S^{-\alpha} = L_1^+(u_1, u_2+\alpha, u_3) - \frac{\alpha}{S}\begin{pmatrix}\theta_2\\ 1\\ \theta_1\end{pmatrix}\begin{pmatrix}-\tilde{\theta}_2 + w\tilde{\theta}_1 & 1 & -\tilde{\theta}_1\end{pmatrix},$$

$$S^\alpha L_2^-(v_1, v_2, v_3)S^{-\alpha} = L_2^-(v_1, v_2-\alpha, v_3) + \frac{\alpha}{S}\begin{pmatrix}\theta_2\\ 1\\ \theta_1\end{pmatrix}\begin{pmatrix}-\tilde{\theta}_2 + w\tilde{\theta}_1 & 1 & -\tilde{\theta}_1\end{pmatrix}.$$

Using the relations

$$L_1^+(u_1, u_2+\alpha, u_3)\begin{pmatrix}\theta_2\\ 1\\ \theta_1\end{pmatrix} = (u_2+\alpha-2)\begin{pmatrix}\theta_2\\ 1\\ \theta_1\end{pmatrix},$$

$$\left(-\tilde{\theta}_2+w\tilde{\theta}_1 \quad 1-\tilde{\theta}_1\right) L_2^-(v_1, v_2-\alpha, v_3) = (v_2-\alpha)\left(-\tilde{\theta}_2+w\tilde{\theta}_1 \quad 1-\tilde{\theta}_1\right),$$

we obtain

$$S^\alpha L_1^+(u_1,u_2,u_3)L_2^-(v_1,v_2,v_3)S^{-\alpha} = L_1^+(u_1,u_2+\alpha,u_3)L_2^-(v_1,v_2-\alpha,v_3) + \frac{\alpha(u_2-v_2-2+\alpha)}{S}\begin{pmatrix}\theta_2\\ 1\\ \theta_1\end{pmatrix}\left(-\tilde{\theta}_2+w\tilde{\theta}_1 \quad 1 \quad -\tilde{\theta}_1\right).$$

Thus, the intertwining relation holds only if $\alpha = v_2 - u_2 + 2$. □

In summary, we conclude that the intertwining operators in the tensor product of the induced representations of $Y(\mathfrak{sl}(2|1))$ are determined by permutations of the representation parameters and changes of the Borel subalgebra from which the representation is induced. However, not all the parameter permutations are permitted—they must be performed in a such way, that they preserve the number of the odd elements in each tensor factor.

§4. The general R-matrix

The general R-matrix acting on the tensor product of representations $T^{\ell,b}\otimes T^{\ell',b'}$ can be determined from the relation:

$$R_{12}(u-v)\,L_1(u_1,u_2,u_3)L_2(v_1,v_2,v_3) = L_2(v_1,v_2,v_3)L_1(u_1,u_2,u_3)\,R_{12}(u-v). \qquad (4.1)$$

It is convenient to pass to the operator $\mathcal{R}_{12}(u-v)$, obtained by composing $R_{12}(u-v)$ with the permutation operator P_{12},

$$P_{12}\,\Phi(z,\theta_1,\theta_2\,,w,\tilde{\theta}_1,\tilde{\theta}_2) = \Phi(w,\tilde{\theta}_1,\tilde{\theta}_2\,,z,\theta_1,\theta_2)\,. \qquad (4.2)$$

Introduce the notation

$$R_{12}(u-v) = P_{12}\,\mathcal{R}(u-v). \qquad (4.3)$$

The defining relation for $\mathcal{R}$ then becomes

$$\mathcal{R}(u-v)\,L_1(u_1,u_2,u_3)L_2(v_1,v_2,v_3) = L_1(v_1,v_2,v_3)L_2(u_1,u_2,u_3)\,\mathcal{R}(u-v).$$

Thus, the operator $\mathcal{R}$ can be interpreted as an operator permuting simultaneously all the parameters, $u_i \leftrightarrow v_i$. This operator can be realized as a product of three commuting intertwining operators, each implementing an elementary permutation, namely, for $i = 1, 2, 3$,

$$\mathcal{R}_i : u_i \leftrightarrow v_i,$$

so that

$$\mathcal{R}(u - v) = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3.$$

Using the set of intertwining operators computed above, we can explicitly construct the operators $\mathcal{R}_i$.

Proposition 7. *The intertwining operator $\mathcal{R}_1$, defined by:*

$$\mathcal{R}_1(u_1|v_1, v_2, v_3) = \mathcal{S}_{1\tilde{3}}(v_3 - v_1)\,\mathcal{D}_{\tilde{1}\tilde{3}}(v_1, v_2, u_1)\,\mathcal{S}_{1\tilde{3}}(u_1 - v_3) \tag{4.4}$$

$$= \mathcal{D}_{\tilde{1}\tilde{3}}(v_1, v_2, u_1)\mathcal{S}_{1\tilde{3}}(u_1 - v_1)\mathcal{D}_{\tilde{1}\tilde{3}}(v_1, v_2, v_3), \tag{4.5}$$

satisfies the relation

$$\mathcal{R}_1 L_1(u_1, u_2, u_3)L_2(v_1, v_2, v_3) = L_1(v_1, u_2, u_3)L_2(u_1, v_2, v_3)\mathcal{R}_1.$$

The operator $\mathcal{R}_1$ is explicitly expressed in terms of the operator-valued gamma functions,

$$(-1)^{v_1-u_1} S_1\mathcal{R}_1 S_1^{-1} = \frac{\Gamma(w\partial_w + u_1 - v_3 + 1)}{\Gamma(w\partial_w + v_1 - v_3 + 1)}\left(1 - \frac{v_1 - u_1}{v_1 - v_2}\tilde{\theta}_2\partial_{\tilde{\theta}_2}\right) + \frac{v_1 - u_1}{v_1 - v_2}\frac{\Gamma(w\partial_w + u_1 - v_3)}{\Gamma(w\partial_w + v_1 - v_3 + 1)} w\partial_{\tilde{\theta}_1}\partial_{\tilde{\theta}_2},$$

where the shift operator S_1 is defined by the relation

$$S_1 = e^{\frac{\tilde{\theta}_1\tilde{\theta}_2}{2}\partial_w} e^{\theta_1(\partial_{\tilde{\theta}_1} + \frac{1}{2}\tilde{\theta}_2\partial_w)} e^{\theta_2(\partial_{\tilde{\theta}_2} + \frac{1}{2}\tilde{\theta}_1\partial_w)} e^{\left(z + \frac{\theta_1\theta_2}{2}\right)\partial_w}.$$

Proof. The construction (4.4) of the operator $\mathcal{R}_1$ is defined by the following diagram

$$L_1(u_1, u_2, u_3)L_2(v_1, v_2, v_3) \xrightarrow{\mathcal{S}_{1\tilde{3}}(u_1 - v_3)} L_1(v_3, u_2, u_3)L_2(v_1, v_2, u_1)$$

$$\xrightarrow{\mathcal{D}_{\tilde{1}\tilde{3}}(v_1, v_2, u_1)} L_1(v_3, u_2, u_3)L_2(u_1, v_2, v_1)$$

$$\xrightarrow{\mathcal{S}_{1\tilde{3}}(v_3 - v_1)} L_1(v_1, u_2, u_3)L_2(u_1, v_2, v_3).$$

This diagram shows the needed commutation relation for intertwining operators $\mathcal{S}_{1\tilde{3}}$ and $\mathcal{D}_{\tilde{1}\tilde{3}}$, which results in the permutation $u_1 \rightleftarrows v_1$.

The construction (4.5) of the operator $\mathcal{R}_1$ is defined by a different diagram

$$
\begin{aligned}
L_1(u_1,u_2,u_3)L_2(v_1,v_2,v_3) &\xrightarrow{\mathcal{D}_{\bar{1}\bar{3}}(v_1,v_2,v_3)} L_1(u_1,u_2,u_3)L_2(v_3,v_2,v_1)\\
&\xrightarrow{\mathcal{S}_{1\bar{3}}(u_1-v_1)} L_1(v_1,u_2,u_3)L_2(v_1,v_2,u_1)\\
&\xrightarrow{\mathcal{D}_{\bar{1}\bar{3}}(v_1,v_2,u_1)} L_1(v_1,u_2,u_3)L_2(u_1,v_2,v_3),
\end{aligned}
$$

which results in the same permutation $u_1 \rightleftarrows v_1$. The two considered diagrams show that the product of intertwining operators (4.4) and (4.5) performs the same permutation $u_1 \rightleftarrows v_1$ in the product of the two L-operators. Below we prove its coincidence by a direct computation.

Note that the shift operator S_1 is determined by the conditions

$$
S_1\left(z-w+\frac{\theta_2\tilde{\theta}_1+\theta_1\tilde{\theta}_2}{2}+\frac{(\theta_1-\tilde{\theta}_1)(\theta_2-\tilde{\theta}_2)}{2}\right)S_1^{-1}=-w,
$$

$$
S_1\left(\partial_{\tilde{\theta}_2}-\frac{1}{2}\tilde{\theta}_1\partial_w\right)S_1^{-1}=\partial_{\tilde{\theta}_2},
$$

$$
S_1\left(\partial_{\tilde{\theta}_1}-\frac{1}{2}\tilde{\theta}_2\partial_w\right)S_1^{-1}=\partial_{\tilde{\theta}_1}-\tilde{\theta}_2\partial_w,
$$

so that all needed intertwining operators

$$
\mathcal{D}_{\bar{1}\bar{3}}(v_1,v_2,u_1)=\partial_w^{u_1-v_1}+\frac{v_1-u_1}{v_1-v_2}\,\partial_w^{u_1-v_1-1}\left(\partial_{\tilde{\theta}_1}-\frac{1}{2}\tilde{\theta}_2\partial_w\right)\left(\partial_{\tilde{\theta}_2}-\frac{1}{2}\tilde{\theta}_1\partial_w\right),
$$

$$
\mathcal{S}_{1\bar{3}}(\alpha)=\left(z-w+\frac{\theta_2\tilde{\theta}_1+\theta_1\tilde{\theta}_2}{2}+\frac{(\theta_1-\tilde{\theta}_1)(\theta_2-\tilde{\theta}_2)}{2}\right)^{\alpha}
$$

transform to a much simpler form, and we obtain

$$
\begin{aligned}
&S_1\mathcal{R}_1S_1^{-1}\\
&=(-1)^{u_1-v_1}w^{v_3-v_1}\left(\partial_w^{u_1-v_1}+\frac{v_1-u_1}{v_1-v_2}\,\partial_w^{u_1-v_1-1}\left(\partial_{\tilde{\theta}_1}-\tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\right)w^{u_1-v_3}\\
&\qquad=(-1)^{u_1-v_1}w^{v_3-v_1}\Big(\partial_w^{u_1-v_1}\left(1-\frac{v_1-u_1}{v_1-v_2}\tilde{\theta}_2\partial_{\tilde{\theta}_2}\right)\\
&\qquad\qquad+\frac{v_1-u_1}{v_1-v_2}\partial_w^{u_1-v_1-1}\partial_{\tilde{\theta}_1}\partial_{\tilde{\theta}_2}\Big)w^{u_1-v_3}.
\end{aligned}
$$

It remains to transform this expression to a final form, using the following formula:

$$
w^{\beta}\partial_w^{\alpha+\beta}\,w^{\alpha}=\frac{\Gamma(w\partial_w+1+\alpha)}{\Gamma(w\partial_w+1-\beta)}\,.
$$

Now, we are switching to the relation

$$\mathcal{S}_{1\tilde{3}}(v_3 - v_1)\,\mathcal{D}_{\tilde{1}\tilde{3}}(v_1, v_2, u_1)\,\mathcal{S}_{1\tilde{3}}(u_1 - v_3) = \mathcal{D}_{\tilde{1}\tilde{3}}(v_3, v_2, u_1)\mathcal{S}_{1\tilde{3}}(u_1 - v_1)\mathcal{D}_{\tilde{1}\tilde{3}}(v_1, v_2, v_3),$$

which results in turn to

$$w^{v_3-v_1}\left(\partial_w^{u_1-v_1} + \frac{v_1-u_1}{v_1-v_2}\,\partial_w^{u_1-v_1-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\right)w^{u_1-v_3}$$
$$= \left(\partial_w^{u_1-v_3} + \frac{v_3-u_1}{v_3-v_2}\,\partial_w^{u_1-v_3-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\right)w^{u_1-v_1}$$
$$\times\left(\partial_w^{v_3-v_1} + \frac{v_1-v_3}{v_1-v_2}\,\partial_w^{v_3-v_1-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\right),$$

after a similarity transformation using the operator S_1. After a multiplication, one obtains

$$w^{v_3-v_1}\,\partial_w^{u_1-v_1}\,w^{u_1-v_3} + \frac{v_1-u_1}{v_1-v_2}\,w^{v_3-v_1}\,\partial_w^{u_1-v_1-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\,w^{u_1-v_3}$$
$$= \partial_w^{u_1-v_3}\,w^{u_1-v_1}\,\partial_w^{v_3-v_1} + \frac{v_3-u_1}{v_3-v_2}\,\partial_w^{u_1-v_3-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\,w^{u_1-v_1}\,\partial_w^{v_3-v_1}$$
$$+ \frac{v_1-v_3}{v_1-v_2}\,\partial_w^{u_1-v_3}\,w^{u_1-v_1}\partial_w^{v_3-v_1-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}$$
$$+ \frac{v_3-u_1}{v_3-v_2}\,\frac{v_1-v_3}{v_1-v_2}\,\partial_w^{u_1-v_3-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\,w^{u_1-v_1}\,\partial_w^{v_3-v_1-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}.$$

The first terms cancel each other due to the star-triangle relation in the form $w^a\,\partial_w^{a+b}\,w^b = \partial_w^b\,w^{a+b}\,\partial_w^a$, and, in fact, the second and the last terms in the RHS coincide. After a simple transformation, one obtains

$$\frac{v_1-u_1}{v_1-v_2}\,w^{v_3-v_1}\,\partial_w^{u_1-v_1-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\,w^{u_1-v_3}$$
$$= \frac{v_3-u_1}{v_1-v_2}\,\partial_w^{u_1-v_3-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2}\,w^{u_1-v_1}\,\partial_w^{v_3-v_1}$$
$$+ \frac{v_1-v_3}{v_1-v_2}\,\partial_w^{u_1-v_3}\,w^{u_1-v_1}\partial_w^{v_3-v_1-1}\left(\partial_{\tilde{\theta}_1} - \tilde{\theta}_2\partial_w\right)\partial_{\tilde{\theta}_2},$$

that results in two relations, namely,

$$\frac{v_1-u_1}{v_1-v_2}\,w^{v_3-v_1}\,\partial_w^{u_1-v_1-1}\,w^{u_1-v_3} = \frac{v_3-u_1}{v_1-v_2}\,\partial_w^{u_1-v_3-1}\,w^{u_1-v_1}\,\partial_w^{v_3-v_1}$$
$$+ \frac{v_1-v_3}{v_1-v_2}\,\partial_w^{u_1-v_3}\,w^{u_1-v_1}\partial_w^{v_3-v_1-1}$$

and

$$\frac{v_1-u_1}{v_1-v_2}\, w^{v_3-v_1}\, \partial_w^{u_1-v_1}\, w^{u_1-v_3} = \left(\frac{v_3-u_1}{v_1-v_2}+\frac{v_1-v_3}{v_1-v_2}\right) \partial_w^{u_1-v_3}\, w^{u_1-v_1} \partial_w^{v_3-v_1}.$$

The last relation is reduced to the star-triangle relation in the form $w^a\, \partial_w^{a+b}\, w^b = \partial_w^b\, w^{a+b}\, \partial_w^a$. The first relation can be rewritten in a simpler form

$$(a+b)\, w^a\, \partial_w^{a+b-1}\, w^{b-1}\, w = b\, \partial_w^{b-1}\, w^{a+b}\, \partial_w^a + a\, \partial_w^b\, w^{a+b}\, \partial_w^{a-1},$$

where $a = v_3 - v_1$ and $b = u_1 - v_3$, and then, using the star-triangle relation in the LHS, it can be transformed as follows:

$$\begin{aligned}
&(a+b)\, \partial_w^{b-1}\, w^{a+b-1}\, \partial_w^a\, w = b\, \partial_w^{b-1}\, w^{a+b}\, \partial_w^a + a\, \partial_w^b\, w^{a+b}\, \partial_w^{a-1} \\
&\to (a+b)\, w^{a+b-1}\, \partial_w^a\, w = b\, w^{a+b}\, \partial_w^a + a\, \partial_w\, w^{a+b}\, \partial_w^{a-1} \\
&\to (a+b)\, w^{a+b-1}\, \left(w\, \partial_w^a + a \partial_w^{a-1}\right) \\
&\quad = b\, w^{a+b}\, \partial_w^a + a\, \left(w^{a+b}\, \partial_w^a + (a+b)\, w^{a+b-1}\, \partial_w^{a-1}\right) . \qquad \square
\end{aligned}$$

The same line of reasoning establishes the following proposition.

Proposition 8. *The intertwining operator $\mathcal{R}_3$, defined by*

$$\begin{aligned}
\mathcal{R}_3(u_1, u_2, u_3|v_3) &= \mathcal{S}_{1\tilde{3}}(u_3 - u_1)\mathcal{D}_{13}(v_3, u_2, u_3)\mathcal{S}_{1\tilde{3}}(u_1 - v_3) \\
&= \mathcal{D}_{13}(v_3, u_2, u_1)\mathcal{S}_{1\tilde{3}}(u_3 - v_3)\mathcal{D}_{13}(u_1, u_2, u_3),
\end{aligned}$$

satisfies the relation

$$\mathcal{R}_3\, L_1(u_1, u_2, u_3)\, L_2(v_1, v_2, v_3) = L_1(u_1, u_2, v_3)\, L_2(v_1, v_2, u_3)\, \mathcal{R}_3.$$

The intertwining operator $\mathcal{R}_3$ is explicitly expressed in terms of the operator-valued gamma functions,

$$\begin{aligned}
S_3 \mathcal{R}_3 S_3^{-1} = \frac{\Gamma(z\partial_z + u_1 - v_3 + 1)}{\Gamma(z\partial_z + u_1 - u_3 + 1)} &\left(1 - \frac{v_3 - u_3}{v_3 - u_2}\theta_2 \partial_{\theta_2}\right) \\
&+ \frac{v_3 - u_3}{v_3 - u_2} \frac{\Gamma(z\partial_z + u_1 - v_3)}{\Gamma(z\partial_z + u_1 - u_3 + 1)} z \partial_{\theta_1} \partial_{\theta_2},
\end{aligned}$$

where the shift operator S_3 is defined by

$$S_3 = e^{\frac{\theta_1 \theta_2}{2} \partial_z} e^{\tilde{\theta}_1 (\partial_{\theta_1} + \frac{1}{2}\theta_2 \partial_z)} e^{\tilde{\theta}_2 (\partial_{\tilde{\theta}_2} + \frac{1}{2}\theta_1 \partial_z)} e^{\left(w + \frac{\tilde{\theta}_1 \tilde{\theta}_2}{2}\right)\partial_z}.$$

Proof. The construction of the operator $\mathcal{R}_3$ is defined by the following diagram:

$$L_1(u_1,u_2,u_3)L_2(v_1,v_2,v_3) \xrightarrow{\mathcal{S}_{1\tilde{3}}(u_1-v_3)} L_1(v_3,u_2,u_3)L_2(v_1,v_2,u_1)$$
$$\xrightarrow{\mathcal{D}_{13}(v_3,u_2,u_3)} L_1(u_3,u_2,v_3)L_2(v_1,v_2,u_1)$$
$$\xrightarrow{\mathcal{S}_{1\tilde{3}}(u_3-u_1)} L_1(u_1,u_2,v_3)L_2(v_1,v_2,u_3).$$

This diagram shows the expected commutation relation for the intertwining operators $\mathcal{S}_{1\tilde{3}}$ and $\mathcal{D}_{13}$, which results in the permutation $u_3 \rightleftarrows v_3$. The second formula for the operator $\mathcal{R}_3$ is defined by the diagram

$$L_1(u_1,u_2,u_3)L_2(v_1,v_2,v_3) \xrightarrow{\mathcal{D}_{13}(u_1,u_2,u_3)} L_1(u_3,u_2,u_1)L_2(v_1,v_2,v_3)$$
$$\xrightarrow{\mathcal{S}_{1\tilde{3}}(u_3-v_3)} L_1(v_3,u_2,u_1)L_2(v_1,v_2,u_3)$$
$$\xrightarrow{\mathcal{D}_{13}(v_3,u_2,u_1)} L_1(u_1,u_2,v_3)L_2(v_1,v_2,u_3).$$

Note that the shift operator S_3 is obtained from the operator S_1 by the simple change of the variables $z,\theta_1,\theta_2 \rightleftarrows w,\tilde{\theta}_1,\tilde{\theta}_2$, so that, due to an evident symmetry, one obtains

$$S_3\left(z-w+\frac{\theta_2\tilde{\theta}_1+\theta_1\tilde{\theta}_2}{2}+\frac{(\theta_1-\tilde{\theta}_1)(\theta_2-\tilde{\theta}_2)}{2}\right)S_3^{-1}=z,$$
$$S_3\left(\partial_{\theta_2}-\frac{1}{2}\theta_1\partial_z\right)S_3^{-1}=\partial_{\theta_2},$$
$$S_3\left(\partial_{\theta_1}-\frac{1}{2}\theta_2\partial_z\right)S_3^{-1}=\partial_{\theta_1}-\theta_2\partial_z\,.$$

All needed intertwining operators

$$\mathcal{D}_{13}(v_3,u_2,u_3)=\partial_z^{u_3-v_3}+\frac{v_3-u_3}{v_3-u_2}\,\partial_z^{u_3-v_3-1}\left(\partial_{\theta_1}-\frac{1}{2}\theta_2\partial_z\right)\left(\partial_{\theta_2}-\frac{1}{2}\theta_1\partial_z\right),$$
$$\mathcal{S}_{1\tilde{3}}(\alpha)=\left(z-w+\frac{\theta_2\tilde{\theta}_1+\theta_1\tilde{\theta}_2}{2}+\frac{(\theta_1-\tilde{\theta}_1)(\theta_2-\tilde{\theta}_2)}{2}\right)^{\alpha}$$

transform to a much simpler form, and we obtain

$$S_3\mathcal{R}_3S_3^{-1} = z^{u_3-u_1}\left(\partial_z^{u_3-v_3} + \frac{v_3-u_3}{v_3-u_2}\,\partial_z^{u_3-v_3-1}\left(\partial_{\theta_1} - \theta_2\partial_w\right)\partial_{\theta_2}\right) z^{u_1-v_3}$$
$$= z^{u_3-u_1}\left(\partial_z^{u_3-v_3}\left(1 - \frac{v_3-u_3}{v_3-u_2}\theta_2\partial_{\theta_2}\right) + \frac{v_3-u_3}{v_3-u_2}\,\partial_z^{u_3-v_3-1}\partial_{\theta_1}\partial_{\theta_2}\right) z^{u_1-v_3}.$$

It remains to transform this expression to a final form using the following formula:

$$z^{\beta}\partial_z^{\alpha+\beta}\,z^{\alpha} = \frac{\Gamma(z\partial_z+1+\alpha)}{\Gamma(z\partial_z+1-\beta)}\,.$$

The proof of an equivalence of the two representations for the operator $\mathcal{R}_3$ is very similar to that in the case of the operator $\mathcal{R}_1$. □

While the operators $\mathcal{R}_1$ and $\mathcal{R}_3$ replicate the well-known structure of the intertwiners in $\mathrm{SL}(2,\mathbb{C})$, where the key relation for understanding their integrability is the star-triangle relation (STR), the remaining operator $\mathcal{R}_2$ is based on fermionic-type relations discovered in [17].

Proposition 9. *The intertwining operator $\mathcal{R}_2$, defined by*

$$\mathcal{R}_2(u_1,u_2|v_2,v_3) = \mathcal{D}^{-1}_{\tilde{2}\tilde{3}}(v_3-u_2)\mathcal{D}^{-1}_{12}(u_1-v_2)\mathcal{S}_{2\tilde{2}}(v_2-u_2)$$
$$\times\mathcal{D}_{12}(u_1-u_2)\mathcal{D}_{\tilde{2}\tilde{3}}(v_3-v_2) \qquad (4.6)$$

satisfies the relation

$$\mathcal{R}_2\,L_1(u_1,u_2,u_3)\,L_2(v_1,v_2,v_3) = L_1(u_1,v_2,u_3)\,L_2(v_1,u_2,v_3)\,\mathcal{R}_2. \qquad (4.7)$$

The intertwining operator $\mathcal{R}_2$ can be explicitly expressed as a differential operator

$$\frac{(v_3-u_2)(u_1-v_2)}{u_2-v_2}\,S_2\mathcal{R}_2S_2^{-1} = \frac{(u_1-u_2)(v_3-v_2)}{u_2-v_2} + (v_3-v_2)\,\theta_2\partial_{\theta_2}$$
$$-(u_1-u_2)\,\tilde{\theta}_1\partial_{\tilde{\theta}_1} - (u_2-v_2)\tilde{\theta}_1\theta_2\partial_{\theta_2}\partial_{\tilde{\theta}_1} - (z-w+\theta_1\tilde{\theta}_2)\partial_{\theta_2}\partial_{\tilde{\theta}_1}, \qquad (4.8)$$

where the shift operator S_2 is defined by the formula

$$S_2 = e^{\tilde{\theta}_2\partial_{\theta_2}}e^{\theta_1\partial_{\tilde{\theta}_1}}e^{\frac{\theta_1\theta_2}{2}\partial_z}e^{-\frac{\tilde{\theta}_1\tilde{\theta}_2}{2}\partial_w}.$$

Proof. The construction of the operator $\mathcal{R}_2$ is defined by the following diagram:

$$
\begin{aligned}
&L_1(u_1,u_2,u_3)L_2(v_1,v_2,v_3)\\
&\xrightarrow{\mathcal{D}_{12}(u_1-u_2)\mathcal{D}_{\tilde{2}\tilde{3}}(v_3-v_2)} L_1^+(u_1-1,u_2+1,u_3)L_2^-(v_1,v_2-1,v_3+1)\\
&\xrightarrow{\mathcal{S}_{2\tilde{2}}(v_2-u_2)} L_1^+(u_1-1,v_2+1,u_3)L_2^-(v_1,u_2-1,v_3+1)\\
&\xrightarrow{\mathcal{D}^{-1}_{\tilde{2}\tilde{3}}(v_3-u_2)\mathcal{D}^{-1}_{12}(u_1-v_2)} L_1(u_1,v_2,v_3)L_2(v_1,u_2,u_3).
\end{aligned}
$$

We have

$$
\begin{aligned}
&(v_3-u_2)(u_1-v_2)\,\mathcal{R}_2\\
&\quad= e^{\frac{\tilde{\theta}_1\tilde{\theta}_2}{2}\partial_w} e^{-\frac{\theta_1\theta_2}{2}\partial_z}\left(\partial_{\tilde{\theta}_1}+(v_3-u_2)\tilde{\theta}_1\right)(\partial_{\theta_2}+(u_1-v_2)\theta_2)\\
&\qquad\times\left(1+\theta_2\tilde{\theta}_2+\theta_1\tilde{\theta}_1+(z-w)\theta_2\tilde{\theta}_1\right)^{v_2-u_2}\\
&\qquad\times(\partial_{\theta_2}+(u_1-u_2)\theta_2)\left(\partial_{\tilde{\theta}_1}+(v_3-v_2)\tilde{\theta}_1\right) e^{\frac{\theta_1\theta_2}{2}\partial_z} e^{-\frac{\tilde{\theta}_1\tilde{\theta}_2}{2}\partial_w}.
\end{aligned}
$$

In the next step we use the formulae

$$
\begin{aligned}
(1+\tilde{\theta}\theta)^{\lambda}\,(\partial_\theta+a\theta) &= (\partial_\theta+a\theta)\left(1-a^{-1}\tilde{\theta}\partial_\theta\right)^{\lambda},\\
(\partial_\theta+a\theta)(\partial_\theta+b\theta) &= b+(a-b)\theta\partial_\theta
\end{aligned}
$$

and rewrite everything in the following form:

$$
\begin{aligned}
&(v_3-u_2)(u_1-v_2)\,S_2\,\mathcal{R}_2\,S_2^{-1} = e^{\tilde{\theta}_2\partial_{\theta_2}}\,e^{\theta_1\partial_{\tilde{\theta}_1}}\\
&\quad\times(u_1-u_2+(u_2-v_2)\theta_2\partial_{\theta_2})\left(v_3-v_2-(u_2-v_2)\tilde{\theta}_1\partial_{\tilde{\theta}_1}\right)\\
&\times\left(1+\frac{1}{u_1-u_2}\tilde{\theta}_2\partial_{\theta_2}-\frac{1}{v_3-v_2}\theta_1\partial_{\tilde{\theta}_1}+\frac{z-w}{(u_1-u_2)(v_3-v_2)}\partial_{\theta_2}\partial_{\tilde{\theta}_1}\right)^{v_2-u_2}\\
&\qquad\times e^{-\tilde{\theta}_2\partial_{\theta_2}}e^{-\theta_1\partial_{\tilde{\theta}_1}}.
\end{aligned}
$$

Let us introduce a compact notation for a simplicity, $u=u_1-u_2\,, v=v_3-v_2\,,\lambda=u_2-v_2$. After expansion in a series we obtain

$$
\begin{aligned}
&\left(1+\frac{1}{u}\tilde{\theta}_2\partial_{\theta_2}-\frac{1}{v}\theta_1\partial_{\tilde{\theta}_1}+\frac{z-w}{uv}\partial_{\theta_2}\partial_{\tilde{\theta}_1}\right)^{-\lambda}\\
&\quad=1-\frac{\lambda}{u}\tilde{\theta}_2\partial_{\theta_2}+\frac{\lambda}{v}\theta_1\partial_{\tilde{\theta}_1}-\frac{\lambda(z-w)}{uv}\partial_{\theta_2}\partial_{\tilde{\theta}_1}+\frac{\lambda(\lambda+1)}{uv}\tilde{\theta}_2\theta_1\partial_{\theta_2}\partial_{\tilde{\theta}_1}
\end{aligned}
$$

and after a multiplication, one further concludes that

$$
\begin{aligned}
&(u+\lambda\theta_2\partial_{\theta_2})\left(v-\lambda\tilde{\theta}_1\partial_{\tilde{\theta}_1}\right)\\
&\quad\times\left(1-\frac{\lambda}{u}\tilde{\theta}_2\partial_{\theta_2}+\frac{\lambda}{v}\theta_1\partial_{\tilde{\theta}_1}-\frac{\lambda(z-w)}{uv}\partial_{\theta_2}\partial_{\tilde{\theta}_1}+\frac{\lambda(\lambda+1)}{uv}\tilde{\theta}_2\theta_1\partial_{\theta_2}\partial_{\tilde{\theta}_1}\right)\\
&=\left(uv+\lambda v\theta_2\partial_{\theta_2}-\lambda u\tilde{\theta}_1\partial_{\tilde{\theta}_1}+\lambda^2\theta_2\tilde{\theta}_1\partial_{\theta_2}\partial_{\tilde{\theta}_1}\right)+\left(-\lambda v\tilde{\theta}_2\partial_{\theta_2}-\lambda^2\tilde{\theta}_2\tilde{\theta}_1\partial_{\theta_2}\partial_{\tilde{\theta}_1}\right)\\
&\quad+\left(\lambda u\theta_1\partial_{\tilde{\theta}_1}+\lambda^2\theta_1\theta_2\partial_{\theta_2}\partial_{\tilde{\theta}_1}\right)-\lambda(z-w)\partial_{\theta_2}\partial_{\tilde{\theta}_1}+\lambda(\lambda+1)\tilde{\theta}_2\theta_1\partial_{\theta_2}\partial_{\tilde{\theta}_1}\\
&\quad=uv+\lambda v(\theta_2-\tilde{\theta}_2)\partial_{\theta_2}+\lambda u(\theta_1-\tilde{\theta}_1)\partial_{\tilde{\theta}_1}+\lambda^2(\theta_1-\tilde{\theta}_1)(\theta_2-\tilde{\theta}_2)\partial_{\theta_2}\partial_{\tilde{\theta}_1}\\
&\qquad\qquad-\lambda(z-w+\theta_1\tilde{\theta}_2)\partial_{\theta_2}\partial_{\tilde{\theta}_1}.
\end{aligned}
$$

The last similarity transformation is equivalent to a simple shift $\theta_2\to\theta_2+\tilde{\theta}_2\,,\tilde{\theta}_1\to\tilde{\theta}_1+\theta_1$, so that finally we obtain

$$
\begin{aligned}
&(v_3-u_2)(u_1-v_2)\,S_2\,\mathcal{R}_2\,S_2^{-1}\\
&\quad=uv+\lambda v\,\theta_2\partial_{\theta_2}-\lambda u\,\tilde{\theta}_1\partial_{\tilde{\theta}_1}-\lambda^2\tilde{\theta}_1\theta_2\partial_{\theta_2}\partial_{\tilde{\theta}_1}-\lambda(z-w+\theta_1\tilde{\theta}_2)\partial_{\theta_2}\partial_{\tilde{\theta}_1}\\
&\quad=\lambda\left(\frac{uv}{\lambda}+v\theta_2\partial_{\theta_2}-u\tilde{\theta}_1\partial_{\tilde{\theta}_1}-\lambda\tilde{\theta}_1\theta_2\partial_{\theta_2}\partial_{\tilde{\theta}_1}-(z-w+\theta_1\tilde{\theta}_2)\partial_{\theta_2}\partial_{\tilde{\theta}_1}\right).\quad\square
\end{aligned}
$$

§5. Conclusion

This work presents a novel approach to factorizing R-matrices invariant under the action of the superalgebra $\mathfrak{sl}(2|1)$. By applying representation-theoretic techniques we have specialized this approach to the first non-trivial case where both bosonic and fermionic degrees of freedom appear.

Significantly, the set of relations ensuring the integrability of a supersymmetric spin chain comprises both the generalized star-triangle relation and the fermionic relation discovered in [17]. Our formulae naturally reduce to the well-known ones, where they reproduce the operators $\mathcal{R}_i$ previously obtained in [4, 7, 8].

The next natural step is a generalization of the present approach to the general case of the superalgebra $\mathfrak{sl}(n|m)$ and, in particular, to representations of the supergroup $\mathrm{SL}(n|n)$, which have different structures, in comparison with the Verma modules.

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