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# FACTORIZATION OF THE R-MATRIX AND BAXTER Q-OPERATORS IN THE $\mathfrak{gl}(1|1)$ SPIN CHAIN

Abstract. We study the $\mathfrak{gl}(1|1)$-invariant R-matrix acting on the tensor product of Verma modules. Rather than solving the Yang-Baxter equation directly, we construct the R-matrix from elementary intertwining operators. Our analysis begins with a study of operators intertwining Verma modules. In contrast to the Lie algebra case, a new type of intertwining operator emerges in the super case, related to so-called odd reflections. We then extend our analysis to tensor products and introduce elementary intertwining operators that act by multiplication by a function. Using these intertwining operators, we construct the R-matrix as a product of two commuting operators. A consequence of this local factorization is the factorization of the transfer matrix into a product of Q-operators and the derivation of the TQ-relation.

**Dedicated to N. M. Bogoliubov**
**on the occasion of his 75th birthday**

## §1. Introduction

Integrable spin chains endowed with supergroup symmetries occupy a central role in diverse areas of theoretical physics, ranging from condensed matter systems [1] to the integrable structures of supersymmetric gauge theories [2–4].

Among these, models with GL(1|1) supergroup invariance have been the subject of extensive analysis [5]. A key result establishes that the GL(1|1)-invariant Wess–Zumino–Novikov–Witten model constitutes a logarithmic

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conformal field theory. The representation theory of GL(1|1) is instrumental in elucidating this logarithmic behavior. Concurrently, a profound connection between Yangian symmetry and integrability in conformal field theory has been uncovered [6].

The objective of the present work is to investigate $\mathfrak{gl}(1|1)$ invariant spin chain in a setting where both the auxiliary and quantum spaces are realized as non-degenerate Verma modules of Lie superalgebra $\mathfrak{gl}(1|1)$. We propose a method to generalize the approach developed for $\mathfrak{sl}(N, \mathbb{C})$ spin chains constructed from Verma modules [7–9] to the case of the superalgebra $\mathfrak{gl}(1|1)$. Within this framework, the solution to the quantum Yang–Baxter equation factorizes into a product of elementary intertwining operators that implement permutations of representation parameters. A direct consequence of this local factorization is the derivation of global relations connecting the $Q$-operators with the transfer matrix.

Previous results concerning the factorization of the R-matrix for the $\mathfrak{sl}(2|1)$ case were obtained through direct computation [10]. This factorization was subsequently used to construct Q-operators for the $\mathfrak{sl}(2|1)$ spin chain [11]. However, a decomposition into genuinely elementary operators remained elusive.

This article is structured as follows. In Section 2, we recount essential facts regarding the superalgebra $\mathfrak{gl}(1|1)$ and undertake a detailed study of the structure of its intertwining operators. In Section 3, we identify operators that intertwine representations with respect to the Yangian $Y(\mathfrak{gl}(1|1))$, a condition equivalent to satisfying certain relations with products of L-operators. We demonstrate that these operators are characterized by permutations of representation parameters and transformations of the Borel subalgebra. As a corollary, we construct R-operator as a product of two commuting operators. We conclude the section by considering global objects–namely, Baxter operators and transfer matrices. From the local factorization, we explicitly construct the Q-operators, compute their action, and prove the global factorization of the transfer matrix alongside the TQ relation.

## §2. Representations of $\mathfrak{gl}(1|1)$ and intertwining operators

Let us write out the basis of the superalgebra $\mathfrak{gl}(1|1)$:

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The first set of relations exactly repeats those of $\mathcal{N} = 2$ supersymmetric quantum mechanics

$$Q^2 = {Q^\dagger}^2 = 0, \quad [Q, H] = [Q^\dagger, H] = 0, \quad \{Q, Q^\dagger\} = H,$$

the second set defines the action of the Cartan subalgebra

$$[G, H] = 0, \ \ [G, Q] = -Q, \ \ [G, Q^\dagger] = Q^\dagger.$$

Another set of generators in this superalgebra is given by standard matrices $e_{ij}$:

$$[e_{ij}, e_{kl}] = \delta_{kj} e_{il} - (-1)^{([i]+[j])([k]+[l])} \delta_{il} e_{kj}.$$

The Casimir operators are defined by the [12]:

$$C_1 = \mathrm{str}\ E, \quad C_2 = \mathrm{str}\ E^2,$$

where the matrix $E = \sum\limits_{i,j=1}^{2} (-1)^{[j]} e_{ij} \otimes E_{ji} = \begin{pmatrix} E_{11} & E_{21} \\ E_{12} & -E_{22} \end{pmatrix}$.

**2.1. Verma modules.** Every irreducible finite-dimensional representation of $\mathfrak{gl}(1|1)$ can be obtained as a quotient of a Verma module by its maximal invariant subspace [13, 14]. A Verma module is constructed by induction from a Borel subalgebra. In contrast to semisimple Lie algebras, $\mathfrak{gl}(1|1)$ admits two non-conjugate Borel subalgebras. Namely, the subalgebras of upper-triangular and lower-triangular matrices [15,16]. As we shall see, reflection in the odd root of $\mathfrak{gl}(1|1)$ interchanges these two induction procedures.

We realize a holomorphic representation of the Verma module on the space of functions of a single Grassmann variable $\theta$. Depending on the choice of Borel subalgebra, we obtain the following realizations. For induction from the upper-triangular subalgebra $\mathfrak{b}_\varnothing$:

$$E^\varnothing_{11} = \theta\partial_\theta + l_1, \quad E^\varnothing_{22} = -\theta\partial_\theta + l_2, \quad E^\varnothing_{12} = -(l_1 + l_2)\theta, \quad E^\varnothing_{21} = -\partial_\theta.$$

For the lower-triangular subalgebra $\mathfrak{b}_\square$:

$$E^\square_{11} = -\theta\partial_\theta + l_2, \quad E^\square_{22} = \theta\partial_\theta + l_1, \quad E^\square_{12} = -\partial_\theta, \quad E^\square_{21} = -(l_1 + l_2)\theta.$$

We denote these representations by $V_\varnothing^{(l_1,l_2)}$ and $V_\square^{(l_1,l_2)}$, respectively. The notation $\varnothing$ and $\square$ are chosen to be consistent with the notation used in [15, 16] in the classification of non-conjugate Borel subalgebras.

**Proposition 1.** *There exists no nontrivial intertwining operator*

$$\mathcal{S} \colon V_\lambda^{(l_1',l_2')} \to V_\lambda^{(l_1,l_2)},$$

*where* $\lambda \in \{\varnothing, \Box\}$. *In particular, the representations* $V_\lambda^{(l_1',l_2')}$ *are indecomposable.*

**Proof.** Every operator could be realized in the form

$$(\mathcal{S}f)(\theta) = \int d\theta'\, K(\theta,\theta')f(\theta').$$

Then the intertwining conditions impose the following constraints on the kernel $K$

$$\begin{aligned} K(\theta,\theta') &= K(\theta-\varepsilon,\theta'-\varepsilon), \\ K(\lambda\theta,\lambda\theta') &= \lambda^{l_1'-l_1+1}K(\theta,\theta') = \lambda^{l_2-l_2'+1}K(\theta,\theta'), \\ K(\theta,\theta') &= (1-\varepsilon\theta')^{l_2'+l_1'}(1+\varepsilon\theta)^{l_2+l_1}K(\theta,\theta'). \end{aligned}$$

These relations require $l_1 = l_1'$, $l_2 = l_2'$, and furthermore imply

$$K(\theta,\theta') = C(\theta-\theta'),$$

whence the integral operator is proportional to the identity. □

In the representation theory of semisimple Lie algebras, a necessary condition for the existence of homomorphisms between Verma modules is that the corresponding highest weights be related by an element of the Weyl group. For the superalgebra $\mathfrak{gl}(1|1)$, the Weyl group is trivial, which would suggest the absence of such homomorphisms. However, for Lie superalgebras, the appropriate algebraic structure is not the conventional Weyl group but rather the Weyl groupoid, which incorporates odd reflections [13, 15, 16]. As we will demonstrate, the representations $V_\Box$ and $V_\varnothing$ are interrelated by a nontrivial intertwining operator that precisely corresponds to an odd reflection in this extended framework.

**Proposition 2.** *There exists a nontrivial intertwining operator*

$$\mathcal{S}_1 \colon V_\varnothing^{(l_1',l_2')} \to V_\Box^{(l_2'-1,l_1'+1)},$$

*given explicitly by*

$$\mathcal{S}_1 = \partial_\theta - (l_1'+l_2')\theta. \tag{1}$$

**Proof.** Proceeding analogously to Proposition 1, we reduce the problem to studying the properties of an integral kernel. The resulting constraints are

$$K(\theta,\theta') = (1+\varepsilon\theta')^{l_2'+l_1'}K(\theta-\varepsilon,\theta'),$$
$$K(\lambda\theta,\lambda^{-1}\theta') = \lambda^{l_2'-l_1+1}K(\theta,\theta') = \lambda^{l_2-l_1'-1}K(\theta,\theta'),$$
$$K(\theta,\theta') = (1-\varepsilon\theta)^{l_2+l_1}K(\theta,\theta'-\varepsilon).$$

Applying the first and last identities sequentially yields

$$K(\theta,\theta') = (1+\theta\theta')^{l_2'+l_1'}K(0,\theta') = (1+\theta\theta')^{l_2'+l_1'}K(0,0).$$

This kernel is invariant under the scaling $\theta \mapsto \lambda\theta$, $\theta' \mapsto \lambda^{-1}\theta'$. The scaling relation then implies

$$l_1 = l_2' - 1, \quad l_2 = l_1' + 1.$$

Rewriting the integral operator as a differential operator yields the expression (1). □

When $l_1 + l_2 = 0$, this intertwining operator degenerates to a derivative, and an invariant subspace generated by the constant function appears. Consequently, this representation constitutes a nontrivial extension of two one-dimensional $\mathfrak{gl}(1|1)$-modules; the Verma module itself is indecomposable.

For $l_1 + l_2 \neq 0$, the operator is invertible

$$(\partial_\theta - (l_1+l_2)\theta)^2 = -(l_1+l_2),$$

and the representation $V_\varnothing^{(l_1,l_2)}$ is irreducible.

**2.2. Tensor product of Verma modules.** It is known [17] that under the conditions $l_1 + l_2 \neq 0$, $l_1' + l_2' \neq 0$, $l_1 + l_2 + l_1' + l_2' \neq 0$, the tensor product decomposes into a direct sum

$$V_\varnothing^{(l_1,l_2)} \otimes V_\varnothing^{(l_1',l_2')} = V_\varnothing^{(l_1+l_1',l_2+l_2')} \oplus V_\varnothing^{(l_1+l_1'+1,l_2+l_2'-1)}. \quad (2)$$

Let the polynomials in the first space depend on $\theta_1$ and those in the second on $\theta_2$. The space of the tensor product is identified with polynomials in two Grassmann variables $\theta_1$ and $\theta_2$. The generators of the representation act as a sum of differential operators correspinding to element of $\mathfrak{gl}(1|1)$ in each tensor multiplier.

The Casimir operators act as follows:

$$\begin{aligned}C_1 &= l_1 + l_2 + l_1' + l_2',\\ C_2 &= -2(l_1' + l_2')(\theta_2 - \theta_1)\partial_{\theta_1} + 2(l_1 + l_2)(\theta_2 - \theta_1)\partial_{\theta_2}\\ &\quad + (l_1 + l_2 + l_1' + l_2')(l_1' + l_1' - l_2 - l_2' + 1).\end{aligned}$$

There are two eigenspaces of $C_2$

$$\begin{aligned}V_{\varnothing}^{(l_1+l_1',l_2+l_2')} &= \langle 1,\ (l_1 + l_2)\theta_1 + (l_1' + l_2')\theta_2\rangle\,,\\ V_{\varnothing}^{(l_1+l_1'+1,l_2+l_2'-1)} &= \langle \theta_1 - \theta_2,\ (l_1 + l_2 + l_1' + l_2')\theta_1\theta_2\rangle\,.\end{aligned}$$

In the degenerate case $l_1+l_2+l_1'+l_2' = 0$, both Casimir operators possess zero eigenvalues, and $C_2$ becomes non-diagonalizable. Such representations are denoted $\mathcal{P}(l_2 + l_2' - 1)$ and realize a nontrivial extension

$$0 \longrightarrow V_{\varnothing}^{(-l_2-l_2',l_2+l_2')} \longrightarrow \mathcal{P}(l_2 + l_2' - 1) \longrightarrow V_{\varnothing}^{(-l_2-l_2'+1,l_2+l_2'-1)} \longrightarrow 0.$$

It can be shown that $\mathcal{P}(l_2+l_2'-1)$ admits no further nontrivial extensions.

Not all indecomposable representations are captured by this description; there exists a series of indecomposable reducible representations of unbounded dimension. However, representations on Verma modules and $\mathcal{P}(h)$ possess the crucial property that their tensor products decompose into direct sums of Verma modules and $\mathcal{P}(h)$; see [17] for details.

From the decomposition (2), we observe that the tensor product depends only on the sums $l_1 + l_1'$ and $l_2 + l_2'$. This implies the existence of an intertwining operator

$$V_{\varnothing}^{(l_1,l_2)} \otimes V_{\varnothing}^{(l_1',l_2')} \to V_{\varnothing}^{(l_1-u,l_2+v)} \otimes V_{\varnothing}^{(l_1'+u,l_2'-v)}$$

for any $u, v \in \mathbb{C}$. This operator is simply a function of Casimir operator $C_2$.

The action of the Casimir operator simplifies considerably when we switch from tensor products of similarly induced representations to those induced from different subalgebras.

**Proposition 3.** *There exists an operator intertwining the tensor product*

$$\mathcal{S}_2^{(1)}(u)\colon V_{\varnothing}^{(l_1,l_2)} \otimes V_{\square}^{(l_1',l_2')} \to V_{\emptyset}^{(l_1-u,l_2)} \otimes V_{\square}^{(l_1',l_2'+u)}, \tag{3}$$

*which acts by multiplication*

$$f(\theta_1, \theta_2) \mapsto (1 + \theta_2\theta_1)^u f(\theta_1, \theta_2).$$

**Proof.** Differentiation yields

$$\partial_{\theta_1}(1+\theta_2\theta_1)^u = -u\theta_2(1+\theta_2\theta_1)^{u-1}, \quad \partial_{\theta_2}(1+\theta_2\theta_1)^u = u\theta_1(1+\theta_2\theta_1)^{u-1}.$$

Acting with the representation generators on $(1+\theta_2\theta_1)^u$ modifies the weights $l_1, l_2$ in accordance with formula (3). □

## §3. The invariant R-matrix

**3.1. Interwiners of Yangian.** We construct the L-operator using the standard formula [10, 18]:

$$L(u) = u + \sum_{i,j}(-1)^{[j]} e_{ij} \otimes E_{ji} = \begin{pmatrix} u + E_{11} & E_{21} \\ E_{12} & u - E_{22} \end{pmatrix}.$$

It is convenient to introduce the variables

$$u_1 = u + l_1, \quad u_2 = u - l_2, \quad v_1 = v + l'_1, \quad v_2 = v - l'_2.$$

The L-operators acting in $V_{\varnothing}^{(l_1,l_2)}$ and $V_{\varnothing}^{(l'_1,l'_2)}$ are then given by

$$L_1^{\varnothing}(u_1,u_2) = \begin{pmatrix} u_1 + \theta_1\partial_{\theta_1} & -\partial_{\theta_1} \\ (u_2-u_1)\theta_1 & u_2 + \theta_1\partial_{\theta_1} \end{pmatrix},$$

$$L_2^{\varnothing}(v_1,v_2) = \begin{pmatrix} v_1 + \theta_2\partial_{\theta_2} & -\partial_{\theta_2} \\ (v_2-v_1)\theta_2 & v_2 + \theta_2\partial_{\theta_2} \end{pmatrix}.$$

Define extensions of the intertwining operator (1):

$$\mathcal{S}_1(u_2-u_1) = S_1(u_2-u_1)\otimes 1, \quad \mathcal{S}_{1'}(v_2-v_1) = 1\otimes S_1(v_2-v_1).$$

Conjugating the L-operators by $\mathcal{S}_1$ or $\mathcal{S}_{1'}$ yields L-operators in the $V_{\square}$ representations

$$L_2^{\square}(v_2,v_1) = \mathcal{S}_{1'}(v_2-v_1)L_2^{\varnothing}(v_1,v_2)\mathcal{S}_{1'}(v_2-v_1)^{-1} = \begin{pmatrix} v_1+1-\theta_2\partial_{\theta_2} & (v_2-v_1)\theta_2 \\ -\partial_{\theta_2} & v_2+1-\theta_2\partial_{\theta_2} \end{pmatrix},$$

$$L_1^{\square}(u_2,u_1) = \mathcal{S}_1(u_2-u_1)L_1^{\varnothing}(u_1,u_2)\mathcal{S}_1(u_2-u_1)^{-1} = \begin{pmatrix} u_1+1-\theta_1\partial_{\theta_1} & (u_2-u_1)\theta_1 \\ -\partial_{\theta_1} & u_2+1-\theta_1\partial_{\theta_1} \end{pmatrix}.$$

**Proposition 4.** *The operator* $\mathcal{S}_2^{(1)}(u_1-v_1)$ (3) *satisfies the relation*

$$\mathcal{S}_2^{(1)}(u_1-v_1)L_1^{\varnothing}(u_1,u_2)L_2^{\square}(v_2,v_1) = L_1^{\varnothing}(v_1,u_2)L_2^{\square}(v_2,u_1)\mathcal{S}_2^{(1)}(u_1-v_1). \quad (4)$$

**Proof.** Conjugating the left-hand side of (4) by $\mathcal{S}_2(u_1-v_1)$ (whose inverse is $\mathcal{S}_2^{(1)}(v_1-u_1)$), we find

$$\mathcal{S}_2^{(1)} L_1^{\varnothing}(u_1,u_2)(\mathcal{S}_2^{(1)})^{-1} = L_1^{\varnothing}(v_1,u_2) - \frac{v_1-u_1}{1+\theta_2\theta_1}\begin{pmatrix}-\theta_1\\ 1\end{pmatrix}\otimes\begin{pmatrix}-\theta_2 & 1\end{pmatrix},$$

$$\mathcal{S}_2^{(1)} L_2^{\square}(v_2,v_1)(\mathcal{S}_2^{(1)})^{-1} = L_2^{\square}(v_2,u_1) + \frac{v_1-u_1}{1+\theta_2\theta_1}\begin{pmatrix}-\theta_1\\ 1\end{pmatrix}\otimes\begin{pmatrix}-\theta_2 & 1\end{pmatrix}.$$

Multiplying these expressions, the cross terms cancel, yielding the right-hand side of (4):

$$\begin{aligned}&\mathcal{S}_2^{(1)} L_1^{\varnothing}(u_1,u_2)(\mathcal{S}_2^{(1)})^{-1}\cdot\mathcal{S}_2^{(1)} L_2^{\square}(u_1,u_2)(\mathcal{S}_2^{(1)})^{-1}\\ &\quad= L_1^{\varnothing}(v_1,u_2)L_2^{\square}(v_2,u_1) + \left(\frac{(v_1-u_1)^2}{1+\theta_2\theta_1} - \frac{(v_1-u_1)^2}{1+\theta_2\theta_1}\right)\begin{pmatrix}-\theta_1\\ 1\end{pmatrix}\otimes(1-\theta_2)\\ &\quad= L_1^{\varnothing}(v_1,u_2)L_2^{\square}(v_2,u_1).\end{aligned}$$

□

A similar argument establishes the following proposition.

**Proposition 5.** *The operator* $\mathcal{S}_2^{(2)}(v_2-u_2)$*, defined by the mapping*

$$\mathcal{S}_2^{(2)}(u)\colon\; V_{\square}^{(l_2-1,l_1+1)}\otimes V_{\varnothing}^{(l_1,l_2)} \to V_{\square}^{(l_2-1-u,l_1)}\otimes V_{\varnothing}^{(l_1',l_2'+u)},$$

*and realized as the multiplicative action specified in* (3)*, satisfies the intertwining relation*

$$\mathcal{S}_2^{(2)}(v_2-u_2)L_1^{\square}(u_2,u_1)L_2^{\varnothing}(v_1,v_2) = L_1^{\square}(v_2,u_1)L_2^{\varnothing}(v_1,u_2)\mathcal{S}_2^{(2)}(v_2-u_2).$$

These operators implement parameter permutations between tensor factors while preserving the number of 'fermionic' parameters in both tensor factors. At the same time, no operators permutes parameters $u_1$ with $v_2$ or $u_2$ with $v_1$.

Propositions 4 and 5 allow the construction of operators implementing permutations $(u_i \leftrightarrow v_i)$ without altering the subalgebra.

**Proposition 6.** *The operator* $\mathcal{R}_1(u_1|v_1,v_2)$

$$\mathcal{R}_1\colon V_{\varnothing}^{(l_1,l_2)}\otimes V_{\varnothing}^{(l_1',l_2')} \to V_{\varnothing}^{(l_1'-u+v,l_2)}\otimes V_{\varnothing}^{(l_1+u-v,l_2')},$$

*defined by*

$$\begin{aligned}\mathcal{R}_1(u_1|v_1,v_2) &= \frac{1}{v_2-v_1}\mathcal{S}_{1'}(v_2-u_1)\mathcal{S}_2^{(1)}(u_1-v_1)\mathcal{S}_{1'}(v_2-v_1)\\ &= 1+\frac{u_1-v_1}{v_2-v_1}(\theta_1-\theta_2)\partial_{\theta_2},\end{aligned}$$

*satisfies the relation*

$$\mathcal{R}_1 L_1^{\varnothing}(u_1,u_2)L_2^{\varnothing}(v_1,v_2) = L_1^{\varnothing}(v_1,u_2)L_2^{\varnothing}(u_1,v_2)\mathcal{R}_1. \tag{5}$$

We employ the notation $\mathcal{R}_1(u) = \mathcal{R}_1(u_1|v_1,u_2)|_{v_1=0}$. This operator admits an integral representation

$$\begin{aligned}
&\mathcal{R}_1 f(\theta_1,\theta_2)\\
&=\frac{1}{v_2-v_1}\int d\theta_2' d\theta_2''(1+\theta_2'\theta_2)^{v_2-u_1}(1+\theta_2'\theta_1)^{u_1-v_1}(1+\theta_2''\theta_2')^{v_2-v_1}f(\theta_1,\theta_2'')\\
&=\frac{1}{v_2-v_1}\int d\theta_2''\,[\theta_2''(v_2-v_1)-\theta_2(v_2-u_1)-\theta_1(u_1-v_1)]\,f(\theta_1,\theta_2''). \qquad (6)
\end{aligned}$$

**Proposition 7.** *The operator* $\mathcal{R}_2(u_1,u_2|v_2)$

$$\mathcal{R}_2\colon V_{\varnothing}^{(l_1,l_2)}\otimes V_{\varnothing}^{(l_1',l_2')}\to V_{\varnothing}^{(l_1,l_2'+v-u)}\otimes V_{\varnothing}^{(l_1',l_2+u-v)},$$

*defined by*

$$\begin{aligned}
\mathcal{R}_2(u_1,u_2|v_2) &= \frac{1}{u_2-u_1}\mathcal{S}_1(v_2-u_1)\mathcal{S}_2^{(2)}(v_2-u_2)\mathcal{S}_1(u_2-u_1)\\
&= 1+\frac{v_2-u_2}{u_2-u_1}(\theta_1-\theta_2)\partial_{\theta_1},
\end{aligned}$$

*satisfies the relation*

$$\mathcal{R}_2 L_1^{\varnothing}(u_1,u_2)L_2^{\varnothing}(v_1,v_2) = L_1^{\varnothing}(u_1,v_2)L_2^{\varnothing}(v_1,u_2)\mathcal{R}_2. \tag{7}$$

We use the notation $\mathcal{R}_2(u) = \mathcal{R}_2(u_1,u_2|v_2)|_{v_2=0}$. Its integral representation is

$$\begin{aligned}
&\mathcal{R}_2 f(\theta_1,\theta_2)\\
&=\frac{1}{u_2-u_1}\int d\theta_1' d\theta_1''(1+\theta_1'\theta_1)^{v_2-u_1}(1+\theta_1'\theta_2)^{u_2-v_2}(1+\theta_1''\theta_1')^{u_2-u_1}f(\theta_1'',\theta_2)\\
&=\frac{1}{u_2-u_1}\int d\theta_1''\,[\theta_1''(u_2-u_1)+\theta_2(v_2-u_2)-\theta_1(v_2-u_1)]\,f(\theta_1'',\theta_2). \qquad (8)
\end{aligned}$$

**Proposition 8.** *The operators* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *commute*

$$\mathcal{R}_2(v_1,u_2|v_2)\mathcal{R}_1(u_1|v_1,v_2) = \mathcal{R}_1(u_1|v_1,u_2)\mathcal{R}_2(u_1,u_2|v_2).$$

**Proof.** This can be verified by direct computation. On a conceptual level, the normalization $\mathcal{R}_1|_{u_1=v_1} = \mathcal{R}_2|_{u_2=v_2} = 1$ is fixed. Both $\mathcal{R}_1\mathcal{R}_2$ and $\mathcal{R}_2\mathcal{R}_1$ satisfy the same relations with the L-operators and share this normalization. Since operators commuting with $L_1^{\varnothing}(u_1,u_2)L_2^{\varnothing}(v_1,v_2)$ are proportional to the identity, they must coincide. □

**3.2. Factorization of the R-matrix.** To compute the R-matrix acting on $V_\varnothing^{(l_1,l_2)} \otimes V_\varnothing^{(l'_1,l'_2)}$, we solve the RLL-relation:

$$\mathrm{R}(u-v)L_1^\varnothing(u_1,u_2)L_2^\varnothing(v_1,v_2) = L_2^\varnothing(v_1,v_2)L_1^\varnothing(u_1,u_2)\mathrm{R}(u-v).$$

Introduce the permutation operator

$$\Pi_{12} : f(\theta_1,\theta_2) \mapsto f(\theta_2,\theta_1).$$

The operator $\mathcal{R}' = \Pi_{12}\mathrm{R}$ then satisfies:

$$\mathcal{R}' L_1^\varnothing(u_1,u_2)L_2^\varnothing(v_1,v_2) = L_1^\varnothing(v_1,v_2)L_2^\varnothing(u_1,u_2)\mathcal{R}'.$$

The solution to this relation is the product $\mathcal{R}_1\mathcal{R}_2$.

**Proposition 9.** *The R-matrix factorizes into a product of three operators*

$$\mathrm{R}(u-v) = \Pi_{12}\mathcal{R}_1(u_1|v_1,u_2)\mathcal{R}_2(u_1,u_2|v_2).$$

*In compact notation*

$$\mathrm{R}(u) = \Pi_{12}\mathcal{R}_1(u-l_1)\mathcal{R}_2(u+l_2).$$

The matrix elements in the basis $\{1,\theta_1,\theta_2,\theta_1\theta_2\}$ are:

$$\mathrm{R}(u-v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u-v+l_1-l'_1}{u-v-l_2-l'_1} & \frac{-l_1-l_2}{u-v-l_2-l'_1} & 0 \\ 0 & \frac{-l'_1-l'_2}{u-v-l_2-l'_1} & \frac{u-v+l'_2-l_2}{u-v-l_2-l'_1} & 0 \\ 0 & 0 & 0 & \frac{u-v+l'_2+l_1}{u-v-l_2-l'_1} \end{pmatrix}.$$

The fundamental representation corresponds to $l_1 = l'_1 = 0$, $l_2 = l'_2 = -1$. In this case, the expression is proportional to the Yang R-matrix.

Introduce the operators:

$$\begin{aligned} \mathrm{R}^1_{12}(u_1|v_1,v_2) &= \Pi_{12}\mathcal{R}_1(u_1|v_1,v_2), \\ \mathrm{R}^2_{12}(u_1,u_2|v_2) &= \Pi_{12}\mathcal{R}_2(u_1,u_2|v_2), \\ \mathrm{R}_{12}(u_1,u_2|v_1,v_2) = \mathrm{R}(u-v) &= \mathrm{R}^1_{12}(u_1|v_1,u_2)\Pi_{12}\mathrm{R}^2_{12}(u_1,u_2|v_2), \end{aligned}$$

where the subscripts denote the numbers of spaces in which the operators act. For special parameter values, the R-matrix degenerates:

$$\begin{aligned} \mathrm{R}_{12}(u_1,u_2|v_1,v_2)|_{u_1=v_1} &= \mathrm{R}^2_{12}(u_1,u_2|v_2), \\ \mathrm{R}_{12}(u_1,u_2|v_1,v_2)|_{u_2=v_2} &= \mathrm{R}^1_{12}(u_1|v_1,u_2), \\ \mathrm{R}^1_{12}(u_1|v_1,v_2)|_{u_1=v_1} &= \Pi_{12}, \\ \mathrm{R}^2_{12}(u_1,u_2|v_2)|_{u_2=v_2} &= \Pi_{12}. \end{aligned} \tag{9}$$

Additional relations arise in the triple tensor product $V_\varnothing^{(l_1,l_2)} \otimes V_\varnothing^{(l_1',l_2')} \otimes V_\varnothing^{(l_1'',l_2'')}$ with spectral parameters $u, v, w$. Operators normalized identically and permuting parameters in the following triple product [7]

$$L_1^\varnothing(u_1,u_2)L_2^\varnothing(v_1,v_2)L_3^\varnothing(w_1,w_2)$$

in a consistent manner are found to be equal.

From this general principle, one may derive triple relations for the operators introduced previously. The primary consequence is the Yang–Baxter equation,

$$\begin{aligned} &\mathrm{R}_{23}(v_1,v_2|w_1,w_2)\mathrm{R}_{13}(u_1,u_2|w_1,w_2)\mathrm{R}_{12}(u_1,u_2|v_1,v_2) \\ &\quad = \mathrm{R}_{12}(u_1,u_2|v_1,v_2)\mathrm{R}_{13}(u_1,u_2|w_1,w_2)\mathrm{R}_{23}(v_1,v_2|w_1,w_2), \end{aligned}$$

which emerges as a consistency condition between the two distinct representations of the permutation

$$L_1^\varnothing(u_1,u_2)L_2^\varnothing(v_1,v_2)L_3^\varnothing(w_1,w_2) \to L_3^\varnothing(w_1,w_2)L_2^\varnothing(v_1,v_2)L_1^\varnothing(u_1,u_2).$$

Furthermore, the following relations involving the operator $\mathrm{R}^2_{12}(u_1,u_2|v_2)$ hold

$$\begin{aligned} &\mathrm{R}^2_{23}(v_1,v_2|w_2)\mathrm{R}_{13}(u_1,u_2|w_1,w_2)\mathrm{R}_{12}(u_1,u_2|v_1,v_2) \\ &\quad = \mathrm{R}_{12}(u_1,u_2|w_1,w_2)\mathrm{R}_{13}(u_1,u_2|w_1,v_2)\mathrm{R}^2_{23}(v_1,v_2|w_2), \end{aligned} \tag{10}$$

as a consequence of the two representations of the permutation and relation (7)

$$L_1^\varnothing(u_1,u_2)L_2^\varnothing(v_1,v_2)L_3^\varnothing(w_1,w_2) \to L_3^\varnothing(v_1,w_2)L_2^\varnothing(w_1,v_2)L_1^\varnothing(u_1,u_2).$$

Analogously, with the operator $\mathrm{R}^1_{12}(u_1|v_1,v_2)$, we have

$$\begin{aligned} &\mathrm{R}^1_{23}(v_1|w_1,w_2)\mathrm{R}_{13}(u_1,u_2|w_1,w_2)\mathrm{R}_{12}(u_1,u_2|v_1,v_2) \\ &\quad = \mathrm{R}_{12}(u_1,u_2|w_1,v_2)\mathrm{R}_{13}(u_1,u_2|v_1,w_2)\mathrm{R}^1_{23}(v_1|w_1,w_2), \end{aligned} \tag{11}$$

stemming from the permutation and relation (5)

$$L_1^{\varnothing}(u_1,u_2)L_2^{\varnothing}(v_1,v_2)L_3^{\varnothing}(w_1,w_2)\to L_3^{\varnothing}(w_1,v_2)L_2^{\varnothing}(v_1,w_2)L_1^{\varnothing}(u_1,u_2).$$

**3.3. Global objects: transfer matrix and Q-operators.** The factorization structure permits a natural extension to the computation of global objects, namely the transfer matrix and the $Q$-operators.

Consider the tensor product of typical representations ($l_1+l_2\neq 0$); we denote $V_{\varnothing}^{(l_1,l_2)}$ simply by $V^{(l_1,l_2)}$ (and the same for L-operators — we will use the notation $L(u_1,u_2)$ instead of $L^{\varnothing}(u_1,u_2)$).

The transfer matrix is defined by

$$T_{l'}(u_1,u_2|v_1,v_2)=\mathrm{str}_{V_0}\left[\mathrm{R}_{10}(u_1,u_2|v_1,v_2)\cdots\mathrm{R}_{n0}(u_1,u_2|v_1,v_2)\right].$$

We introduce the operators $Q_1$ and $Q_2$ as follows:

$$Q_1(u_1|v_1,u_2)=\mathrm{str}\left[\mathrm{R}^1_{10}(u_1|v_1,u_2)\cdots\mathrm{R}^1_{n0}(u_1|v_1,u_2)\right],$$
$$Q_2(u_1,u_2|v_2)=\mathrm{str}\left[\mathrm{R}^2_{10}(u_1,u_2|v_2)\cdots\mathrm{R}^2_{n0}(u_1,u_2|v_2)\right].$$

We will also employ the abbreviated notation

$$T_{l'}(u)=T_{l'}(u_1,u_2|v_1,v_2)|_{v=0},$$
$$Q_1(u)=Q_1(u_1|v_1,v_2)|_{v_1=0},\quad Q_2(u)=Q_2(u_1,u_2|v_2)|_{v_2=0}.$$

**Proposition 10.** *The operator $Q_1(u_1|v_1,u_2)$ acts as follows*

$$Q_1(u_1|v_1,u_2)F(\theta_1,\ldots,\theta_n)=\frac{(u_2-v_1)^n-(u_1-v_1)^n}{(u_2-v_1)^n}F\left(\widetilde{\theta}_1\ldots,\widetilde{\theta}_n\right),\quad (12)$$

*where*

$$\widetilde{\theta}_k=\frac{u_1-u_2}{(u_1-v_1)^n-(u_2-v_1)^n}\sum_{i=0}^{n-1}(u_1-v_1)^{n-1-i}(u_2-v_1)^i\,\theta_{i+k-1}$$

*and the indices are identified periodically: $\theta_{a+n}\sim\theta_a$.*

**Proof.** The explicit form of the operator $\mathrm{R}^1_{k0}(u_1|v_1,u_2)$ is known (6):

$$\mathrm{R}^1_{k0}(u_1|v_1,u_2)f(\theta_k,\theta_0)$$
$$=\int\frac{d\theta'_k d\theta'_0}{u_2-v_1}(\theta'_k-\theta_0)\left(\theta'_0(u_2-v_1)+\theta_k(u_1-u_2)-\theta_0(u_1-v_1)\right)f(\theta'_k,\theta'_0).$$

The kernel of the operator $Q_1(u_1|v_1, u_2)$ is therefore given by

$$\int \frac{d\theta_0^{(0)} d\theta_0^{(1)} \cdots d\theta_0^{(n)}}{(u_2 - v_1)^n} (\theta_0^{(0)} - \theta_0^{(n)})$$
$$\times \prod_{k=1}^{n} (\theta_0^{(k-1)} - \theta_k') \left( \theta_0^{(k)}(u_2 - v_1) + \theta_k(u_1 - u_2) - \theta_0^{(k-1)}(u_1 - v_1) \right).$$

This results in the following expression for the kernel of $Q_1$:

$$\frac{1}{(u_2 - v_1)^n} \prod_{k=1}^{n} \left( \theta_{k+1}'(u_2 - v_1) + \theta_k(u_1 - u_2) - \theta_k'(u_1 - v_1) \right),$$

with the periodic condition $\theta_{n+1}' = \theta_1'$. Solving the resulting linear system for $\theta_k'$ yields

$$\theta_k' = \frac{u_1 - u_2}{(u_1 - v_1)^n - (u_2 - v_1)^n} \sum_{i=0}^{n-1} (u_1 - v_1)^{n-1-i} (u_2 - v_1)^i \, \theta_{i+k-1},$$

with $\theta_{a+n} \sim \theta_a$. Incorporating all normalization factors yields the operator action specified in equation (12). □

**Proposition 11.** *The operator $Q_2(u_1, u_2|v_2)$ acts as follows:*

$$Q_2(u_1, u_2|v_2) F(\theta_1, \ldots, \theta_n)$$
$$= F\left( \frac{\theta_1(u_2 - v_2) + \theta_n(v_2 - u_1)}{u_2 - u_1}, \ldots, \frac{\theta_n(u_2 - v_2) + \theta_{n-1}(v_2 - u_1)}{u_2 - u_1} \right). \tag{13}$$

**Proof.** The operator $\mathrm{R}_{k0}^2(u_1, u_2|v_2)$ acts according to (8):

$$\mathrm{R}_{k0}^2 f(\theta_k, \theta_0)$$
$$= \int \frac{d\theta_k' d\theta_0'}{u_2 - u_1} (\theta_0' - \theta_k) \left( \theta_k'(u_2 - u_1) - \theta_0(v_2 - u_1) + \theta_k(v_2 - u_2) \right) f(\theta_k', \theta_0').$$

The kernel of the operator $Q_2$ is then

$$\int \frac{d\theta_0^{(0)} d\theta_0^{(1)} \cdots d\theta_0^{(n)}}{(u_2 - u_1)^n} (\theta_0^{(0)} - \theta_0^{(n)})$$
$$\times \prod_{k=1}^{n} (\theta_0^{(k)} - \theta_k) \left( \theta_k'(u_2 - u_1) - \theta_k(u_2 - v_2) - \theta_0^{(k-1)}(v_2 - u_1) \right).$$

Evaluating the integrals over $\theta_0^{(i)}$ gives

$$\frac{1}{(u_2-u_1)^n}\prod_{k=1}^{n}\left(\theta'_k(u_2-u_1)-\theta_k(u_2-v_2)-\theta_{k-1}(v_2-u_1)\right),$$

with the periodic condition $\theta_0=\theta_n$. The integral operator with this kernel acts as specified in equation (13). □

**Proposition 12.** *The operators $T_{l'}(u)$, $Q_1(u)$, and $Q_2(u)$ satisfy the commutation relations*

$$[T_{l'}(u),Q_i(v)]=[Q_i(u),Q_j(v)]=0 \tag{14}$$

*for any $i,j\in\{1,2\}$.*
*Furthermore, the transfer matrix factorizes as follows*:

$$T_{l'}(u_1,u_2|v_1,v_2)=\Pi\,Q_1(u_1|v_1,u_2)\,Q_2(u_1,u_2|v_2), \tag{15}$$

*where $\Pi=\Pi_{12}\cdots\Pi_{n1}$ is the cyclic shift operator. In a more compact notation, this formula becomes*

$$T_{l'}(u)=\Pi\,Q_1(u-l_1)Q_2(u+l_2). \tag{16}$$

**Proof.** It is a standard result that transfer matrices constructed from different auxiliary spaces $V_0^{(l'_1,l'_2)}$ and $V_{0'}^{(l''_1,l''_2)}$ commute for all values of the spectral parameter:

$$[T_{l'}(u),T_{l''}(v)]=0. \tag{17}$$

We define parameters $l''_1,l''_2$ such that the transfer matrix $T_{l''}(v)$ degenerates into a Q-operator (9). For the degeneration to $Q_2$, we set

$$T_{l''}(u_1,u_2|u_1,v_2)=Q_2(u_1,u_2|v_2),\qquad l''_1=u-v+l_1.$$

For the degeneration to $Q_1$, we set

$$T_{l''}(u_1,u_2|u_1,v_2)=Q_1(u_1|v_1,u_2),\qquad l''_2=l_2-u+v.$$

Equation (14) then follows directly from (17).

We now turn to the factorization relation (15). Consider equation (10):

$$\begin{aligned}&\mathrm{R}^2_{00'}(v_1,v_2|w_2)\mathrm{R}_{k0'}(u_1,u_2|w_1,w_2)\mathrm{R}_{k0}(u_1,u_2|v_1,v_2)\\&\qquad=\mathrm{R}_{k0}(u_1,u_2|v_1,w_2)\mathrm{R}_{k0'}(u_1,u_2|w_1,v_2)\mathrm{R}^2_{00'}(v_1,v_2|w_2).\end{aligned}$$

Substitute $w_1 = u_1$, $w_2 = u_2$ to obtain:

$$\mathrm{R}^2_{00'}(v_1|u_1,u_2)\Pi_{k0'}\mathrm{R}_{k0}(u_1,u_2|v_1,v_2) = \mathrm{R}^1_{k0}(u_1|v_1,u_2)\mathrm{R}^2_{k0'}(u_1,u_2|v_2)\mathrm{R}^1_{00'}(v_1|u_1,u_2).$$

Promoting this to a global relation and taking supertraces yields:

$$\begin{aligned}\mathrm{str}_{V_{0'}}\left[\Pi_{10'}\cdots\Pi_{n0'}\right]\mathrm{str}_{V_0}\left[\mathrm{R}_{10}(u_1,u_2|v_1,v_2)\cdots\mathrm{R}_{n0}(u_1,u_2|v_1,v_2)\right] \\ = \mathrm{str}_{V_0}\left[\mathrm{R}^1_{10}(u_1|v_1,u_2)\cdots\mathrm{R}^1_{n0}(u_1|v_1,u_2)\right] \\ \times\,\mathrm{str}_{V_{0'}}\left[\mathrm{R}^2_{10'}(u_1,u_2|v_2)\cdots\mathrm{R}^2_{n0'}(u_1,u_2|v_2)\right].\end{aligned}$$

The first factor on the left-hand side is identically the cyclic shift operator $\Pi$, which completes the proof. □

**Proposition 13.** *The operators $Q_i$ satisfy the TQ-relations*:

$$u_2^n Q_1(u)t(u) = (u_1^n - u_2^n)(u_2-1)^n Q_1(u-1), \tag{18}$$

$$Q_2(u)t(u) = (u_1^n - u_2^n)Q_2(u+1), \tag{19}$$

*where $t(u)$ denotes the fundamental transfer matrix (i.e., the auxiliary space is the fundamental representation).*

**Proof.** We follow the method outlined in [7, 8]. A direct computation shows:

$$\left.\begin{pmatrix}1 & 0\\ \theta_1 & 1\end{pmatrix}\mathcal{R}_2(u_1,u_2|v_2)L_1(u_1,u_2)\begin{pmatrix}1 & 0\\ -\theta_2 & 1\end{pmatrix}\right|_{v_2=0} = \begin{pmatrix}u_1\mathcal{R}_2(u+1) & -\mathcal{R}_2(u)\partial_{\theta_2}\\ 0 & u_2\mathcal{R}_2(u+1)\end{pmatrix}.$$

Interpreting $V_1$ as the $k$-th site and $V_2 = V_0$ as the auxiliary space, we derive a relation for $\mathrm{R}^2_{k0}(u)$:

$$\mathrm{R}^2_{k0}(u)L_k(u_1,u_2) = \begin{pmatrix}1 & 0\\ -\theta_0 & 1\end{pmatrix}\begin{pmatrix}u_1\mathrm{R}^2_{k0}(u+1) & -\mathrm{R}^2_{k0}(u)\partial_{\theta_2}\\ 0 & u_2\mathrm{R}^2_{k0}(u+1)\end{pmatrix}\begin{pmatrix}1 & 0\\ \theta_0 & 1\end{pmatrix}.$$

Let us move to the global relation

$$\begin{aligned}\mathrm{str}_{V_0\otimes V_{0'}}\ \mathrm{R}^2_{10}(u)L_1(u_1,u_2)\ldots\mathrm{R}^2_{n0}(u)L_n(u_1,u_2) \\ = \mathrm{str}_{V_0\otimes V_{0'}}\begin{pmatrix}1 & 0\\ -\theta_0 & 1\end{pmatrix}\begin{pmatrix}u_1\mathrm{R}^2_{10}(u+1) & -\mathrm{R}^2_{10}(u)\partial_{\theta_2}\\ 0 & u_2\mathrm{R}^2_{10}(u+1)\end{pmatrix} \\ \times\ \ldots\times\begin{pmatrix}u_1\mathrm{R}^2_{n0}(u+1) & -\mathrm{R}^2_{n0}(u)\partial_{\theta_2}\\ 0 & u_2\mathrm{R}^2_{n0}(u+1)\end{pmatrix}\begin{pmatrix}1 & 0\\ \theta_0 & 1\end{pmatrix},\end{aligned}$$

where $V_{0'}$ is the fundamental representation. Rewriting in terms of $t(u)$ and $Q_2(u)$, we obtain (19):

$$Q_2(u)t(u) = (u_1^n - u_2^n)Q_2(u+1).$$

The relation for $Q_1(u)$, equation (18), is derived analogously from a similar local computation

$$\begin{pmatrix} 1 & 0 \\ \theta_1 & 1 \end{pmatrix} L_2(u_1, u_2)\mathcal{R}_1(u_1|v_1, u_2) \begin{pmatrix} 1 & 0 \\ -\theta_2 & 1 \end{pmatrix}\Bigg|_{v_1=-1}$$
$$= \begin{pmatrix} \frac{u_2(u_1+1)}{u_2+1}\mathcal{R}_1(u) & -\left(\frac{u_2-u_1}{u_2+1}\right)\mathcal{R}_1(u)\partial_{\theta_2} \\ 0 & u_2\mathcal{R}_1(u) \end{pmatrix}.$$

Again, let $V_1 = V_k,\ V_2 = V_0$, and we obtain the relation

$$L_k(u_1, u_2)\mathrm{R}^1_{k0}(u+1)$$
$$= \begin{pmatrix} 1 & 0 \\ \theta_0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{u_2(u_1+1)}{u_2+1}\mathrm{R}^1_{k0}(u) & -\frac{u_2-u_1}{u_2+1}\mathrm{R}^1_{k0}(u)\partial_{\theta_2} \\ 0 & u_2\mathrm{R}^1_{k0}(u) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \theta_0 & 1 \end{pmatrix}.$$

Promoting this local relation to a global one gives

$$t(u)Q_1(u+1) = u_2^n\left(\frac{(u_1+1)^n}{(u_2+1)^n} - 1\right)Q_1(u),$$

which is equivalent to equation (18). □

**Remark 1.** Equations (15), (18), and (19) provide the standard Nested Bethe Ansatz equation (e.g., given in [3]) for a supersymmetric spin chain under the assumption of polynomial behavior of the eigenvalues of the Q-operators, i.e., in the joint diagonalizing basis of the Q-operator:

$$Q_2(u) = \prod_{i=1}^{n}(u - \sigma_i),$$

where $\sigma_i$ are the standard Bethe roots.

## §4. Conclusion

In this work, we have demonstrated a new method for constructing factorizations of R-matrices invariant under the superalgebra $\mathfrak{gl}(1|1)$. Beginning from the theory of intertwining operators, we were led to several interesting objects: odd intertwining operators and operators that permute bosonic and fermionic parameters across tensor factors.

Using these operators, we constructed the factorization of the R-matrix and verified that our formulas reproduce the known Yang R-matrix. Proceeding to global objects–transfer matrices and Q-operators, we employed the factorization to derive explicit expressions for the Baxter Q-operators, establish the factorization of the transfer matrix, and obtain the TQ-relations.

Let us note several characteristic features of the $\mathfrak{gl}(1|1)$ case. This superalgebra is remarkable due to the finite-dimensionality of its Verma modules, which allowed us to develop the theory using methods pertinent to infinite-dimensional representations. A limitation of the approaches in [7–9] is the absence of a direct pathway from infinite-dimensional to finite-dimensional representations. We hope that our research will help clarify the structure of compact spin chains possessing symmetry under a Lie group.

A detailed analysis of the structure of the $\mathfrak{gl}(1|1)$ spin chain also sheds light on the more general case of the $\mathfrak{gl}(m|n)$ spin chain: to construct a solution to the quantum Yang-Baxter equation, it is necessary to use a new type of intertwining operator – operators corresponding to odd reflection.

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