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EXTENSIONS OF POONEN'S THEOREMS

Abstract. We prove some extensions of Poonen's form of Bertini type theorems over a finite field. These extensions are stated and proved in section 2. First applications are given in section 3. Further applications will be given in a next paper.

§1. Introduction

Let X be a quasi-projective subscheme of $\mathbf{P}^n$ of dimension $m \geqslant 0$ over $\mathbb{F}_q$. Let

$$\varnothing = X_{n+1} \subset \ldots \subset X_2 \subset X_1 = X$$

be a filtration of X by closed subvarieties such that $X_i - X_{i+1}$ is smooth equidimensional of dimension $m_i \geqslant 0$. Then there exist homogeneous polynomials f over $\mathbb{F}_q$ such that for each i the intersection of $X_i - X_{i+1}$ and the hypersurface $f = 0$ is smooth of dimension $m_i - 1$. In fact, the set of such f has a positive density, equal to $\prod_{i=1}^{n} \zeta_{X_i - X_{i+1}}(m_i + 1)^{-1}$, where for an $\mathbb{F}_q$-scheme Y the function $\zeta_Y(s) = Z_Y(q^{-s})$ is the zeta function of Y.

As applications we prove Corollary 3.3 and Theorem 3.4. Further applications will be given in a next paper. Particularly, we expect to give a detailed proof of existences of an elementary fibration for smooth varieties over a finite field. The latter statement was formulated in [2, Proposition 2.3] and a sketch of its proof was given there [2, Appendix B].

§2. Poonen's type theorems

Let $\mathbb{F}_q$ be a finite field of $q = p^a$ elements. Let $S = \mathbb{F}_q[x_0, \ldots, x_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$, let $S_d \subset S$ be the $\mathbb{F}_q$-subspace of homogeneous polynomials of degree d, and let $S_{\text{hom}} = \cup_{d=0}^{\infty} S_d$. For each $f \in S_d$, let H_f be the subscheme $\text{Proj}(S/(f)) \subset \mathbb{P}^r$. Typically (but not always), H_f is a hypersurface of dimension $r - 1$ defined by the equation

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$f = 0$. Define the density of a subset $\mathcal{P} \subseteq S_{\text{hom}}$ by

$$\mu(\mathcal{P}) := \lim_{d\to\infty} \frac{\sharp(\mathcal{P} \cap S_d)}{\sharp(S_d)}$$

if the limit exists.

Notation 2.1. Let $U \subset \mathbb{P}^r$ be a smooth quasi-projective equidimensional subscheme of dimension m. Let $f \in S_{\text{hom}}$. We write $H_f \pitchfork U$ if the scheme $H_f \cap U$ is smooth of dimension $m-1$.

Let $P \in U$. We write $(H_f \pitchfork U)_P$ if $f(P) = 0$ and the scheme $H_f \cap U$ is smooth of dimension $m-1$ at the point P.

Theorem 2.2 (Bertini type theorem)**.** *Let X be an arbitrary quasi-projective subscheme of $\mathbb{P}^r$ over $\mathbb{F}_q$. Let*

$$\varnothing = X_{n+1} \subset \ldots \subset X_1 \subset X_0 = X \tag{1}$$

be a filtration of X by closed subvarieties such that $U_i := X_i - X_{i+1}$ is smooth equidimensional of dimension $m_i \geqslant 0$. Define

$$\mathcal{P} := \{f \in S_{\text{hom}} : \textit{for each } i \textit{ one has } H_f \pitchfork U_i\}.$$

Then $\mu(\mathcal{P}) = \prod_i \zeta_{U_i}(m_i+1)^{-1}$ and $\mu(\mathcal{P}) > 0$.

Let $W \subseteq \mathbb{P}^r$ be a finite subscheme of $\mathbb{P}^r$. The following notation is taken from [3, Theorem 1.2]. Given $g \in R_d$ let $g|_W$ be the element of $H^0(W, \mathcal{O}_W)$ that on each connected component W_i equals the restriction of g/x_j^d to W_i, where $j = j(i)$ is the smallest $j \in \{0, \ldots, r\}$ such that the coordinate x_j is invertible on Z_j. Theorem 2.2 is a partial case of the following result. Indeed, taking $W_i = \varnothing$ for all i and $B = \varnothing$ in Theorem 2.3 we get Theorem 2.2.

Theorem 2.3 (Bertini type theorem)**.** *Let X be an arbitrary quasi-projective subscheme of $\mathbb{P}^r$ over $\mathbb{F}_q$. Let $m = \dim X \geqslant 2$. Let*

$$\varnothing = X_{n+1} \subset \ldots \subset X_1 \subset X_0 = X \tag{2}$$

be a filtration of X by closed subvarieties and $B \subset X$ is a closed subset with dim$B \leqslant m-2$. Let $Y_i = X_i - B$. Let $W \subset \mathbb{P}^r$ be a finite subscheme and $T \subset H^0(W, \mathcal{O}_W)$ a nonempty subset. Suppose $V_i := Y_i - Y_{i+1}$ is smooth equidimensional of dimension $m_i \geqslant 0$. Put $U_i = V_i - W$ and consider a set

$$\mathcal{P}^T := \{f \in S_{\text{hom}} : f|_W \in T \textit{ and for each } i \textit{ one has } H_f \pitchfork U_i\}.$$

Then $\mu(\mathcal{P}^T) = \frac{\sharp(T)}{\sharp H^0(W,\mathcal{O}_W)} \prod_{i=0}^{n} \zeta_{U_i}(m_i+1)^{-1}$ *and* $\mu(\mathcal{P}^T) > 0$.

Let U be a smooth equidimensional of dimension m quasi-projective subscheme of $\mathbb{P}^r$ over $\mathbb{F}_q$ and $r > 0$ be an integer. Define Let $U_{<r}$ be the set of closed points of U of degree $< r$. Similarly define $U_{>r}$. Let W be a finite subscheme of $\mathbb{P}^r$ with $W \cap U = \varnothing$. Let T be a subset in $H^0(W,\mathcal{O}_W)$. Define

$$\mathcal{P}_r^U := \{f \in S_{\mathrm{hom}} : (H_f \pitchfork U)_P \text{ at all } P \in U_{<r}\},$$
$$\mathcal{P}_r^{U,T} := \{f \in S_{\mathrm{hom}} : H_f \pitchfork U \text{ at all } P \in U_{<r} \text{ and } f|_W \in T\},$$
$$\mathcal{P}^U := \{f \in S_{\mathrm{hom}} : H_f \pitchfork U\}.$$
$$\mathcal{P}^{U,T} := \{f \in S_{\mathrm{hom}} : H_f \pitchfork U \text{ and } f|_W \in T\}.$$

Lemma 2.4 (Lemma 2.2 of [3]). $\mu(\mathcal{P}_r^U) = \prod_{P \in U_{<r}} (1 - q^{-(m+1)\deg P})$;
$\mu(\mathcal{P}_r^{U,T}) = \frac{\sharp(T)}{\sharp H^0(W,\mathcal{O}_W)} \prod_{P \in U_{<r}} (1 - q^{-(m+1)\deg P})$.

As mentioned in the proof of [3, Lemma 2.4], the number of closed points of degree r in U is $O(qrm)$; this guarantees that the product defining $\zeta_U(s)^{-1}$ converges at $s = m+1$. By Lemma 2.4

$$\lim_{r\to\infty} \mu(\mathcal{P}_r^U) = \zeta_U(m+1)^{-1}$$

and

$$\lim_{r\to\infty} \mu(\mathcal{P}_r^{U,T}) = \frac{\sharp(T)}{\sharp H^0(W,\mathcal{O}_W)} \zeta_U(m+1)^{-1}.$$

Define

$$\mathcal{P}_r^T := \{f \in S_{\mathrm{hom}} : \text{for each } \ i \ (H_f \pitchfork U_i)_P \text{ at all } P \in (U_i)_{<r} \text{ and} f|_W \in T\}.$$
$$\mathcal{P}^T := \{f \in S_{\mathrm{hom}} : \text{for each } \ i \ H_f \pitchfork U_i \text{ and } f|_W \in T\}.$$

Clearly, $\mathcal{P}_r^T = \mathcal{P}_r^{U_0,T} \cap \bigcap_{i=1}^{n} \mathcal{P}_r^{U_i}$ and $\mathcal{P}^T = \mathcal{P}^{U_0,T} \cap \bigcap_{i=1}^{n} \mathcal{P}^{U_i}$. Hence Lemma 2.4 yields the following

Lemma 2.5. *One has*

$$\mu(\mathcal{P}_r^T) = \Big[\frac{\sharp(T)}{\sharp H^0(W,\mathcal{O}_W)} \prod_{P \in (U_0)_{<r}} (1 - q^{-(m_0+1)\deg P})\Big]$$
$$\times \prod_{i=1}^{n} \prod_{P \in (U_i)_{<r}} (1 - q^{-(m_i+1)\deg P}).$$

Corollary 2.6.

$$\lim_{r\to\infty}\mu(\mathcal{P}^T_r)=\Big[\frac{\sharp(T)}{\sharp H^0(W,\mathcal{O}_W)}\zeta_{U_0}(m_0+1)^{-1}\Big]\prod_{i=1}^{n}[\zeta_{U_i}(m_i+1)^{-1}].$$

Define

$$Q^{U_i}_{\mathrm{med},r}:=\cup_{d\geqslant 0}\{f\in S_d:\exists P\in U_i \text{ with } r\leqslant \deg P\leqslant d/(m_i+1)$$
$$\text{such that } H_f\cap U_i \text{ is not smooth of dimension } m_i-1 \text{ at } P\},$$
$$Q^{U_i}_{\mathrm{high}}:=\cup_{d\geqslant 0}\{f\in S_d:\exists P\in (U_i)_{>d/(m_i+1)} \text{ such that}$$
$$H_f\cap U_i \text{ is not smooth of dimension } m_i-1 \text{ at } P\}$$

The following inclusions are obvious

$$\mathcal{P}^T\subseteq\mathcal{P}^T_r\subseteq\mathcal{P}^T\cup\bigcup_{i=0}^{n}Q^{U_i}_{\mathrm{med},r}\cup\bigcup_{i=0}^{n}Q^{U_i}_{\mathrm{high}}\tag{3}$$

Proof of Theorem 2.3. Due to inclusions (3) as $\bar{\mu}(\mathcal{P}^T)$, so $\underline{\mu}(\mathcal{P}^T)$ each differ from $\mu(\mathcal{P}^T_r)$ by at most $\sum\limits_{i=0}^{n}\bar{\mu}(Q^{U_i}_{\mathrm{med},r})+\sum\limits_{i=0}^{n}\bar{\mu}(Q^{U_i}_{\mathrm{high}})$. By [3, Lemma 2.4] for each $i\in\{0,1,\dots,n\}$ one has $\lim\limits_{r\to\infty}\bar{\mu}(Q^{U_i}_{\mathrm{med},r})=0$. By [3, Lemma 2.6] for each $i\in\{0,1,\dots,n\}$ one has $\bar{\mu}(Q^{U_i}_{\mathrm{high}})=0$. By Corollary 2.6 the limit $\lim\limits_{r\to\infty}\mu(\mathcal{P}^T_r)$ exists. Thus letting r tend to infinity, we obtain

$$\mu(\mathcal{P}^T)=\lim_{r\to\infty}\mathcal{P}^T_r=\frac{\sharp(T)}{\sharp H^0(W,\mathcal{O}_W)}\prod_{i=0}^{n}\zeta_{U_i}(m_i+1)^{-1}.\tag{4}$$

□

§3. First applications

Proposition 3.1. *Let X be a projective equidimensional subscheme of $\mathbb{P}^r$ over $\mathbb{F}_q$. Let $m\geqslant 2$ be the dimension of X. Let $f_0\in S_{\mathrm{hom}}$ be such that $Y:=H_{f_0}\cap X$ is equidimensional of dimension $m-1$. Put $X_0=Y_{\mathrm{red}}$. Let $B\subset X_0$ be a closed subset with $\dim B\leqslant m-2$ and such that $X-B$ and X_0-B are smooth. Let $\underline{x}=\{x_1,\dots,x_l\}$ be a finite set of closed points in $\mathcal{X}:=X-X_0$. Then there exists a homogeneous polynomial $f_1\in S_{\mathrm{hom}}$ such that*

1) *for the scheme $X_1:=H_{f_1}\cap X$ the one X_1-B is smooth of dimension $m-1$;*

2) *the scheme $(X_0\cap X_1)-B$ is smooth of dimension $m-2$;*

3) $\underline{x} \cap (X_0 \cup X_1) = \varnothing$;

4) $\deg f_0$ *divides* $\deg f_1$.

Let $\{Z_c\}_{c \in C}$ *be any finite family of closed irreducible subsets in* X. *Then one can choose* $f_1 \in S_{\text{hom}}$ *such that additionally for each* $c \in C$ *one has* $\dim(X_1 \cap Z_c) \leqslant \dim Z_c - 1$. *Particularly, one can choose* $f_1 \in S$ *such that*

5) $\dim(X_1 \cap X_0) \leqslant m - 2$;

6) $\dim(X_1 \cap B) \leqslant m - 3$.

If X *is absolutely irreducible then one can choose* $f_1 \in S_{\text{hom}}$ *such that additionally*

7) *the scheme* X_1 *is absolutely irreducible.*

Proof. (of Proposition 3.1). Take the filtration $\varnothing \subset X_0 \subset X$. Put $Y_1 = X - B$, $Y_0 = X_0 - B$. Clearly, Y_0 and Y_1 are smooth. Consider a finite family W of closed points in X such that W contains at least one point of each irreducible component of X_0 and B and contains at least one point of each Z_c ($c \in C$). Suppose also that W contains $\underline{x}$. Consider W as a closed subscheme of $\mathbb{P}^r$ with the reduced structure. For each $w \in W$ set $T_w = \mathbb{F}_q - \{0\}$ and take $T = \prod_{w \in W} T_w \subset \Gamma(W, \mathcal{O}_W)$. Put $U_{(0)} = Y_1 - Y_0$ and $U_{(1)} = Y_0$, and

$$\mathcal{P}^T := \{f \in S_{\text{hom}} : f|_W \in T \text{ and for each } s \text{ one has } H_f \pitchfork (U_{(s)} - W)\}.$$

Clearly, for each $f_1 \in \mathcal{P}^T$ the assertions (1)–(3),(5) and (6) are true. By Theorem 2.3 the density $\mu(\mathcal{P}^T)$ of $\mathcal{P}^T$ is well-defined and it is positive. Thus we may choose $f_1 \in \mathcal{P}^T$ such that conditions (1)–(6) and (4) are satisfied.

Suppose now that X is absolutely irreducible. Consider a set

$$\mathcal{P}_1 := \{f \in S_{\text{hom}} : \text{the scheme } H_f \cap X \text{ is absolutely irreducible}\}.$$

Clearly, for each $f_1 \in \mathcal{P}^T \cap \mathcal{P}_1$ the assertions (1)–(3), (5), (6), and (7) are true. By [1, Theorem 1.1] the density of $\mathcal{P}_1$ is 1. Thus the density of $\mathcal{P}^T \cap \mathcal{P}_1$ is well-defined and positive. So, we may choose $f_1 \in \mathcal{P}^T \cap \mathcal{P}_1$ such that conditions (1)–(3), (5)–(7) are satisfied and the condition (4) is satisfied too. □

Theorem 3.2. *Let* $X \subset \mathbb{P}^r$, $m \geqslant 2$, $f_0 \in S_{\text{hom}}$, Y, $X_0 \subset X$, $B \subset X_0$, $\mathcal{X} = X - X_0$ *and* $\underline{x} \subset \mathcal{X}$ *be the same as in Proposition* 3.1. *Then for each integer* n *with* $1 \leqslant n \leqslant m$ *there exist homogeneous polynomials* $f_1, \ldots, f_n$ *such that for schemes* $X_i := H_{f_i} \cap X$ *and any subset* I *in* $\{0, 1, \ldots, n\}$ *and the scheme intesection* $X_I := \bigcap_{i \in I} X_i$ *we have*

1) *the scheme* $X_I - B$ *is smooth of dimension* $m - |I|$;
2) $\underline{x} \cap (X_0 \cup X_1) = \varnothing$;
3) $\underline{x}$ *is contained in* $\bigcap_{i=2}^{n} X_i$;
4) $\deg f_i$ *divides* $deg f_{i+1}$ *for each* $i \in \{0, 1, \dots, n-1\}$;

Let $\{Z_c\}_{c \in C}$ *be any finite family of closed irreducible subsets in* X. *Then one can choose* $f_1, \dots, f_n \in S$ *such that additionally for each* $c \in C$ *and any subset* I *in* $\{1, \dots, n\}$ *one has* $\dim(X_I \cap Z_c) \leqslant \dim Z_c - |I|$. *Particularly, one can choose* $f_1, \dots, f_n \in S$ *such that*
5) *for any subset* I *in* $\{0, 1, \dots, n\}$ *the scheme* X_I *has dimension* $m - |I|$;
6) *for any subset* I *in* $\{1, \dots, n\}$ *one has* $\dim(X_I \cap B) \leqslant \dim B - |I|$.

If X *is absolutely irreducible then one can choose* $f_1, \dots, f_n \in S$ *such that additionally*
7) *for any* I *in* $\{1, \dots, n\}$ *with* $|I| < m$ *the scheme* X_I *is absolutely irreducible.*

Proof. Assuming the theorem is true for all integers strictly less than n prove it for the integer n. Proposition 3.1 shows that we may assume $n \geqslant 2$. Thus we are given with $f_1, \dots, f_{n-1}$ which enjoy properties (1)–(6). Our aim is to find $f_n \in S_{\mathrm{hom}}$ such that $f_1, \dots, f_n$ enjoy properties (1)–(6).

For each $s \in \{1, \dots, n\}$ put $X_{(s)} := \bigcup_{\mathrm{Card}(I)=s} X_I$, where $I \subset \{0, 1, \dots, n-1\}$. Put $X_{(0)} := X$. Consider the following filtration on X via closed subschemes

$$\varnothing \subset X_{(n)} \subset \ldots \subset X_{(1)} \subset X_{(0)} = X \tag{5}$$

Put $Y_{(s)} = X_{(s)} - B$. Clearly, $U_s := X_{(s)} - X_{(s+1)}$ is smooth equidimensional of dimension $m_s = m - s \geqslant 0$. If $n = 2$, then put $\mathbb{X} = X$. If $n > 2$, then put $\mathbb{X} = \bigcap_{i=2}^{n-1} X_i$.

Consider all closed sets of the form $Z_c \cap X_I$ ($c \in C$ and $I \subset \{0, 1, \dots, n-1\}$). Consider all closed sets of the form X_I with $I \subset \{0, 1, \dots, n-1\}$ and all closed sets of the form $B \cap X_J$ with $J \subset \{1, \dots, n-1\}$. Let E be a set enumerating all irreducible components of these sets. So, for each $e \in E$ there is a unique irreducible component Z_e of one of the mentioned closed sets. For each e in E choose one point $z_e \in Z_e$. If $\dim Z_e > 0$, then choose $z_e \in Z_e - \underline{x}$. For each $x \in \underline{x}$ set $x^{(2)} = \mathrm{Spec}(\mathcal{O}_{\mathbb{X}}/m^2_{\mathbb{X},x})$. Put $W'_n = (\sqcup_{e \in E} z_e) - \underline{x}$ and $W_n = W'_n \sqcup (\sqcup_{x \in \underline{x}} x^{(2)})$. For each $x \in \underline{x}$ put $T_x = (m_{\mathbb{X},x}/m^2_{\mathbb{X},x}) - \{0\}$. If $w \in W'_n$, then put $T_w = \mathbb{F}_q(w) - \{0\}$. Put $T = (\prod_{w \in W'_n} T_w) \times (\prod_{x \in \underline{x}} T_x) \subset \Gamma(W, \mathcal{O}_W)$. Put $U_{(s)} = Y_{(s)} - Y_{(s+1)}$ and consider

$$\mathcal{P}^T := \{f \in S_{\text{hom}} : f|_W \in T \text{ and for each } s \text{ one has } H_f \pitchfork (U_{(s)} - W)\}.$$

It is straight forward to check that for each $f_n \in \mathcal{P}^T$ the polynomials $f_1, \dots, f_n$ enjoy the properties (1)–(5) and (6). By Theorem 2.3 the density $\mu(\mathcal{P}^T)$ of $\mathcal{P}^T$ is well-defined and it is positive. Thus we may choose $f_n \in \mathcal{P}^T$ such that $f_1, \dots, f_n$ enjoy properties (1)–(6) and (4).

Suppose now that X is absolutely irreducible. Let I be a subset of $\{1, \dots, n-1\}$. If $|I| < m-1$, put $\mathcal{P}_I := \{f \in S_{\text{hom}}$: the scheme $H_f \cap X_I$ is absolutely irreducible$\}$. Write $\text{Sub}(n-1)$ for the set of all subsets I of $\{1, \dots, n-1\}$ with $|I| < m-1$. Clearly, for each $f_n \in \mathcal{P}^T \cap \bigcap_{I \in \text{Sub}(n-1)} \mathcal{P}_I$ the assertions (1)–(3),(5)–(6) and (7) are true. By [1, Theorem 1.1] for each $I \in \text{Sub}(n-1)$ the density of $\mathcal{P}_I$ is 1. Thus the density of $\mathcal{P}^T \cap \bigcap_{I \in \text{Sub}(n-1)} \mathcal{P}_I$ is well-defined and positive. So, we may choose $f_n \in \mathcal{P}^T \cap \bigcap_{I \in \text{Sub}(n-1)} \mathcal{P}_I$ such that conditions (1)–(3), (5)–(6), (7) are satisfied and the condition (4) is satisfied too. The proof is completed.

□

Corollary 3.3. *Let $X \subset \mathbb{P}^r$, $m \geqslant 2$, $f_0 \in S_{\text{hom}}$, Y, $X_0 \subset X$, $B \subset X_0$, $\mathcal{X} = X - X_0$ and $\underline{x} \subset \mathcal{X}$ be the same as in Proposition* 3.1*. Then there exist homogeneous polynomials $f_1, \dots, f_m$ such that for $X_i := H_{f_i} \cap X$ and any subset I in $\{0, 1, \dots, m\}$ and the scheme intesection $X_I := \bigcap_{i \in I} X_i$ we have*

1) $X_{\{0,1,\dots,m\}} = \varnothing$;
2) $X_{\{1,\dots,m\}}$ *is smooth of dimension* 0;
3) $X_{\{0,2,\dots,m\}}$ *is smooth of dimension* 0;
4) $X_{\{2,\dots,m\}}$ *is smooth of dimension* 1;
5) $\underline{x}$ *is contained in* $X_{\{2,\dots,m\}}$;
6) $\underline{x} \cap (X_0 \cup X_1) = \varnothing$;
7) $\deg f_i$ *divides* $\deg f_{i+1}$ *for each* $i \in \{0, 1, \dots, m-1\}$.

Let Z be a closed subset in X with $\dim Z \leqslant m-1$ such that $(Z \cap X_0) \subset B$. Then one can choose $f_1, \dots, f_m \in S_{\text{hom}}$ such that additionally

8) $X_{\{1,\dots,m\}} \cap Z = \varnothing$;
9) $X_{\{0,2,\dots,m\}} \cap Z = \varnothing$.

If X is absolutely irreducible then one can choose $f_1, \dots, f_m \in S$ such that additionally

10) *the scheme* $X_{\{2,\dots,m\}}$ *is absolutely irreducible.*

Also, Theorem 3.2 easily implies the following result.

Theorem 3.4 (Bertini type theorem). *Let X be a smooth projective equidimensional subscheme of $\mathbb{P}^r$ over $\mathbb{F}_q$. Let $\underline{x} = \{x_1, \ldots, x_l\}$ be a finite set of closed points in X. Let $m \geqslant 0$ be the dimension of X. There exist homogeneous polynomials $f_0, f_1, \ldots, f_m$ of degrees $e_0, e_1, \ldots, e_m$ respectively such that the subschemes $X_i := H_{f_i} \cap X$ enjoy the following properties:*

1) for any $I \subset \{0, 1, \ldots, m\}$ the intesection $X_I := \cap_{i \in I} X_i$ is smooth of dimension $m - |I|$;

2) X_0 and X_1 do not contain any point of $\underline{x}$;

3) for any $i > 1$ the scheme X_i contains the set $\underline{x}$; 4) for each $i = 0, 1, \ldots, m-1$ the number e_i divides e_{i+1}.

Let Z be a closed subset in X with $dimZ \leqslant m-1$. Then one can choose $f_0, \ldots, f_m \in S_{\text{hom}}$ such that additionally $(\cap_{i=1}^m X_i) \cap Z = \varnothing$ and $(\cap_{i=2}^m X_i) \cap (Z \cap X_0) = \varnothing$

We will write $\mathcal{O}(e)$ for $\mathcal{O}_{\mathbb{P}^r}(e)|_X$ and $s_i \in \Gamma(X, \mathcal{O}(e_i))$ for $f_i|_X$.

Proposition 3.5. *Let $\mathbb{P}^{m,w}$ be the weighted projective space with homogenneous coordinates $[t_0 : t_1 : \ldots : t_m]$ of weights $1, e_1/e_0, \ldots, e_m/e_0$ respectively. Then under the notation of Theorem 3.4 the morphism*

$$\pi = [s_0 : s_1 : \ldots : s_m] : X \to \mathbb{P}^{m,w} \tag{6}$$

is well-defined and finite.

Proof. One has $X = \cup_{j=0}^m X^{(j)}$, where $X^{(j)} := X_{s_j \neq 0}$. Let $\mathbb{P}_j^{m,w}$ be the open subscheme of $\mathbb{P}^{m,w}$, where the j-th weighted coordinate does not vanish. Then $\pi^{-1}(\mathbb{P}_j^{m,w}) = X^{(j)}$. Since each $X^{(j)}$ is affine, the morphism π is affine. At the same time π is projective. Thus, π is finite. □

Let $\mathbb{P}^{m-1,w}$ be the weighted projective space with homogeneous coordinates $[x_1 : \ldots : x_m]$ of weights $1, e_2/e_1, \ldots, e_m/e_1$ respectively.

Proposition 3.6. *Under the hypotheses of Corollary 3.3 let $M \subset X$ be a closed subscheme such that $X_{1,\ldots,m} \cap M = \varnothing$. Then the morphism $[s_1 : \ldots : s_m] : M \to \mathbb{P}^{m-1,w}$ is finite. Particularly, morphisms*
$[s_1 : \ldots : s_m] : Y \to \mathbb{P}^{m-1,w}$,
$[s_1 : \ldots : s_m] : B \to \mathbb{P}^{m-1,w}$,
$[s_1 : \ldots : s_m] : Z \to \mathbb{P}^{m-1,w}$
are finite.

Proof. They are affine and projective. □

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