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ELEMENTARY FIBRATIONS OVER FINITE FIELDS

Abstract. In 1973 M. Artin introduced a notion of an elementary fibration. It is a very useful tool for various applications. In this paper we give a detailed proof of existences of an elementary fibration for smooth affine varieties over a finite field. A short version of that has been published in a paper of mine (Izvestia RAS, 2019). To give the detailed proof we use our extension of Poonen's form of Bertini type theorems over a finite field.

§1. Introduction

Let Y be a quasi-projective subscheme of $\mathbf{P}^n$ of dimension $m \geqslant 0$ over $\mathbb{F}_q$. Let

$$\varnothing = Y_{n+1} \subset \cdots \subset Y_2 \subset Y_1 = Y$$

be a filtration of Y by closed subvarieties such that $Y_i - Y_{i+1}$ is smooth equidimensional of dimension $m_i \geqslant 0$. Then there exist homogeneous polynomials f over $\mathbb{F}_q$ such that for each i the intersection of $Y_i - Y_{i+1}$ and the hypersurface $f = 0$ is smooth of dimension $m_i - 1$. In fact, the set of such f has a positive density, equal to $\prod_{i=1}^{n} \zeta_{Y_i - Y_{i+1}}(m_i + 1)^{-1}$, where for an $\mathbb{F}_q$-scheme X the function $\zeta_X(s) = Z_X(q^{-s})$ is the zeta function of X. These yields two results proven in [4]. Namely, [4, Corollary 3.3] and [4, Theorem 3.4].

Using these two results we give in the present paper a detailed proof of existences of an elementary fibration for smooth affine varieties over a finite field (Theorem 5.3). A proof of Theorem 5.3 was sketched in [5, Appendix B]. Since this Theorem was used for various applications we decided to give its detailed proof in this paper.

The paper is organized as follows. In §2 we recall the notion of a weak elementary fibration. In §3 we work with a weighted blow up to construct a weak elementary fibration (Theorem 3.16). In §4 we construct the required weak elementary fibration (Theorem 4.8). Finally, in §5 we recall Artin's notion of elementary fibration and construct such a fibration in Theorem 5.3.

Key words and phrases: Bertini type theorems, elementary fibrations, finite fields.

The present paper is inspired by the one [3]. The *major difference* is that our projective variety X (a compactification of our affine smooth $\mathcal{X}$) is normal rather than smooth as in [3]. So, most of arguments used in the present paper are essentially *more refined*.

§2. Weak elementary fibrations

It turns out that the following notion (introduced in [3]) is useful. It is weaker than the notion of an elementary fibration introduced by Artin in [1, Exp. XI, Déf. 3.1]. It coincides with the notion of an elementary fibration if in the condition (iii) we require that $\widehat{q}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one.

Definition 2.1. *A weak elementary fibration is a morphism of schemes $q : X \to S$ which can be included in a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "q"'] & \widehat{X} \arrow[d, "\widehat{q}"] & X_\infty \arrow[l, "i"'] \arrow[dl, "q_\infty"] \\ & S & \end{tikzcd} \tag{1}$$

of morphisms satisfying the following conditions:

(i) *i is a closed embedding*;

(ii) *j is an open immersion dense at each fibre of $\widehat{q}$, and $X = \widehat{X} - X_\infty$*;

(iii) *$\widehat{q}$ is smooth projective all of whose fibres are equidimensional of dimension one*;

(iv) *q_∞ is finite étale all of whose fibres are non-empty.*

Remark 2.2. Under the hypotheses and notation of [4, Corollary 3.3] the variety
$X_{f_1} := \{x \in X | f_1(x) \neq 0\}$ is affine. Consider the morphism
$q = (f_2/f_1^{e_2/e_1}, \dots, f_m/f_1^{e_m/e_1}) : X_{f_1} \to \mathbb{A}^{m-1}$.
For an open neighborhood $S \subset \mathbb{A}^{m-1}$ of the origin let $X_S = q^{-1}(S)$ and write q for $q|_{X_S} : X_S \to S$. It will be proved below that for each sufficiently small S

1) the morphism $q : X_S \to S$ is a weak elementary fibration;

2) moreover the scheme $Z_S := Z \cap X_S$ is finite over S.

However we need often a bit more. For an open neighborhood $S \subset \mathbb{A}^{m-1}$ of the origin put $\mathcal{X}_S = \mathcal{X} \cap X_S$ and write q for $q|_{\mathcal{X}_S} : \mathcal{X}_S \to S$. It will be proved below that for each sufficiently small S

1) the morphism $q : \mathcal{X}_S \to S$ is a weak elementary fibration;
2) moreover the scheme $\mathcal{Z}_S := Z \cap \mathcal{X}_S$ is finite over S.

§3. Applications of weighted blowups

Let X, $\mathbb{P}^r$ over $\mathbb{F}_q$, $x \in X$, $m \geqslant 0$, closed subsets B and Z in X, homogeneous polynomials $f_0, f_1, \dots, f_m$, integers $e_0, e_1, \dots, e_m$, subschemes $X_i := H_{f_i} \cap X$ be enjoying the conclusion of [4, Theorem 3.2]. Write $\underline{w}$ for the subscheme $\bigcap_{i=1}^m X_i$ in X. It is finite étale over $\mathbb{F}_q$ and $\underline{w} \cap Z = \varnothing = \underline{w} \cap X_0$, $X_{2,\dots,m} \cap B = \varnothing$ and $\underline{w} \cap B = \varnothing$.

Notation 3.1. Let $\mathbb{P}^{m,w}$ be the weighted projective space with homogeneous coordinates $[t_0 : t_1 : \dots : t_m]$ of weights $1, e_1/e_0, \dots, e_m/e_0$ respectively. Let $\mathbb{P}^{m-1,w}$ be the weighted projective space with homogeneous coordinates $[x_1 : \dots : x_m]$ of weights $1, e_2/e_1, \dots, e_m/e_1$ respectively.

We will write $\mathcal{O}(e)$ for $\mathcal{O}_{\mathbb{P}^r}(e)|_X$ and $s_i \in \Gamma(X, \mathcal{O}(e_i))$ for $f_i|_X$. We will write X_{s_i} for $X - X_i$, $\mathbb{P}^{m,w}_{t_i}$ for $\mathbb{P}^{m,w} - \{t_i = 0\}$ and $\mathbb{P}^{m-1,w}_{x_i}$ for $\mathbb{P}^{m-1,w} - \{x_i = 0\}$.

For all $i, j = 0, \dots, m$ with $j > i$ put $d_{i,j} = e_j/e_i$. Note that $\mathbb{P}^{m-1,w}_{x_1}$ is the affine space $\mathbb{A}^{m-1}$. The identification is given as $[x_1 : x_2 : \dots : x_m] \mapsto (x_2/x_1^{d_{1,2}}, \dots, x_m/x_1^{d_{1,m}})$.

Note that $\mathbb{P}^{m,w}_{t_0}$ is the affine space $\mathbb{A}^m$. The identification is given as $[t_0 : t_1 : \dots : t_m] \mapsto (t_1/t_0^{d_{0,1}}, \dots, t_m/t_0^{d_{0,m}})$.

Construction 3.2. Define $\widehat{X}$ as a closed subscheme of $X \times \mathbb{P}^{m-1,w}$ given by equations $s_j x_i^{d_{i,j}} = s_i^{d_{i,j}} x_j$ $(j > i)$. We regard $\widehat{X}$ as a weighted blowup of X at the subscheme $\underline{w}$.

If $X = \mathbb{P}^{m,w}$ is the weighted projective space as in Notation 3.1 then $\widehat{\mathbb{P}}^{m,w}$ is a closed subscheme of $\mathbb{P}^{m,w} \times \mathbb{P}^{m-1,w}$ given by equations $t_j x_i^{d_{i,j}} = t_i^{d_{i,j}} x_j$. We regard $\widehat{\mathbb{P}}^{m,w}$ as a weighted blowup of $\mathbb{P}^{m,w}$ at the point $\mathbf{0} := [1 : 0 : \dots : 0]$.

Projections $X \times \mathbb{P}^{m-1,w} \to \mathbb{P}^{m-1,w}$ and $\mathbb{P}^{m,w} \times \mathbb{P}^{m-1,w} \to \mathbb{P}^{m-1,w}$ induces morphisms $\widehat{q} : \widehat{X} \to \mathbb{P}^{m-1,w}$ and $\widehat{p} : \widehat{\mathbb{P}}^{m,w} \to \mathbb{P}^{m-1,w}$ allowing to consider the schemes $\widehat{X}$ and $\widehat{\mathbb{P}}^{m,w}$ as $\mathbb{P}^{m-1,w}$-schemes. The morphism $\pi \times id : X \times \mathbb{P}^{m-1,w} \to \mathbb{P}^{m,w} \times \mathbb{P}^{m-1,w}$ induces a morphism

$$\widehat{\pi} : \widehat{X} \to \widehat{\mathbb{P}}^{m,w}.$$

This is a morphism of the $\mathbb{P}^{m-1,w}$-schemes. Put $\widehat{X}_{x_1} = q^{-1}(\mathbb{P}^{m-1,w}_{x_1})$ and $\widehat{\mathbb{P}}^{m,w}_{x_1} = p^{-1}(\mathbb{P}^{m-1,w}_{x_1})$. Recall that $\mathbb{P}^{m-1,w}_{x_1}$ is the affine space $\mathbb{A}^{m-1}$.

Consider the weighted $\mathbb{P}^{1,w}$ with weighted homogeneous coordinates $[t_0 : t_1]$ of weights $1, e_1/e_0$ respectively. Put $d_1 = e_1/e_0$. It is known that $\mathbb{P}^{1,w}$ is isomorphic to $\mathbb{P}^1$ with homogeneous coordinates $[t_0^{d_1} : t_1]$. Particularly, $\mathbb{P}^{1,w}$ is smooth. Consider a morphism

$$\varphi : \widehat{\mathbb{P}}^{m,w}_{x_1} \to \mathbb{P}^{1,w} \times \mathbb{A}^{m-1}$$

given by $([t_0 : \cdots : t_m], [x_1 : \cdots : x_m]) \mapsto ([t_0 : t_1], (x_2/x_1^{d_{1,2}}, \ldots, x_m/x_1^{d_{1,m}}))$. Consider a morphism

$$\psi : \mathbb{P}^{1,w} \times \mathbb{A}^{m-1} \to \widehat{\mathbb{P}}^{m,w}_{x_1}$$

given by

$$([t_0 : t_1], (y_2, \ldots, y_m)) \mapsto ([t_0 : t_1 : t_1^{d_{1,2}} y_2 : \cdots : t_1^{d_{1,m}} y_m], [1 : y_2 : \cdots : y_m]).$$

Lemma 3.3. *The morphisms φ and ψ are well-defined and they are mutually inverse isomorphisms. Moreover, they are isomorphisms of the $\mathbb{A}^{m-1}$-schemes.*

Corollary 3.4. *The scheme $\widehat{\mathbb{P}}^{m,w}_{x_1}$ is smooth. The morphism*

$$\widehat{p} : \widehat{\mathbb{P}}^{m,w}_{x_1} \to \widehat{\mathbb{P}}^{m-1,w}_{x_1} = \mathbb{A}^{m-1}$$

is smooth projective with $\mathbb{P}^{1,w} \cong \mathbb{P}^1$ as a fiber.

Notation 3.5. Projections $X \times \mathbb{P}^{m-1,w} \to X$ and $\mathbb{P}^{m,w} \times \mathbb{P}^{m-1,w} \to \mathbb{P}^{m,w}$ induces morphisms $\tau : \widehat{X} \to X$ and $\sigma : \widehat{\mathbb{P}}^{m,w} \to \mathbb{P}^{m,w}$ respectively. Put $E_w = \tau^{-1}(\underline{w})$, $E_0 = \sigma^{-1}(\mathbf{0})$.
Put $\widehat{B} = \tau^{-1}(B)$. Recall that $B \cap \underline{w} = \varnothing$. Thus, τ identifies $\widehat{B}$ with B.

Put

$$\widehat{X}^\circ = \widehat{X} - \widehat{q}^{-1}\widehat{q}(\widehat{B}), \quad \widehat{\mathbb{P}}^{m,w,\circ} = \widehat{\mathbb{P}}^{m,w} - \widehat{p}^{-1}\widehat{q}(\widehat{B}), \quad \mathbb{P}^{m-1,w,\circ} = \mathbb{P}^{m-1,w} - \widehat{q}(\widehat{B}).$$

Finally, put $X^\circ = X - \tau(\widehat{q}^{-1}\widehat{q}(\widehat{B}))$, $\mathbb{P}^{m,w,\circ} = \mathbb{P}^{m,w} - \sigma(\widehat{p}^{-1}\widehat{q}(\widehat{B}))$, $\mathbb{A}^{m-1,\circ} := \mathbb{P}^{m-1,w,\circ}_{x_1}$.

For each locally closed subset S in $\widehat{X}$ put $S^\circ = S \cap \widehat{X}^\circ$. For each locally closed subset M in X put $M^\circ = M \cap X^\circ$.

Remark 3.6. Note that $\sigma \circ \widehat{\pi} = \pi \circ \tau$. Clearly, the morphisms

$$\tau : \widehat{X}^\circ - E_w \to X^\circ - \underline{w}$$

and

$$\sigma : \widehat{\mathbb{P}}^{m,w,\circ} - E_0 \to \mathbb{P}^{m,w,\circ} - \{0\}$$

are isomorphisms.

Note that $X^\circ = (X^\circ - \underline{w}) \cup X^\circ_{s_0}$ and $\mathbb{P}^{m,w,\circ} = (\mathbb{P}^{m,w,\circ} - \mathbf{0}) \cup \mathbb{P}^{m,w,\circ}_{t_0}$.

Proposition 3.7. *The following are true*

1) *the morphism* $\tau : \widehat{X}^\circ_{x_1} - (E_w)_{x_1} \to X^\circ_{s_1}$, $\sigma : \widehat{\mathbb{P}}^{m,w,\circ}_{x_1} - (E_0)_{x_1} \to \mathbb{P}^{m,w,\circ}_{t_1}$ *are isomorphisms*;

2) *the schemes* $\widehat{X}^\circ_{x_1} - (E_w)_{x_1}$, $\widehat{\mathbb{P}}^{m,w,\circ}_{x_1} - (E_0)_{x_1}$, $X^\circ_{s_1}$, $\mathbb{P}^{m,w,\circ}_{t_1}$ *are smooth.*

Proof. The first assertion is clear. Prove the second one. The scheme X° is smooth. Hence so are the schemes $X^\circ_{s_1}$ and $\widehat{X}^\circ_{x_1} - (E_w)_{x_1}$. By Corollary 3.4 the scheme $\widehat{\mathbb{P}}^{m,w}_{x_1}$ is smooth. Hence so are the schemes $\widehat{\mathbb{P}}^{m,w,\circ}_{x_1}$, $\widehat{\mathbb{P}}^{m,w,\circ}_{x_1} - (E_0)_{x_1}$ and $\mathbb{P}^{m,w,\circ}_{t_1}$. □

Lemma 3.8. *Put* $\mathbb{A}^{m,\circ} = \mathbb{P}^{m,w,\circ}_{t_0}$. *Let* $X^\circ_{s_0} \xrightarrow{\pi_0} \mathbb{P}^{m,w,\circ}_{t_0} = \mathbb{A}^{m,\circ}$ *be the base change of the morphism* $\pi : X \to \mathbb{P}^{m,w}$. *Then* π_0 *is finite flat.*

Proof. By [4, Proposition 3.5] the morphism π is finite surjective. Thus the morphism π_0 is finite surjective as the base change of π. Since $X^\circ_{s_0}$ and $\mathbb{A}^{m,\circ}$ are smooth it follows that π_0 is flat. And it is also finite. □

Lemma 3.9. *Put* $\widehat{\pi}^\circ_{x_1} = \widehat{\pi}_{x_1}|_{\widehat{X}^\circ_{x_1}}$. *Then the morphism* $\widehat{\pi}^\circ_{x_1} : \widehat{X}^\circ_{x_1} \to \widehat{\mathbb{P}}^{m,w,\circ}_{x_1}$ *is finite flat.*

Proof. One has $\widehat{X}^\circ_{x_1} = (\widehat{X}^\circ - E_w)_{x_1} \cup (\widehat{X}^\circ_{s_0})_{x_1}$, $\widehat{\mathbb{P}}^{m,w,\circ}_{x_1} = (\widehat{\mathbb{P}}^{m,w,\circ} - E_0)_{x_1} \cup (\widehat{\mathbb{P}}^{m,w,\circ}_{t_0})_{x_1}$. Thus, it is sufficient to check that morphisms $(\widehat{X}^\circ - E_w)_{x_1} \xrightarrow{\widehat{\pi}_{x_1}} (\widehat{\mathbb{P}}^{m,w,\circ} - E_0)_{x_1}$ and $(\widehat{X}^\circ_{s_0})_{x_1} \xrightarrow{\widehat{\pi}_{x_1}} (\widehat{\mathbb{P}}^{m,w,\circ}_{t_0})_{x_1}$ are finite flat.

The morphism π_0 is finite flat by Lemma 3.8. The morphism $\widehat{X}^\circ_{s_0} \xrightarrow{\widehat{\pi}_0} \widehat{\mathbb{P}}^{m,w,\circ}_{t_0}$ is a base change of π_0. Thus it is finite flat too. Eventually the morphism $(\widehat{X}^\circ_{s_0})_{x_1} \to (\widehat{\mathbb{P}}^{m,w,\circ}_{t_0})_{x_1}$ is a base change of the morphism $\widehat{\pi}_0$. Thus it is finite flat as well.

The morphism

$$X^\circ - \underline{w} = \widehat{X}^\circ - E_w \to \widehat{\mathbb{P}}^{m,w,\circ} - E_0 = \mathbb{P}^{m,w,\circ} - [1 : 0 : \cdots : 0]$$

is finite surjective. Thus the morphism $\widehat{X}_{x_1} - (E_w)_{x_1} \to \widehat{\mathbb{P}}^{m,w}_{x_1} - (E_0)_{x_1}$ is finite surjective. By the second item of Proposition 3.7 the source and the target are smooth schemes. Thus, the morphism $\widehat{X}^\circ_{x_1} - (E_w)_{x_1} \to \widehat{\mathbb{P}}^{m,w,\circ}_{x_1} - (E_0)_{x_1}$ is flat and finite. Hence $\widehat{\pi}^\circ_{x_1}$ is finite flat. □

Lemma 3.10. *The morphism* $\widehat{q} : \widehat{X}^{\circ}_{x_1} \to \widehat{\mathbb{P}}^{m-1,w,\circ}_{x_1} = \mathbb{A}^{m-1,\circ}$ *is flat projective.*

Proof. We know that $\widehat{q} = \widehat{p} \circ \widehat{\pi}_{x_1}$. Apply now Lemma 3.9 and Corollary 3.4. $\square$

Proposition 3.11. *The scheme* $\widehat{X}^{\circ}_{x_1}$ *is smooth.*

Proof. By Lemma 3.8 the morphism π_0 is finite flat. The scheme $\pi_0^{-1}(\{0\}) = X_{1,\dots,m} = \underline{w}$ is smooth of dimension zero by [4, Corollary 3.3]. Thus, π_0 is étale over the origin $\{0\}$. So, we can take a Zariski open $U \subset \mathbb{A}^{m,\circ}$ containing $\{0\}$ such that for $W = \pi_0^{-1}(U)$ the morphism $\pi_0 : W \to U$ is finite étale. Then the morphisms $\widehat{\pi}_0 : \widehat{W} \to \widehat{U}$ and $(\widehat{\pi}_0)_{x_1} : \widehat{W}_{x_1} \to \widehat{U}_{x_1}$ are finite étale as base change of π_0. Note that $\widehat{U}_{x_1}$ is open in $\widehat{\mathbb{P}}^{m,w,\circ}_{x_1}$ and the latter is smooth by Corollary 3.4. Hence $\widehat{U}_{x_1}$ is smooth and so is $\widehat{W}_{x_1}$ as an its étale cover.

One has $\widehat{X}^{\circ}_{x_1} = (\widehat{X}^{\circ} - E_w)_{x_1} \cup \widehat{W}_{x_1}$. By Proposition 3.7 $(\widehat{X}^{\circ} - E_w)_{x_1}$ is smooth. We know already that $\widehat{W}_{x_1}$ is smooth. Thus, $\widehat{X}^{\circ}_{x_1}$ is smooth. $\square$

Recall that $\widehat{p} : \widehat{\mathbb{P}}^{m,w}_{x_1} \to \mathbb{P}^{m-1,w}_{x_1} = \mathbb{A}^{m-1}$ is smooth projective with the projective line $\mathbb{P}^1$ as fibres. For the morphism $\widehat{q} : \widehat{X}_{x_1} \to \mathbb{P}^{m-1,w}_{x_1}$ one has $\widehat{q} = \widehat{\pi}_{x_1} \circ \widehat{p}$.

Proposition 3.12. *Let* $C = \widehat{q}^{-1}(\{0\})$*. Then*

1) C *is a smooth projective curve*;

2) $\tau|_C : C \to X$ *is a closed embedding*;

3) $\tau(C)$ *coincides with the smooth closed dimension one subscheme* $X_{2,\dots,m}$ *in* X.

Proof. Let U and W be as in the proof of Proposition 3.11. Prove the first assertion. One has an open cover $C = (C - E_{\underline{w}}) \cup (C \cap \widehat{W})$. Note that $\tau : C - E_{\underline{w}} \to X_{2,\dots,m} - \underline{w}$ is an isomorphism. Since $X_{2,\dots,m}$ is smooth, thus so is $C - E_{\underline{w}}$.

The $\widehat{\pi}_0 : \widehat{W} \to \widehat{U}$ is finite étale. Hence so is the one $C \cap \widehat{W} \to (\mathbb{P}^1 \times \{0\}) \cap \widehat{U}$. Since $\mathbb{P}^1$ is smooth, hence so is $C \cap \widehat{W}$. Thus C is smooth. Since C is closed in $\widehat{X}$ it is projective.

Prove the second assertion. Put $l := \widehat{p}^{-1}(\{0\})$. By Corollary 3.4 $\sigma|_l : l \to \mathbb{P}^{m,w}$ identifies l with the closed subscheme $\mathbb{P}^{m,w}_{2,\dots,m} := \{t_2 = t_3 = \cdots = t_m = 0\}$. Particularly, $\sigma|_l : l \to \mathbb{P}^{m,w}$ is a closed embedding and $\sigma|_l : l \cap \widehat{U} \to U$ is a closed embedding as well. Put $\tau_C := \tau|_C$. One has

$X = (X - \underline{w}) \cup W$. Thus $C = \tau_C^{-1}(X - \underline{w}) \cup \tau_C^{-1}(W) = (C - E_{\underline{w}}) \cup (C \cap \widehat{W})$. Since $\tau_C : C - E_{\underline{w}} \to X - \underline{w}$ is a closed embedding it remains to check that $\tau_C : (C \cap \widehat{W}) \to W$ is a closed embedding. But this morphism is a base change of the closed embedding $\sigma|_l : (l \cap \widehat{U}) \to U$. Thus, the morphism $\tau_C : (C \cap \widehat{W}) \to W$ is a closed embedding indeed.

Prove the third assertion. The morphism $\tau : \widehat{X} \to X$ is the base change of the one $\sigma : \widehat{\mathbb{P}}^{m,w} \to \mathbb{P}^{m,w}$ by means of π. The closed embedding $j_C : C \hookrightarrow \widehat{X}$ is the base change of $j_l : l \hookrightarrow \widehat{\mathbb{P}}^{m,w}$ by means of $\widehat{\pi}$. Thus, $\tau_C : C \hookrightarrow X$ is a base change of $\sigma|_l : l \hookrightarrow \mathbb{P}^{m,w}$ by means of π. Recall that $\sigma|_l$ identifies l with $\mathbb{P}^{m,w}_{2,\dots,m}$. Thus τ_C identifies C with the subscheme $\pi^{-1}(\mathbb{P}^{m,w}_{2,\dots,m}) = X_{2,\dots,m}$ of the scheme X. $\square$

Corollary 3.13. *There exists a Zariski open neighborhood S of the origin $0 \in \mathbb{A}^{m-1,\circ}$ such that for $\widehat{X}^\circ_S := \widehat{q}^{-1}(S)$ the morphism $\widehat{q} : \widehat{X}^\circ_S \to S$ is smooth projective.*

Proof. Lemma 3.10 and Proposition 3.12 yield this corollary.

$\square$

Notation 3.14. Write j for the composition $X^\circ_{s_1} \xrightarrow{\tau^{-1}} \widehat{X}^\circ_{x_1} - (E_w)_{x_1} \hookrightarrow \widehat{X}^\circ_{x_1}$, where τ is the isomorphism as in Proposition 3.7. Then j is an open embedding.

Write $\widehat{Z}^\circ$ for $\tau^{-1}(Z^\circ) \subset \widehat{X}^\circ$. Since $Z \cap \underline{w} = \varnothing$ the morphism τ identifies $\widehat{Z}^\circ$ with Z°. Also j identifies $Z^\circ_{s_1}$ with $\widehat{Z}^\circ_{x_1}$.

Put $q = \widehat{q} \circ j : X^\circ_{s_1} \to \mathbb{P}^{m-1,w,\circ}_{x_1} = \mathbb{A}^{m-1,\circ}$.

Write i for the closed embedding $\mathbb{A}^{m-1,\circ} \times \underline{w} = (E^\circ_{\underline{w}})_{x_1} \hookrightarrow \widehat{X}^\circ_{x_1}$ and put $q_\infty = \widehat{q} \circ i$.

Below in this paper we also often use Notation 3.5.

Lemma 3.15. *The morphism $q|_{Z^\circ_{s_1}} : Z^\circ_{s_1} \to \mathbb{A}^{m-1,\circ}$ is finite. The morphism q_∞ is finite étale.*

Proof. The morphism $q|_{Z^\circ_{s_1}}$ is affine and projective. Thus it is finite. $\square$

Consider the following commutative diagram

$$\begin{tikzcd} X^\circ_{s_1} \arrow[r, "j"] \arrow[dr, "q"'] & \widehat{X}^\circ_{x_1} \arrow[d, "\widehat{q}"] & \mathbb{A}^{m-1,\circ} \times \underline{w} \arrow[l, "i"'] \arrow[dl, "q_\infty"] \\ & \mathbb{A}^{m-1,\circ} & \end{tikzcd} \qquad (2)$$

The morphism $\widehat{q}$ is flat projective. The morphism q_∞ is finite étale. The morphism i is a closed embedding, the morphism j is an open embedding identifying $X^\circ_{s_1}$ with $\widehat{X}^\circ_{x_1} - i(\mathbb{A}^{m-1,\circ} \times \underline{w})$. The closed set $\underline{x}$ is in $X^\circ_{s_1}$. If $Z \subset X$ is as in [4, Theorem 3.2], then $Z_{s_1} \subset X_{s_1}$ and $Z^\circ_{s_1}$ is finite over $\mathbb{A}^{m-1,\circ}$ by Lemma 3.15.

Theorem 3.16. *Let $S \subset \mathbb{A}^{m-1,\circ}$ be as in Corollary* 3.13. *Then the base change of the diagram* (2) *by means of the open embedding $S \hookrightarrow \mathbb{A}^{m-1,\circ}$*

$$\begin{tikzcd} X^\circ_S \arrow[r, "j"] \arrow[dr, "q"'] & \widehat{X}^\circ_S \arrow[d, "\widehat{q}"] & S \times \underline{w} \arrow[l, "i"'] \arrow[dl, "q_\infty"] \\ & S & \end{tikzcd} \tag{3}$$

is the diagram of a weak elementary fibration and $\underline{x} \subset X^\circ_S$. Moreover, if $Z \subset X$ is as in [4, Theorem 3.2], *then $Z^\circ_S := Z^\circ_{s_1} \cap X^\circ_S$ is finite over S.*

Proof. This follows from Corollary 3.13 and Lemma 3.15. □

§4. Family of curves

In this section we construct a diagram of the form (6) under the hypotheses and notation of [4, Corollary 3.3]. First consider the diagram (2) and call it (4)

$$\begin{tikzcd} X^\circ_{s_1} \arrow[r, "j"] \arrow[dr, "q"'] & \widehat{X}^\circ_{x_1} \arrow[d, "\widehat{q}"] & \mathbb{A}^{m-1,\circ} \times \underline{w} \arrow[l, "i"'] \arrow[dl, "q_\infty"] \\ & \mathbb{A}^{m-1,\circ} & \end{tikzcd} \tag{4}$$

Notation 4.1. Let $M \subset X$ be a closed subscheme such that $M \cap \underline{w} = \varnothing$. Write $\widehat{M}$ for $\tau^{-1}(M) \subset \widehat{X}$. Since $M \cap \underline{w} = \varnothing$ the morphism τ identifies $\widehat{M}$ with M. Also j identifies M_{s_1} with $\widehat{M}_{x_1}$. We use below also Notation 3.5.

Lemma 4.2. *Under the Notation* 4.1 *$M^\circ_{s_1}$ is finite over $\mathbb{A}^{m-1,\circ}$. Particularly, $Y^\circ_{s_1}$, $B^\circ_{s_1}$ and $Z^\circ_{s_1}$ are finite over $\mathbb{A}^{m-1,\circ}$. The schemes $\widehat{Y}^\circ_{x_1}$, $\widehat{B}^\circ_{x_1}$ and $\widehat{Z}^\circ_{x_1}$ are also finite over $\mathbb{A}^{m-1,\circ}$.*

Proof. This follows from [4, Proposition 3.6] and the fact that j identifies $M^\circ_{s_1}$ with $\widehat{M}^\circ_{x_1}$. □

Recall that $X_{2,\dots,m} \cap B = \varnothing$. This yields that $\widehat{q}(\widehat{B}_{x_1})$ does not contain $\{0\} \in \mathbb{A}^{m-1,\circ}$. Thus, $\widehat{q}^{-1}(\{0\}) \cap \widehat{B}_{x_1} = \varnothing$ and $q^{-1}(\{0\}) \cap B_{s_1} = \varnothing$. The following Proposition is proved in Section 3 (Proposition 3.12).

Proposition 4.3. *Let $C = \widehat{q}^{-1}(\{0\})$. Then*

1) *C is a smooth projective curve*;

2) *$\tau|_C : C \to X$ is a closed embedding*;

3) *$\tau(C)$ coincides with the smooth closed dimension one subscheme $X_{2,\dots,m}$ in X.*

The following Corollary is proved in Section 3 (Corollary 3.13).

Corollary 4.4. *There exists a Zariski open neighborhood S of the origin $\{0\} \in \mathbb{A}^{m-1,\circ}$ such that $S \cap \widehat{q}(\widehat{B}_{x_1}) = \varnothing$ and for $\widehat{X}^\circ_S := \widehat{q}^{-1}(S)$ the morphism $\widehat{q} : \widehat{X}^\circ_S \to S$ is smooth projective.*

Theorem 4.5. *Let $S \subset \mathbb{A}^{m-1,\circ}$ be as in Corollary 4.4. Then the base change of the diagram (4) by means of the open embedding $S \hookrightarrow \mathbb{A}^{m-1,\circ}$*

$$\begin{tikzcd} X^\circ_S \arrow[r, "j"] \arrow[dr, "q"'] & \widehat{X}^\circ_S \arrow[d, "\widehat{q}"] & S \times \underline{w} \arrow[l, "i"'] \arrow[dl, "q_\infty"] \\ & S & \end{tikzcd} \tag{5}$$

is the diagram of a weak elementary fibration and $\underline{x} \in X^\circ_S$. Moreover, if $Z \subset X$ is as in [4, Corollary 3.3], *then $Z^\circ_S := Z^\circ_{s_1} \cap X^\circ_S$ is finite over S.*

Proof. This follows from Corollary 4.4 and Lemma 4.2. □

Remark 4.6. Recall that $\mathcal{X} = X - X_0 = X - Y_{\mathrm{red}}$. In Remark 2.2 we promised to prove that the morphism $q|_{\mathcal{X}_S} : \mathcal{X}_S \to S$ is a weak elementary fibration. To achieve that we need to do an additional work.

Lemma 4.7. *There exists a Zariski open neighborhood S of the origin $\{0\} \in \mathbb{A}^{m-1,\circ}$ such that $B^\circ_S := B^\circ_{s_1} \cap X^\circ_S = \varnothing$ and for $Y^\circ_{red,S} := (Y^\circ_{\mathrm{red}})_{s_1} \cap X_S$ the morphism $q : Y^\circ_{red,S} \to S$ is finite étale.*

Proof. First take $S \subset \mathbb{A}^{m-1,\circ}$ as in Corollary 4.4. In this case $B_S = \varnothing$. Thus, $Y^\circ_{red,S} = Y^\circ_{red,S} - B_S$ is smooth. By Lemma 4.2 $(Y^\circ_{\mathrm{red}})_{s_1}$ is finite over $\mathbb{A}^{m-1,\circ}$. Thus, $Y^\circ_{red,S}$ is finite over S. Since the morphism $Y^\circ_{red,S} \to S$ is finite surjective and S is smooth it follows that $Y^\circ_{red,S}$ is finite flat over S. Recall that $X_{\{0,2,\dots,m\}}$ is smooth of dimension 0. Thus, $Y^\circ_{red,S}$ is finite étale over a neighborhood of $\{0\}$ in S. □

Theorem 4.8. *Let S be as in Lemma 4.7. Then for $\mathcal{X}_S^\circ := \mathcal{X} \cap X_S^\circ$ the morphism $q : \mathcal{X}_S^\circ \to S$ is a weak elementary fibration. If $\mathcal{Z}_S^\circ = Z_S^\circ \cap \mathcal{X}^\circ$, then $\mathcal{Z}_S^\circ$ is finite over S.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{X}_S^\circ \arrow[r, "j"] \arrow[dr, "q"'] & \widehat{X}_S^\circ \arrow[d, "\widehat{q}"] & S \times \underline{w} \sqcup Y^\circ_{red,S} \arrow[l, "i"'] \arrow[dl, "q_\infty \sqcup q_{Y^\circ_{red,S}}"] \\ & S & \end{tikzcd} \tag{6}$$

Corollary 4.4 and Lemma 4.7 show that $q : \mathcal{X}_S^\circ \to S$ is a weak elementary fibration. It is easy to see that $Z_S^\circ \cap Y^\circ_{red,S} = \varnothing$. Thus, $Z_S^\circ \subset \mathcal{X}_S^\circ$ and $\mathcal{Z}_S^\circ = Z_S^\circ$. By Theorem 4.5 Z_S° is finite over S. Thus, $\mathcal{Z}_S^\circ$ is finite over S. $\square$

§5. Elementary fibrations

In this Section we extend a result of M. Artin from [1] concerning existence of nice neighborhoods. The following notion is introduced by Artin in [1, Exp. XI, Déf. 3.1].

Definition 5.1. *An elementary fibration is a morphism of schemes $q : X \to S$ which can be included in a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "q"'] & \overline{X} \arrow[d, "\overline{q}"] & Y \arrow[l, "i"'] \arrow[dl, "p"] \\ & S & \end{tikzcd} \tag{7}$$

of morphisms satisfying the following conditions:

(i) *j is an open immersion dense at each fibre of $\overline{q}$, and $X = \overline{X} - Y$*;
(ii) *$\overline{q}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one*;
(iii) *p is finite étale all of whose fibres are non-empty.*

Remark 5.2. Clearly, an elementary fibration is an almost elementary fibration in the sense of [7, Defn. 2.1].

We prove the following result, which is a slight extension of Artin's result [1, Exp. XI, Prop. 3.3]. Its proof is sketched in [5, Appendix B].

Theorem 5.3. *Let $\mathbb{F}$ be a finite field, $\mathcal{X}$ be a smooth geometrically irreducible affine variety over the field $\mathbb{F}$, $x_1, x_2, \ldots, x_n \in \mathcal{X}$ be a family of closed points. Then there exists a Zariski open neighborhood $\mathcal{X}^\circ$ of the family $\{x_1, x_2, \ldots, x_n\}$ and an elementary fibration $q : \mathcal{X}^\circ \to S$, where S is an affine open subscheme of the projective space $\mathbf{P}^{\dim X-1}$.*

If, moreover, $\mathcal{Z}$ is a closed codimension one subvariety in $\mathcal{X}$, then one can choose $\mathcal{X}^\circ$ and q in such a way that $q|_{\mathcal{Z}\bigcap\mathcal{X}^\circ} : \mathcal{Z}\bigcap\mathcal{X}^\circ \to S$ is finite surjective.

Proof. If $\dim\mathcal{X} = 1$, then take $\mathcal{X}^\circ = \mathcal{X}$ and p the structure morphism. Clearly, p is an elementary fibration. So, we may assume that $m=\dim\mathcal{X}\geqslant 2$. Let $\rho : \mathcal{X} \to \mathbb{A}^m$ be a finite surjective morphism. Let $in : \mathbb{A}^m \hookrightarrow \mathbb{P}^m$ be the open embedding. Let X be the normalization of $\mathbb{P}^m$ into $\mathrm{Spec}(\mathbb{F}(\mathcal{X}))$. Since $\mathcal{X}$ is smooth it is Zariski open in X. Since $\mathbb{F}(X) = \mathbb{F}(\mathcal{X})$ it follows that X is geometrically irreducible.

Let $X_0 = X - \mathcal{X}$ be a closed subscheme of X with the reduced scheme structure. Clearly, $\mathcal{X} = X - X_0$. Let $\bar{\rho} : X \to \mathbb{P}^m$ be the canonical finite morphism. It is a unique morphism whose restriction to $\mathcal{X}$ is $in \circ \rho$. Consider the projective subspace $H = \mathbb{P}^m - \mathbb{A}^m$ as a Cartie divisor on $\mathbb{P}^m$. Then $Y := (\bar{\rho})^{-1}(H)$ is a Cartie divisor on X. Since $\bar{\rho}$ is finite the Cartie divisor Y is ample. Thus, for appropriate $N >> 0$ the Cartie divisor $N \cdot Y$ is very ample. Hence there is a closed embedding $X \subset \mathbb{P}^r$ such that $H_{t_0} \cap X = N \cdot Y$. In this case $X_0 = Y_{\mathrm{red}}$.

Let $\mathcal{Z} \subsetneqq \mathcal{X}$ be as above, Z be its closure in X. Let $B = \mathrm{Sing}\,(X_0) \cup \mathrm{Sing}\,(X) \cup (X_0 \cap Z)$. Then B is a closed subset in X_0 and $\dim B \leqslant m-2$. Clearly, $X - B$ and $X_0 - B$ are smooth. Also $\mathcal{X} = X - X_0$.

Put $f_0 = t_0$ and stress that $X_0 = Y_{\mathrm{red}}$. Now by Corollary [4, Corollary 3.3] there exist homogeneous polynomials $f_1, \ldots, f_m$ such that for $X_i := H_{f_i} \cap X$ and any subset I in $\{0, 1, \ldots, m\}$ and the scheme intesection $X_I := \bigcap_{i\in I} X_i$ we have

1) $X_{\{0,1,\ldots,m\}} = \varnothing$;
2) $X_{\{1,\ldots,m\}}$ is smooth of dimension 0;
3) $X_{\{0,2,\ldots,m\}}$ is smooth of dimension 0;
4) $X_{\{2,\ldots,m\}}$ is smooth of dimension 1;
5) $\underline{x}$ is contained in $X_{\{2,\ldots,m\}}$;
6) $\underline{x} \cap (X_0 \cup X_1) = \varnothing$;
7) $\deg f_i$ divides $\deg f_{i+1}$ for each $i \in \{0, 1, \ldots, m-1\}$;
8) $X_{\{1,\ldots,m\}} \cap Z = \varnothing$;

9) $X_{\{0,2,\ldots,m\}} \cap Z = \varnothing$.

10) the scheme $X_{\{2,\ldots,m\}}$ is geometrically irreducible.

Use again Notation 3.5 and 3.14. Let S be as in Lemma 4.7. We claim that the morphism $q : \mathcal{X}^\circ_S \to S$ as in the diagram (6) is an elementary fibration. By Theorem 4.8 the morphism $q : \mathcal{X}^\circ_S \to S$ is a weak elementary fibration. Since the scheme $X_{\{2,\ldots,m\}}$ is geometrically irreducible it follows that $q : \mathcal{X}^\circ_S \to S$ is an elementary fibration.

Also by Theorem 4.8 $\mathcal{Z}^\circ_S := \mathcal{Z} \cap \mathcal{X}^\circ$ is finite over S. The theorem is proved. $\square$

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