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NON-STABLE K_1-FUNCTORS OF DISCRETE VALUATION RINGS CONTAINING A FIELD

Abstract. Let k be a field, and let G be a simply connected semisimple k-group which is isotropic and contains a stricty proper parabolic k-subgroup P. Let D be a discrete valuation ring which is a local ring of a smooth algebraic curve over k. We show that $K_1^G(D) = K_1^G(K)$, where K is the fraction field of D and $K_1^G(-) = G(-)/E_P(-)$ is the corresponding non-stable K_1-functor, also called the Whitehead group of G. As a consequence, $K_1^G(D)$ coincides with the (generalized) Manin's R-equivalence class group of $G(D)$.

§1. Introduction

Let R be a commutative ring with 1. Let G be a reductive group scheme over R in the sense of [4]. For any reductive group G over R and a parabolic subgroup P of G, one defines the elementary subgroup $E_P(R)$ of $G(R)$ as the subgroup generated by the R-points of the unipotent radicals of P and of any opposite parabolic R-subgroup P^-, and considers the corresponding non-stable K_1-functor $K_1^{G,P}(R) = G(R)/E_P(R)$ [14, 17, 18]. It does not depend on the choice of P^- by [4, Exp. XXVI, Corollary 1.8]. In particular, if $A = k$ is a field and P is minimal, $E(k)$ is nothing but the group $G(k)^+$ introduced by J. Tits [19], and $K_1^G(k)$ is the subject of the Kneser–Tits problem [5]. If $G = \mathrm{GL}_n$ and P is a Borel subgroup, then $K_1^G(R) = \mathrm{GL}_n(R)/E_n(R)$, $n \geqslant 1$, are the usual non-stable K_1-functors of algebraic K-theory.

A parabolic R-subgroup P in G is called strictly proper, if it intersects properly every non-trivial semisimple normal R-subgroup of G. If R is semilocal (or, more generally, a local-global ring, see [6]), then $E_P(R)$ is

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the same for all strictly proper parabolic R-subgroups P of G [17, Theorem 2.1], and, in particular, is normal in $G(R)$.

Let R be a regular local ring and K be its field of fractions. The Serre–Grothendieck conjecture ([15, Remarque, p. 31], [8, Remarque 3, p. 26–27], and [10, Remarque 1.11.a]) predicts that for any reductive group G over R, the natural map between the first non-Abelian étale cohomology sets

$$H^1_{ét}(R,G) \to H^1_{ét}(K,G) \tag{1}$$

has trivial kernel. It is known to hold in many cases, in particular, for all regular local rings R which are equicharacteristic, i.e., contain a field [13], and for all isotropic reductive groups over rings which are geometrically regular over a Dedekind ring [1]. The étale cohomology functor $H^1_{ét}(-,G)$ is often viewed as a non-stable, non-Abelian version of the K_0-functor of algebraic K-theory, and the map (1) itself is reminiscent of the first term in the Gersten conjecture. Building on this analogy, it is natural to ask if the map

$$K_1^{G,P}(R) \to K_1^{G,P}(K)$$

is injective under the same assumptions on R.

We say that G has isotropic rank $\geqslant n$ over R, if every non-trivial semisimple normal subgroup of G contains an R-subgroup of the form $\mathbb{G}^n_{m,R}$. It was proved in [18] that if G has isotropic rank $\geqslant 2$ over R, and R is any regular local ring containing a field, then

$$K_1^{G,P}(R) \to K_1^{G,P}(K) \tag{2}$$

is injective, where K is the fraction field of R and P is a strictly proper parabolic R-subgroup. However, the method of proof was definitely inapplicable to groups of isotropic rank 1. Namely, it relied on the injectivity of the map

$$K_1^{G,P}(R[x]) \to K_1^{G,P}(R[x]_f),$$

where $f \in R[x]$ is a monic polynomial; and this injectivity is false for $G = SL_2$, $f = x$ and R any discrete valuation ring [11][1] Thus, the injectivity of (2) is not known in general even for equicharacteristic discrete valuation rings (although for SL_2, as well as for any other simply connected split group, it is trivially true).

[1]In fact, if B is a standard Borel subgroup of SL_2 and R is a discrete valuation ring, then $K_1^{SL_2,B}(R[x])$ is not a group and, in particular, is not trivial [11, Proposition 1.8], while $K_1^{SL_2,B}(R[x]_x) = 1$ [11, Theorem 3.1].

In [7] we extended Manin's notion of R-equivalence of points from algebraic varieties over fields to schemes over commutative rings. The very definition of the R-equivalence implies that $K_1^{G,P}(R) = G(R)/E_P(R)$ surjects onto the R-equivalence class group $G(R)/\mathcal{R}$ for any R [7, 2.1, 4.3]. It has been previously known that for any field K and any simply connected semisimple group G having a strictly parabolic K-subgroup one has $K_1^{G,P}(K) = G(K)/\mathcal{R}$ [5]. In [7, Proposition 8.10] we have extended this equality to henselian discrete valuation rings and concluded that the map (2) is an isomorphism for every simply connected semisimple group G and every henselian discrete valuation ring R. In the present paper we obtain the following results in the non-henselian case.

Theorem 1.1. *Let k be a field, let G be a reductive algebraic group over k having a strictly proper parabolic k-subgroup P. Let D be a discrete valuation ring which is a local ring at a closed point of a smooth algebraic curve over k, and let K be the fraction field of D. Then $K_1^{G,P}(D) \to K_1^{G,P}(K)$ is injective.*

Corollary 1.2. *Let k be a field, let G be a reductive algebraic group over k having a strictly proper parabolic k-subgroup P. Let D be a discrete valuation ring which is a local ring of a finitely generated k-algebra and assume that the residue field of D is a separable extension of k. Let K be the fraction field of D. Then $K_1^{G,P}(D) \to K_1^{G,P}(K)$ is injective.*

Corollary 1.2 is an easy consequence of Theorem 1.1, since one may show that any D as in this Corollary is in fact a local ring of a smooth curve over a suitable transcendental extension of k. This is well-known to specialists, however, we were unable to find an exact reference, so we provide a proof in the end of the paper.

Corollary 1.3. *Let k, G, P and D be as in Theorem* 1.1*, or as in Corollary* 1.2*. Assume, moreover, that G is a simply connected semisimple group. Then $K_1^{G,P}(D) \cong G(D)/\mathcal{R}$.*

Proof. As mentioned above, there is a surjective homomorphism $K_1^{G,P}(D) \to G(D)/\mathcal{R}$. Since $K_1^{G,P}(D) \to K_1^{G,P}(K)$ is injective and $K_1^{G,P}(K) \to G(K)/\mathcal{R}$ is an isomorphism [5], we conclude that $K_1^{G,P}(D) \to G(D)/\mathcal{R}$ is injective. Hence the claim. □

The surjectivity of the map $K_1^{G,P}(D) \to K_1^{G,P}(K)$ may hold only for simply connected semisimple groups, since it fails, for example, for $D = k[[t]]$ and $G = GL_n$ or PGL_n. We settle it in this generality.

Theorem 1.4. *Let D be any discrete valuation ring, let K be the fraction field of D, and let G be a simply connected semisimple group scheme over D having a strictly proper parabolic D-subgroup scheme P. Then the natural map $K_1^{G,P}(D) \to K_1^{G,P}(K)$ is surjective.*

§2. Preliminary lemmas

Lemma 2.1. *Let B be a Noetherian commutative ring, let G be a reductive group over B, let P be a parabolic B-subgroup of G and let U_P be the unipotent radical of P. Let A be a commutative B-algebra and let $C_1, C_2 \subseteq A$ be two B-subalgebras of A such that $A = C_1 + C_2$. Then $U_P(A) = U_P(C_1) \cdot U_P(C_2)$.*

Proof. By [4, Exp. XXVI, Proposition 2.1] there is a sequence

$$U_0 = U_P \supset U_1 \supset U_2 \supset \dots U_n \supset \dots$$

of B-subgroup schemes of U_P with the following properties.

(1) Each U_i is B-smooth, connected and a closed characteristic subgroup of P.
(2) For every B-algebra R one has $[U_i(R), U_j(R)] \subseteq U_{i+j+1}(R)$ for all $i, j \geqslant 0$.
(3) For all $i \geqslant 0$ there is a finitely generated projective B-module V_i and an isomorphism of B-group schemes
$U_i/U_{i+1} \cong W(V_i)$, where $W(V_i)$ is the canonical affine B-scheme corresponding to V_i in the sense of [4].
(4) One has $U_i = 1$, as soon as $i > \dim_s((U_P)_s)$ for all $s \in \operatorname{Spec}(B)$.

Note that since B is Noetherian, the last property implies that there is a finite integer $N \geqslant 0$ such that $U_{N+1} = 1$. Taken together, these properties also imply that $H^1_{\acute{e}t}(R, U_i) = 0$ for any commutative B-algebra R and $i \geqslant 0$, see the proof of [4, Exp. XXVI, Cor. 2.2]. As a consequence, there are short exact sequences of groups

$$1 \to U_{i+1}(R) \to U_i(R) \to (U_i/U_{i+1})(R) \to 1.$$

We prove that $U_i(A) = U_i(C_1) \cdot U_i(C_2)$ by descending induction on i. If $i = N + 1$, this equality is clear. Assume it holds for U_{i+1} and prove it for U_i. We have

$$\begin{aligned}(U_i/U_{i+1})(A) \cong V_i \otimes_B A &= V_i \otimes_B C_1 + V_i \otimes_B C_2 \\ &= (U_i/U_{i+1})(C_1) \cdot (U_i/U_{i+1})(C_2).\end{aligned}$$

Then

$$U_i(A) = U_i(C_1) \cdot U_i(C_2) \cdot U_{i+1}(A).$$

By the inductive assumption, we have $U_{i+1}(A) = U_{i+1}(C_2)U_{i+1}(C_1)$ (note that C_1, C_2 are interchangeable in all statements). Then

$$\begin{aligned} U_i(A) &= U_i(C_1) \cdot U_i(C_2) \cdot U_{i+1}(A) = U_i(C_1)U_i(C_2)U_{i+1}(C_1) \\ &\subseteq U_i(C_1)U_{i+1}(C_1)[U_{i+1}(C_1), U_i(C_2)]U_i(C_2) \subseteq U_i(C_1)U_{2i+2}(A)U_i(C_2) \\ &= U_i(C_1)U_{2i+2}(C_1)U_{2i+2}(C_2)U_i(C_2) = U_i(C_1)U_i(C_2). \end{aligned}$$ □

Lemma 2.2. *Let D be a Dedekind domain, let G be a reductive group over D, let P be a strictly proper parabolic D-subgroup of G. Assume that $(G/P)(D) = G(D)/P(D)$ and $G(D) = E_P(D)P(D)$. Then $G(D_S) = E_P(D) \cdot P(D_S)$ and*

$$E_P(D_S) = E_P(D) \cdot (E_P(D_S) \cap P(D_S))$$

for any multiplicatively closed subset S of D.

Proof. Let K be the fraction field of D. Since D is a Dedekind domain and G/P is a smooth projective D-scheme, we have $(G/P)(D) = (G/P)(K) = (G/P)(D_S)$ (see e.g. [9, Corollaire 7.3.6]). Since $(G/P)(D) = G(D)/P(D)$, we conclude that $G(D_S) = G(D)P(D_S)$. Since $G(D) = E_P(D) \cdot P(D)$ by assumption, it follows that $G(D_S) = E_P(D) \cdot P(D_S)$. Then also $E_P(D_S) = E_P(D) \cdot (E_P(D_S) \cap P(D_S))$. □

Lemma 2.3. *Let $B \subseteq A$ be two commutative rings, let $h \in B$ be such that h is a non-zero divisor in A and $B/hB \cong A/hA$. Let G be a reductive group scheme over B, let P, P^- be two opposite strictly proper parabolic B-subgroups of G. Assume that*

$$E_P(B_h) = E_P(B) \cdot (E_P(B_h) \cap P(B_h)) = E_P(B) \cdot (E_P(B_h) \cap P^-(B_h)). \quad (3)$$

Then $E_P(A_h) = E_P(A)E_P(B_h)$.

Proof. Let P^- be any parabolic B-subgroup of G opposite to P. Let $L = P \cap P^-$ be the Levi subgroup of P. To prove the claim of the Lemma, it is enough to prove that

$$E_P(B_h) \cdot U_Q(A_h) \subseteq E_P(A) \cdot E_P(B_h), \quad (4)$$

where Q is one of P, P^-. Indeed, we have $E_P(A_h) = \langle U_P(A_h), U_{P^-}(A_h) \rangle$, so any fixed $g \in E_P(A_h)$ has a presentation $g = u_1 u_2 \ldots u_n$, $u_i \in U_P(A_h)$ or $u_i \in U_{P^-}(A_h)$. Proceeding by induction on n, we deduce from (4) that

$$g \in E_P(A) \cdot E_P(B_h) \cdot U_Q(A_h) \subseteq E_P(A) \cdot E_P(A) \cdot E_P(B_h) = E_P(A) \cdot E_P(B_h),$$

as required.

To prove (4), we start by applying (3) and obtaining

$$\begin{aligned} E_P(B_h) &= E_P(B) \cdot (E_P(B_h) \cap Q(B_h)) \\ &\subseteq E_P(A) \cdot (E_P(B_h) \cap L(B_h)) \cdot U_Q(B_h). \end{aligned} \tag{5}$$

Since $L(B_h) \leqslant L(A_h)$ normalizes $U_Q(A_h)$, we deduce that

$$\begin{aligned} E_P(B_h) \cdot U_Q(A_h) &\subseteq E_P(A) \cdot (E_P(B_h) \cap L(B_h)) \cdot U_Q(A_h) \\ &\subseteq E_P(A) \cdot U_Q(A_h) \cdot (E_P(B_h) \cap L(B_h)). \end{aligned} \tag{6}$$

By the choice of $B \subseteq A$, we have $A = B + hA$. Replacing the A on the right-hand side of the latter equality by $B + hA$, we deduce that $A = B + h^n A$ for any $n \geqslant 1$. Since h is a non-zero divisor, B, B_h and A are subrings of A_h, and the previous equality implies $A_h = B_h + A$. Then by Lemma 2.1 we have

$$U_Q(A_h) = U_Q(A) \cdot U_Q(B_h).$$

Substituting this equality into (6), we obtain (4). □

Corollary 2.4. *Let k be a field, let G be a reductive algebraic group over k, and let P be a strictly proper parabolic subgroup of G. Then for any two coprime polynomials $f, g \in k[x]$ one has*

$$E_P(k[x]_{fg}) = E_P(k[x]_f) \cdot E_P(k[x]_g).$$

Proof. We check that Lemma 2.2 applies to G over $k[x]$. By the Margaux-Soulé theorem [12] we have $G(k[x]) = G(k) \cdot E_P(k[x])$. By [4, Exp. XXVI, Th. 5.1] $G(k) = E_P(k) \cdot P(k)$, hence $G(k[x]) = E_P(k[x]) \cdot P(k)$. Let L be a Levi subgoup of P. By [2, Prop. 2.2] the map $H^1_{ét}(k[x], L) \to H^1_{ét}(k(x), L)$ has trivial kernel. By [4, Exp. XXVI, Cor. 5.10] the map $H^1_{ét}(k(x), L) \to H^1_{ét}(k(x), G)$ is injective. Hence $H^1_{ét}(k[x], L) \to H^1_{ét}(k[x], G)$ has trivial kernel. Since $H^1_{ét}(k[x], P) = H^1_{ét}(k[x], L)$, the "long" exact sequence of étale cohomology associated to $1 \to P \to G \to G/P \to 1$ then implies that $(G/P)(k[x]) = G(k[x])/P(k[x])$. Then all the conditions of Lemma 2.2 are satisfied for G over $k[x]$, and hence

$$E_P(k[x]_f) = E_P(k[x]) \cdot (E_P(k[x]_f) \cap P(k[x]_f)).$$

The same argument applies to any opposite parabolic subgroup P^- of P. Now we see that Lemma 2.3 applies to G with $B = k[x]$, $A = k[x]_g$ and $h = f$. It follows that $E_P(k[x]_{fg}) = E_P(k[x]_f) E_P(k[x]_g)$. □

Corollary 2.5. *Let $B \subseteq A$ be two discrete valuation rings with a common uniformizer $h \in B$, and such that $B/hB \cong A/hA$. Let G be a reductive group scheme over B having a strictly proper parabolic B-subgroup scheme P. Then $E_P(A_h) = E_P(A) \cdot E_P(B_h)$.*

Proof. As in the proof of Corollary 2.4, it is enough to check that the conditions of Lemma 2.2 hold for G over B, and then apply Lemma 2.3. Since B is a local ring, one has $G(B) = E_P(B) \cdot P(B)$ by [4, Exp. XXVI, Theorem 5.1]. Also, by [4, Exp. XXVI, Corollary 5.10] the map $H^1_{ét}(B,P) \to H^1_{ét}(B,G)$ is injective. Hence the "long" exact sequence of étale cohomology associated to $1 \to P \to G \to G/P \to 1$ implies that $(G/P)(B) = G(B)/P(B)$. Then all the conditions of Lemma 2.2 are satisfied. □

Lemma 2.6. *Let $B = k[x]_p$ be a discrete valuation ring which is a local ring of an affine line over a field k, let $K = k(x)$ be its fraction field. Let G be a simply connected semisimple algebraic group over k, let P be a strictly proper parabolic subgroup of G. Then we have isomorphisms*

$$K_1^{G,P}(k) \cong K_1^{G,P}(k[x]_p) \cong K_1^{G,P}(k(x)).$$

Proof. The isomorphism $K_1^{G,P}(k) \cong K_1^{G,P}(k(x))$ is [5, Theorem 5.8]. Hence the map $K_1^{G,P}(k[x]_p) \to K_1^{G,P}(k(x))$ is onto, and

$$K_1^{G,P}(k[x]_p) = K_1^{G,P}(k) \oplus \ker\big(K_1^{G,P}(k[x]_p) \to K_1^{G,P}(k(x))\big).$$

Let us establish the triviality of the kernel. This reduces to proving that

$$K_1^{G,P}(k[x]_g) \to K_1^{G,P}(k[x]_{gf})$$

has trivial kernel, where $f, g \in k[x]$ are coprime polynomials. Take $a \in G(k[x]_g) \cap E_P(k[x]_{gf})$. We need to show $a \in E_P(k[x]_g)$. By Corollary 2.4 we have that

$$E_P(k[x]_{gf}) = E_P(k[x]_g) \cdot E_P(k[x]_f).$$

Multiplying a by a suitable element of $E_P(k[x]_g)$, we then have $a \in G(k[x]_g) \cap E_P(k[x]_f)$. Since $G(k[x]_f) \cap G(k[x]_g) = G(k[x])$, we conclude that $a \in G(k[x]) \cap E_P(k[x]_f)$. By the Margaux–Soulé theorem [12] we have $G(k[x]) = G(k) \cdot E_P(k[x])$. Therefore

$$a \in G(k) \cdot E_P(k[x]) \cap E_P(k[x]_f). \tag{7}$$

If k is finite, then G is a quasi-split simply connected group over k and $G(k) = E_P(k)$, so $a \in E_P(k[x]) \subseteq E_P(k[x]_g)$, as required. If k is infinite, then there is $u \in k$ such that $f(u) \in k^\times$. Then, taking $x = u$ in (7), we see that $a|_{x=u} \in E_P(k)$, and hence $a \in E_P(k) \cdot E_P(k[x]) = E_P(k[x]) \subseteq E_P(k[x]_g)$ as well. $\square$

§3. Proof of the main results

Proof of Theorem 1.4. Let $\hat{D}$ be the complete discrete valuation ring obtained by completing D with respect to the maximal ideal, and let $\hat{K}$ be the fraction field of $\hat{D}$. Let $h \in D$ be a common uniformizer of D and $\hat{D}$. Then $K = D_h$ and $\hat{K} = \hat{D}_h$. Let $g \in G(K)$ be any element. Since $\hat{D}$ is henselian, the natural map $K_1^{G,P}(\hat{D}) \to K_1^{G,P}(\hat{K})$ is surjective by [7, Prop. 8.10]. Hence $g \in G(\hat{D}) \cdot E_P(\hat{K})$. By Corollary 2.5 we have $E_P(\hat{K}) = E_P(K) \cdot E_P(\hat{D})$. Multiplying $g \in G(K)$ by a suitable element of $E_P(K)$, we then achieve that $g \in G(\hat{D}) \leqslant G(\hat{K})$. Since $G(\hat{D}) \cap G(K) = G(D)$, we conclude that $g \in G(D)$. This proves that $G(K) = G(D) \cdot E_P(K)$, as required. $\square$

Lemma 3.1. *Let D be any discrete valuation ring, let K be the fraction field of D, and let G be a reductive group scheme over D having a strictly proper parabolic D-subgroup scheme P. Let G^{sc} be the simply connected cover of the derived subgroup G^{der} of G over D. Then G^{sc} has a strictly proper parabolic subgroup P^{sc}, and the canonical homomorphism $G^{\mathrm{sc}}(D) \to G^{\mathrm{der}}(D) \to G(D)$ induces a surjection from the kernel of the map*

$$K_1^{G^{\mathrm{sc}},P^{\mathrm{sc}}}(D) \to K_1^{G^{\mathrm{sc}},P^{\mathrm{sc}}}(K) \tag{8}$$

onto the kernel of the map

$$K_1^{G,P}(D) \to K_1^{G,P}(K). \tag{9}$$

Proof. The intersection $P^{\mathrm{der}} = G^{\mathrm{der}} \cap P$ is a strictly proper parabolic subgroup of G^{der} by [4, Exp. XXVI, Proposition 1.19]. Let $\pi : G^{\mathrm{sc}} \to G^{\mathrm{der}}$ be the canonical homomorphism, then $P^{\mathrm{sc}} = \pi^{-1}(P^{\mathrm{der}})$ is a strictly proper parabolic subgroup of G^{sc}.

There is a short exact sequence of algebraic groups

$$1 \to C \xrightarrow{i} G^{\mathrm{sc}} \xrightarrow{\pi} G^{\mathrm{der}} \to 1,$$

where C is a finite group of multiplicative type over D, central in G^{sc}. Write the respective "long" exact sequences over D and K with respect to fppf topology. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C(D) & \xrightarrow{i} & G^{\mathrm{sc}}(D) & \xrightarrow{\pi} & G^{\mathrm{der}}(D) & \xrightarrow{\delta} & H^1_{fppf}(D,C) \\ \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & C(K) & \xrightarrow{i} & G^{\mathrm{sc}}(K) & \xrightarrow{\pi} & G^{\mathrm{der}}(K) & \xrightarrow{\delta_K} & H^1_{fppf}(K,C) \end{array}$$

Here the rightmost vertical arrow is injective by [3, Theorem 4.1]. Take any $g \in G^{\mathrm{der}}(D) \cap E_{P^{\mathrm{der}}}(K)$. Then $\delta_K(g) = 1$, since $E_{P^{\mathrm{sc}}}(K)$ surjects onto $E_{P^{\mathrm{der}}}(K)$. Hence there is $\tilde{g} \in G^{\mathrm{sc}}(D)$ with $\pi(\tilde{g}) = g$. Clearly, $\tilde{g} \in C(K) \cdot E_{P^{\mathrm{sc}}}(K)$, since $\pi(\tilde{g}) \in E_{P^{\mathrm{der}}}(K)$. However, since D is a discrete valuation ring and C is finite, we have $C(D) = C(K)$. (Note that C embeds into into a quasi-split D-torus which is a maximal torus of the unique quasi-split inner D-form of G^{sc} [4, Exp. XXIV, Proposition 3.13], and hence C embeds into some split D-torus.) Therefore,

$$\tilde{g} \in G^{\mathrm{sc}}(D) \cap C(D) \cdot E_{P^{\mathrm{sc}}}(K) = C(D) \cdot (G^{\mathrm{sc}}(D) \cap E_{P^{\mathrm{sc}}}(K)).$$

Since $C(D) \leqslant \ker \pi$, it follows that $G^{\mathrm{sc}}(D) \cap E_{P^{\mathrm{sc}}}(K)$ surjects onto $G^{\mathrm{der}}(D) \cap E_{P^{\mathrm{der}}}(K)$.

Consider now the short exact sequence $1 \to G^{\mathrm{der}} \to G \to \mathrm{corad}(G) \to 1$ [4, Exp. XXIII, 6.2.3]. Taking into account that

$$\mathrm{corad}(G)(D) \to \mathrm{corad}(G)(K)$$

is injective and $E_P(D) = E_{P^{\mathrm{der}}}(D)$, $E_P(K) = E_{P^{\mathrm{der}}}(K)$, we immediately see that $G(D) \cap E_P(K) = G^{\mathrm{der}}(D) \cap E_{P^{\mathrm{der}}}(K)$. Therefore, $G^{\mathrm{sc}}(D) \cap E_{P^{\mathrm{sc}}}(K)$ surjects onto $G(D) \cap E_P(K)$, which is exactly the claim of the lemma. □

Proof of Theorem 1.1. By Lemma 3.1 we can assume that G is simply connected semisimple. If k is a finite field, then G is a quasi-split simply connected semisimple group, and since D is local, we have $K_1^{G,P}(D) = 1$ and there is nothing to prove. Assume k is infinite. We may assume that D is a local ring of a smooth irreducible algebraic curve without loss of generality. By Ojanguren's lemma [2, Lemme 1.2] there is a maximal localization $k[x]_p$ of the polynomial ring $k[x]$ at a maximal ideal $p = (f)$ and an essentially étale local homomorphism $\phi : k[x]_p \to D$ such that $\phi(f)$ is a uniformizer of D and the induced map $k[x]_p/f \cdot k[x]_p \to D/\phi(f) \cdot D$ is an isomorphism. Then the triple $B = k[x]_p$, $A = D$ and $h = f$ is subject to Corollary 2.5 (note that ϕ is injective, since it is flat). Hence $E_P(K) = E_P(D_{\phi(f)}) = E_P(D) \cdot E_P(\phi(k(x)))$. Assume that $g \in G(D)$ is mapped into

$E_P(K) \leqslant G(K)$. Then, multiplying g by an element of $E_P(D)$, we achieve that $g \in G(D) \cap E_P(\phi(k(x))) = \phi(G(k[x]_p) \cap E_P(k(x)))$. By Lemma 2.6 this implies that $g \in \phi(E_P(k[x]_p)) \leqslant E_P(D)$. Then $g \in E_P(D)$, and the injectivity of $K_1^{G,P}(D) \to K_1^{G,P}(K)$ is proved. □

§4. Proof of Corollary 1.2

The following two lemmas are very standard, however, we were unable to find perfectly matching references.

Lemma 4.1. *Let k be a field, and let A be a finitely generated k-algebra. Let D be a local ring of A. Then there is a purely transcendental field extension $l = k(x_1, \dots, x_n)$ of k such that D is a localization of a finitely generated l-algebra at a maximal ideal. Moreover, if the residue field of D is separable over k, we may secure that it is separable over l.*

Proof. Let $q \subset A$ be a prime ideal such that $D = A_q$, and let L be the residue field of D. We have $L = \mathrm{Frac}(A/q)$. Let $L' = k(t_1, \dots, t_n)$ be a purely transcendental field extension of k such that $k \subseteq L' \subseteq L$ and L is a finite extension of L', which is also separable if L is separable over k. Let $x_1, \dots, x_n \in A$ be any lifts of $t_1, \dots, t_n$. Then $l = k(x_1, \dots, x_n)$ is a purely transcendental field extension of k that embeds into $D = A_q$ and is mapped isomorphically onto L' inside $L = A_q/qA_q$. Then $A' = A \otimes_{k[x_1,\dots,x_n]} l$ is a finitely generated l-algebra such that D is a localization of A' at a prime ideal q'. Moreover, L is a finite extension of $l \cong L'$, so, in particular, q' is a maximal ideal of A'. □

Lemma 4.2. *Let k be a field, and let A be a finitely generated k-algebra. Let D be a local ring of A such that D is regular and the residue field L of D is separable over k. Then there is a purely transcendental field extension $l = k(x_1, \dots, x_n)$ of k such that D is a localization of an integral smooth l-algebra at a maximal ideal.*

Proof. Let $q \subset A$ be a prime ideal such that $D = A_q$. By Lemma 4.1 we can assume without loss of generality that q was a maximal ideal of A. Since D is a regular ring and its residue field L is separable over k, and A is of finite type over k, by [16, Tag 00TV] A is smooth over k at q. By definition, it means that there is $g \in A$, $g \notin q$, such that $k \to A_g$ is a smooth ring map (in particular, A_g is of finite type over k). Since A_g is smooth over k, by [16, Tag 00TT] the k-algebra A_g is a regular ring. Hence A_g is a finite direct sum of regular domains, and there is $f \in A$ such that

$f \notin q$ and A_{gf} is a regular domain, so that D is a localization of A_{gf} at qA_{gf}. Thus, D is a localization of an integral smooth k-algebra A_{gf} at a maximal ideal. □

Proof of Corollary 1.2. Follows immediately from Theorem 1.1 and Lemma 4.2. □

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