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SEMIGROUP APPROACH TO ADMISSIBLE REPRESENTATIONS OF THE INFINITE SYMMETRIC GROUP

Abstract. Let $S(\infty)$ denote the group of finitary permutations of the set $\mathbb{N} := \{1, 2, 3, \dots\}$. It is a countable group admitting a lot of different topologies compatible with the group structure. In particular, such topologies arise from partitions of the set $\mathbb{N}$ into blocks of infinite size. The corresponding categories of continuous unitary representations of $S(\infty)$ were studied by Nessonov (Sbornik: Mathematics, 2012). We propose a different approach to his classification results based on the so-called semigroup method. Some additional information is also obtained.

§1. Introduction

1.1. Suppose that we have an infinite-dimensional group G (usually G is an inductive limit of some compact, finite or finite-dimensional groups $G(n)$) and suppose that we fix some subgroup $K \subset G$ which would play the role of a maximal compact subgroup of G. We define the notion of a *tame representation* of a group K, and consider unitary representations of G that become tame after being restricted to K. Such representations are called *admissible* representations of the pair (G, K). We will also call them K-admissible representations of G. Different choices of K may lead to different classes of admissible representations.

Different examples of (G, K)-pairs were studied in [2–5].

The pair (G, K) often allows one to construct a series $\mathfrak{G}(n)$ of finite semigroups in some natural way. This semigroup approach can be extremely useful as there are important connections between the representations of thus obtained semigroups and K-admissible representations of G.

In this paper we consider pairs $(S(\infty), K^\alpha)$, where $S(\infty)$ is the group of all finite permutations of the set $\mathbb{N} := \{1, 2, \dots\}$, and K^α is the Young subgroup of $S(\infty)$ that respects a partition α of $\mathbb{N}$ into countable sets $\mathbb{N} = \bigsqcup \alpha_i$.

Key words and phrases: infinite symmetric group, Young subgroups, admissible representations, semigroups.

The total classification of admissible representations of these types of pairs was firstly obtained by Nessonov in [1]. We will give another proof of these results making use of the semigroup approach.

1.2. Let us denote by $S(X)$ the group of finite permutations on the set X. When $X = \mathbb{N}$ and $X = [n] := \{1, 2, \ldots, n\}$ we will write $S(\infty)$ and $S(n)$ respectively. It will be convenient to us to realise $S(X)$ as the group of all strictly monomial matrices indexed by the set X.

Definition 1.1. A *partition* of a countable set X is a collection of countable sets $\alpha = \{\alpha_i\}_{i\in I}$ such that

$$X = \bigsqcup_{i\in I} \alpha_i.$$

The set of all partitions of X will be denoted by $\mathcal{P}(X)$. When X equals $\mathbb{N}$ we will omit $\mathbb{N}$ from the notation, and sometimes, when we want to fix an index set I, we will write $\mathcal{P}^I(X)$.

Each $\alpha \in \mathcal{P}$ defines a group topology on $S(\infty)$ in the following way.

Definition 1.2. For any $\alpha \in \mathcal{P}^I(X)$ let K^α denote the Young subgroup of $S(X)$ corresponding to α. It consists of all permutations that respect the partition α. It is clear that K^α is a direct sum of symmetric groups on the sets α_i

$$K^\alpha = \sum_{i\in I} S(\alpha_i).$$

For a finite subset $Y \subset X$ let us denote by $\bar{Y}$ the set $\mathbb{N}\backslash Y$, by $S_Y(X)$ the subgroup of $S(\infty)$ of all finite permutations of the set $\bar{Y}$ (in other words, all permutations fixing the set Y pointwise), and set $K^\alpha_Y = K^\alpha \cap S_Y(X)$. When $Y = [n] \subset \mathbb{N}$ we will write $S_n(\infty)$ and K^α_n respectively.

Suppose now that $X = \mathbb{N}$.

Groups K^α_n form a decreasing chain

$$K^\alpha = K^\alpha_0 \supset K^\alpha_1 \supset \cdots$$

and satisfy the following conditions:

(1) $\bigcap_{n\in\mathbb{N}} K^\alpha_n = \{1\}$;
(2) $K^\alpha_m \cap S(n)$ is finite for any $m \leqslant n$;
(3) K^α_n and $S(n)$ commute for all $n \in \mathbb{N}$.

The conditions above guarantee the existence of the unique group topology on $S(\infty)$ for which $\{K^\alpha_n\}$ form a neighbourhood basis of unity. We will call it *α-topology*.

Remark 1.3. The completion of $S(\infty)$ in this topology consists of all bijections g of $\mathbb{N}$ such that for any $i \neq j \in I$ the number $d_{ij}(g) = |\{k \in \alpha_j \mid g(k) \in \alpha_i\}|$ is finite, and for any $i \in I$ we have

$$\sum_{j \neq i} d_{ij}(g) = \sum_{j \neq i} d_{ji}(g).$$

1.3.

Definition 1.4. Let $\alpha \in \mathcal{P}$. A unitary representation T of $S(\infty)$ is said to be α-*admissible* if it is continuous in α-topology.

For any unitary representation T of $S(\infty)$ we will denote by $H(T)$ the Hilbert space of T, and by $H_X^\alpha(T)$ its subspace of all K_X^α-invariant vectors. Once again, when $X = [n]$, we will write just $H_n^\alpha(T)$. The representation T is α-admissible if and only if the space

$$H_\infty^\alpha(T) := \bigcup_{n \in \mathbb{N}} H_n^\alpha(T)$$

is dense in $H(T)$.

When the partition α is fixed we will usually omit the upper index α from the notation.

Remark 1.5. When the representation T is irreducible, it is α-admissible if and only if $H_n^\alpha \neq 0$ for some n.

Note that if we change α in a finite way, the category of admissible representations will not change. More formally, suppose we have two partitions α and β. We are interested in the relations between the corresponding categories $\mathrm{Adm}(\alpha)$ and $\mathrm{Adm}(\beta)$ of equivalence classes of admissible representations.

Notation 1.6. For any finite set $Y \subset \mathbb{N}$ let us denote by $\alpha^{(Y)}$ the partition of Y^c obtained from α by removing the set Y. In other words, $\alpha_i^{(Y)} = \alpha_i \cap \bar{Y}$. When $Y = [n]$ we will write $\alpha^{(n)}$.

Definition 1.7. Let $\alpha \in \mathcal{P}^I$, $\beta \in \mathcal{P}^J$ be two partitions of $\mathbb{N}$.

(1) We say that two partitions α and β are *equivalent*, $\alpha \sim \beta$, if $\alpha^{(Y)} = \beta^{(Y)}$ for some finite set Y.

Note that $\alpha \sim \beta$ if and only if $K_n^\alpha = K_n^\beta$ for all n large enough. It follows that they yield the same categories $\mathrm{Adm}(\alpha) = \mathrm{Adm}(\beta)$.

We will denote by $[\alpha]$ the equivalence class of α.

(2) We say that α is *finer* than β (and that β is *coarser* than α) and write $\alpha \succcurlyeq \beta$ if for any $j \in J$ there exists $I_j \subset I$ such that

$$\beta_j = \bigsqcup_{i \in I_j} \alpha_i.$$

Note that if $\alpha \succcurlyeq \beta$, then $K^\alpha \subset K^\beta$.

We will write $\alpha = \beta$ when $\alpha \succcurlyeq \beta$ and $\beta \succcurlyeq \alpha$, i.e., when β can be obtained from α by re-indexing its parts.

(3) We say that class $[\alpha]$ is *finer* than class $[\beta]$, $[\alpha] \succcurlyeq [\beta]$, if $\alpha^{(Y)} \succcurlyeq \beta^{(Y)}$ for some finite set Y. This definition does not depend on the choice of representatives α and β.

Note that $[\alpha] \succcurlyeq [\beta]$ if and only if $K_n^\alpha \subset K_n^\beta$ for all n large enough, and if $[\alpha] \succcurlyeq [\beta]$ and $[\beta] \succcurlyeq [\alpha]$, then $\alpha \sim \beta$.

It follows that $\mathrm{Adm}(\alpha) \supset \mathrm{Adm}(\beta)$ when $[\alpha] \succcurlyeq [\beta]$.

The structure of this paper is as follows. In Section 2 we introduce semigroups of double cosets $\mathfrak{G}^\alpha(X)$ and describe the connection between their representations and α-admissible representations of $S(\infty)$. In Section 3 we describe some particular subclass of α-admissible representations, namely, spherical representations. The proof of the classification of α-spherical representations using the semigroup approach was done by Neretin in [2] and we will not repeat it. In Section 4 we use the classification of α-spherical representations to give the semigroup-theoretical proof of the classification result of irreducible α-admissible representations, and in Section 5 give the necessary and sufficient conditions on unitary equivalence of two irreducible α-admissible representations. Section 6 is an application of the classification result: we describe irreducible representations that are admissible with respect to two partitions simultaneously.

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§2. Semigroups of double cosets

In this section we describe the set $\mathfrak{G}^\alpha(X)$ of double cosets $K_X^\alpha \backslash S(\infty) / K_X^\alpha$ and endow it with a semigroup structure.

2.1. Firstly, we agree that in this paper when we say "semigroup" we mean "involutive semigroup with unity", i.e., we require a semigroup $\mathfrak{G}$ to possess a unity 1 and an involutive anti-homomorphism $\mathfrak{g} \mapsto \mathfrak{g}^*$.

Definition 2.1. Let X be a set.

(i) A *partial bijection* of X is a bijection $A\colon \mathrm{dom}(A) \to \mathrm{range}(A)$ between two (possible empty) subsets $\mathrm{dom}(A), \mathrm{range}(A) \subset X$. The set of all partial bijections of X will be denoted by $\mathrm{PB}(X)$.

(ii) For $A_1, A_2 \in \mathrm{PB}(X)$ we define their product A_1A_2 in the natural way: A_1A_2 is defined on x whenever both $A_2(x)$ and $A_1(A_2(x))$ are defined, and for any $x \in \mathrm{dom}(A_1A_2)$

$$A_1A_2(x) = A_1(A_2(x)).$$

(iii) Given $A \in \mathrm{PB}(X)$ we define $A^* \in \mathrm{PB}(X)$ such that $\mathrm{dom}(A^*) = \mathrm{range}(A)$, $\mathrm{range}(A^*) = \mathrm{dom}(A)$, and for any $x \in \mathrm{dom}(A^*)$

$$A^*(x) = A^{-1}(x).$$

Under these operations $\mathrm{PB}(X)$ becomes a semigroup with an involution A^* and a unity $\mathrm{id}\colon X \to X$.

Notation 2.2. If X' is a subset of X, we will denote by $1_{X'}$ an idempotent defined by

$$1_{X'}(x) = \begin{cases} x, & x \in X'; \\ \text{not defined}, & x \notin X'. \end{cases}$$

Remark 2.3. It is convenient to realise semigroup $\mathrm{PB}(X)$ as the semigroup of all not strictly monomial matrices, i.e., matrices $A = (A_{xy})_{x,y\in X}$ defined by

$$A_{xy} = \begin{cases} 1 & \text{when } A(y) = x \\ 0 & \text{otherwise.} \end{cases}$$

The product of partial bijections is the matrix product and $A^* = A^t$ is the transpose of A.

Let us now describe the set of double cosets $K_X^\alpha \backslash S(\infty)/K_X^\alpha$ for some finite subset X of $\mathbb{N}$.

Proposition 2.4. *For any $g \in S(\infty)$ the double coset $K_X^\alpha g K_X^\alpha$ is uniquely determined by the data $\mathfrak{g} = (A^X(g), B_i^{\alpha,X}(g), C_j^{\alpha,X}(g), d_{ij}^{\alpha,X}(g) \mid i,j \in I)$ defined as follows:*

(1) *$A^X(g)$ is a partial bijection on the set X obtained from g by taking the submatrix of g corresponding to the set X. More formally,*

$$\operatorname{dom}(A^X(g)) = \{k \in X \mid g(k) \in X\}, \quad \operatorname{range}(A^X(g)) = g(\operatorname{dom}(A^X(g))),$$
$$A^X(g)(k) = g(k);$$

(2) *$B_i^{\alpha,X}(g)$ is a subset of X defined by*

$$B_i^{\alpha,X}(g) = \{k \in X \mid g(k) \in \alpha_i^{(X)}\}, \quad i \in I;$$

(3) *$C_j^{\alpha,X}(g)$ is a subset of X defined by*

$$C_j^{\alpha,X}(g) = \{k \in X \mid g^{-1}(k) \in \alpha_j^{(X)}\}, \quad j \in I;$$

(4) *For $i \neq j$ the number $d_{ij}^{\alpha,X}(g)$ is the size of the finite set*

$$D_{ij}^{\alpha,X}(g) = \{k \in \alpha_j^{(X)} \mid g(k) \in \alpha_i^{(X)}\}, \quad i \neq j$$

and $d_{ij}^{\alpha,X}(g) = \infty$ when $i = j$.

When there is no confusion we will omit α from the notation. In the proof of the proposition we assume that α is fixed.

Proof. It is clear that this data does not change after multiplication by elements of K_X on either side.

Now suppose that $g, h \in S(\infty)$ have the same data. We may assume that g, h lie in $S(M)$ for some M big enough. Let us write $D_{ii}(g)$ for the set $\{k \in \alpha_i^{(X)} \mid g(k) \in \alpha_i^{(X)}\} \cap [M]$. Note that $|D_{ii}(g)| = |D_{ii}(h)|$ for all $i \in I$.

We want to find $k_1, k_2 \in K_X$ such that $k_1 h = g k_2$. Let us fix for any i, j some bijection φ_{ij} from the set $D_{ij}(h)$ to the set $D_{ij}(g)$. Then we set

$$k_1(x) = \begin{cases} (gh^{-1})(x) & \text{for } x \in h(B_i(h)); \\ (g\varphi_{ij}h^{-1})(x) & \text{for } x \in h(D_{ij}(h)); \\ x & \text{otherwise.} \end{cases}$$

$$k_2(k) = \begin{cases} (g^{-1}h)(x) & \text{for } x \in h^{-1}(C_i(h)); \\ \varphi_{ij}(x) & \text{for } x \in D_{ij}(h); \\ x & \text{otherwise.} \end{cases}$$

It is easy to check that k_1, k_2 satisfy the desired properties. □

There is a more convenient way to write this data. Put

$$\mathfrak{g} = (A, (B_i), (C_i), (d_{ij}))$$

as above. We will write A as a finite monomial matrix $A = (A_{kl})_{k,l\in X}$, the set B_i as a row vector indexed by X, $B_i = (B_{ik})_{k\in X}$

$$B_{ik} = \begin{cases} 1 & \text{if } k \in B_i \\ 0 & \text{if } k \notin B_i \end{cases}$$

and the set C_j as a column vector $C_j = (C_{jk})_{k\in X}$ in the similar way.

Now we may write $\mathfrak{g}$ as a following (infinite when I is infinite) matrix with coefficients in $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$:

$$\mathfrak{g} = \left[\begin{array}{c|ccc} A & C_{i_1} & C_{i_2} & \cdots \\ \hline B_{i_1} & \infty & d_{i_1 i_2} & \dots \\ B_{i_2} & d_{i_2 i_1} & \infty & \dots \\ \vdots & \vdots & \vdots & \infty \end{array}\right]. \tag{2.1}$$

More formally, $\mathfrak{g}$ becomes a matrix indexed by the set $X \sqcup I$ defines as follows:

$$\mathfrak{g}_{ab} = \begin{cases} A_{ab} & \text{if } a, b \in X; \\ C_{ba} & \text{if } a \in X, b \in I; \\ B_{ab} & \text{if } a \in I, b \in X; \\ d_{ab} & \text{if } a, b \in I. \end{cases} \tag{2.2}$$

For the sake of brevity, we will write

$$\mathfrak{g} = \begin{bmatrix} A & (C_j) \\ (B_i) & (d_{ij}) \end{bmatrix}.$$

Definition 2.5. For any finite set X and any decomposition $\alpha \in \mathcal{P}^I$ let $\mathfrak{G}^\alpha(X)$ be the set of all matrices $\mathfrak{g}$ indexed by the set $X \sqcup I$ with coefficients in $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ such that

(1) for any $k \in X$ the corresponding row $(\mathfrak{g}_{*k})$ (and column $(\mathfrak{g}_{k*})$) has exactly one entry equal to 1 and all the rest equal to 0;
(2) for any $i \in I$ the entry $\mathfrak{g}_{ii}$ equals ∞;
(3) for any $i \neq j \in I$ the entry $\mathfrak{g}_{ij}$ lies in $\mathbb{Z}_{\geqslant 0}$ and only finitely many of them are not zero;
(4) for any $i \in I$ holds

$$\sum_{k\in X} \mathfrak{g}_{ik} + \sum_{I\ni j\neq i} \mathfrak{g}_{ij} = \sum_{I\ni j\neq i} \mathfrak{g}_{ji} + \sum_{k\in X} \mathfrak{g}_{ki}.$$

It is clear that for any such $\mathfrak{g}$ we can find an element $g \in S(\infty)$ such that $K_X g K_X$ corresponds to $\mathfrak{g}$. So we have the following

Proposition 2.6. *For any partition α of the set $\mathbb{N}$ the map $\theta^{\alpha,X}: g \mapsto \mathfrak{g}$ described in the statement of Proposition* 2.4 *gives a bijection between the set of double cosets $K_X^\alpha \backslash S(\infty)/K_X^\alpha$ and $\mathfrak{G}^\alpha(X)$.*

Now we want to define on $\mathfrak{G}^\alpha(X)$ a structure of a semigroup.

Let us once again omit α from notation.

Definition 2.7. For any permutation g in $S(X)$ its *support* is the set of all points in X that are not fixed under the action of g.

Let $\mathfrak{f}, \mathfrak{h}$ be from $\mathfrak{G}(X)$, and f, h be their corresponding representatives in $S(\infty)$. We say that representatives f and h are *in general position* if the size of the intersection of their supports is as small as possible. We define the multiplication in $\mathfrak{G}(X)$ as

$$\mathfrak{f} \cdot \mathfrak{h} = \theta^X(fh).$$

Proposition 2.8. *The multiplication is correctly defined* (*i.e., doesn't depend on the choice of representatives in general position*) *and is associative.*

Proof. Since f and h are in general position, we have

(i) $h(B_i(h)) \cap \operatorname{supp}(f) = f^{-1}(C_i(f)) \cap \operatorname{supp}(h) = \emptyset$ for any $i \in I$;

(ii) $D_{ij}(f) \cap \operatorname{supp}(h) = D_{ij}(h) \cap \operatorname{supp}(f) = \emptyset$ for any $i \neq j \in I$;

(iii) $f(D_{ij}(f)) \cap \operatorname{supp}(h) = h(D_{ij}(h)) \cap \operatorname{supp}(f) = \emptyset$ for any $i \neq j \in I$.

Let now $g = fh$. Then conditions above give us the following formulas (all multiplications are multiplications as matrices). We give below informal explanations, but the formulas can be checked directly.

1. $A(g) = A(f)A(h)$. To see this, observe that condition (i) ensures that any point $k \in X$ that leaves the set X under the action of h cannot return back under f. So we have $\operatorname{dom}(A(g)) = \operatorname{dom}(A(h)) \cap h^{-1}(\operatorname{dom}(A(f)))$ and $A(g) = A(f) \cdot A(h)$ as partial bijections.
2. $B_i(g) = B_i(h) + B_i(f)A(h)$. An element $k \in X$ moves to the set α_i^X under the action of g in exactly two cases: either it is moved there by h (in which case f does not move it again), or it stays in X under h and is then moved to $\alpha_i^{(X)}$ by f.
3. $C_j(g) = C_j(f) + A(f)C_j(h)$. To see this, observe that $C_j(g) = B_j(g^{-1})^t$.
4. $d_{ij}(g) = d_{ij}(h) + d_{ij}(f) + B_i(f)C_j(h)$. An element $k \in \alpha_j^{(X)}$ moves to $\alpha_i^{(X)}$ in three cases: it was moved there by h (and is left there by f); it

was left in α_j^X by h, and then moved by f; or it was moved by h to X and then moved to $\alpha_i^{(X)}$ by f.

So we have the following formula for the multiplication:

$$\begin{bmatrix} A(f) & (C_j(f)) \\ (B_i(f)) & (d_{ij}(f)) \end{bmatrix} \cdot \begin{bmatrix} A(h) & (C_j(h)) \\ (B_i(h)) & (d_{ij}(h)) \end{bmatrix} = \begin{bmatrix} A(f)A(h) & (C_j(f) + A(f)C_j(h)) \\ (B_i(h) + B_i(f)A(h)) & (d_{ij}(f) + d_{ij}(h) + B_i(f)C_j(h)) \end{bmatrix}. \quad (2.3)$$

From this formula the correctness follows automatically and the associativity can be checked directly. □

When X is the empty set, we will write $\mathfrak{G}^\alpha$ instead of $\mathfrak{G}^\alpha(\emptyset)$.

Example 2.9. The semigroup $\mathfrak{G}^\alpha$ is realised as the semigroup of matrices $\mathfrak{g} = [d_{ij}]_{i \neq j \in I}$ with $d_{ij} \in \mathbb{Z}_{\geqslant 0}$ satisfying the condition

$$\sum_{j \neq i} d_{ij} = \sum_{j \neq i} d_{ji}$$

and with semigroup operation given by matrix addition. It is therefore a commutative semigroup.

We define an involution on $\mathfrak{G}(X)$ by matrix transposition. One can see that $(\theta_X(g))^* = \theta_X(g^{-1})$.

There is another equivalent way to define multiplication on $\mathfrak{G}(X)$.

Notation 2.10. Let Y be some finite subset of $\mathbb{N}$, and let $Y = X \sqcup X'$. We fix some element $w^\alpha_{Y,X}$ of $S(\infty)$ such that

(i) $w^\alpha_{Y,X}$ lies in K_X;
(ii) $w^\alpha_{Y,X}(X') \subset \bar{Y}$;

Proposition 2.11. *The multiplication in $\mathfrak{G}^\alpha(X)$ can be defined in the following way: let f, h be arbitrary elements of $S(\infty)$ and Y be such that $X \subset Y$ and $f, h \in S(Y)$. Then*

$$\theta^{\alpha,X}(f) \cdot \theta^{\alpha,X}(h) = \theta^{\alpha,X}(f w^\alpha_{Y,X} h).$$

The result does not depend on the choice of Y and $w^\alpha_{Y,X}$.

The proof of this proposition is a straightforward, but rather unpleasant check.

We will denote the image of $w^\alpha_{Y,X}$ under $\theta^{\alpha,Y}$ by $\epsilon^\alpha_{Y,X}$. One can easily see that

$$\epsilon^\alpha_{Y,X} = \begin{bmatrix} 1_X & (X' \cap \alpha_i) \\ (X' \cap \alpha_i) & (0) \end{bmatrix}.$$

2.2. Let H be a complex Hilbert space of finite or countable dimension. A *contraction* on H is an operator with norm ≤ 1. Let $C(H)$ be the set of contractions. It is a semigroup with involution (the conventional conjugation of bounded operators). If H has countable dimension, we endow $C(H)$ with the weak operator topology.

By a *representation* of a semigroup $\mathfrak{G}$ on H we mean a homomorphism $\mathfrak{T}\colon \mathfrak{G} \to C(H)$ which preserves the unity and is compatible with the involution, that is, $\mathfrak{T}(\mathfrak{g}^*) = (\mathfrak{T}(\mathfrak{g}))^*$ for all $\mathfrak{g} \in \mathfrak{G}$.

Let T be some unitary representation of $S(\infty)$. For any partition $\alpha \in \mathcal{P}$ and a finite set X, let P^α_X denote the orthogonal projection to the space $H^\alpha_X(T)$ of K^α_X-invariant vectors. Then for any $g \in S(\infty)$ the operator

$$P^\alpha_X\, T(g)|_{H^\alpha_X(T)} : H^\alpha_X(T) \to H^\alpha_X(T)$$

depends only on the double coset $K^\alpha_X g K^\alpha_X$. Thus we have a correctly defined map

$$\mathfrak{T}^\alpha_X : \mathfrak{G}^\alpha(X) \to \operatorname{End}(H^\alpha_X(T)), \quad \mathfrak{T}^\alpha_X : \theta^{\alpha,X}(g) \mapsto P^\alpha_X\, T(g)|_{H^\alpha_X(T)}.$$

Let us again omit α from notations for the rest of this section.

Proposition 2.12. *For any unitary representation T of $S(\infty)$ and any finite set X the corresponding map $\mathfrak{T}_X$ is a representation of a semigroup $\mathfrak{G}(X)$. It is irreducible when T is irreducible, and if T is also admissible, then T is uniquely defined by $\mathfrak{T}_X$ for any X s.t. $H_X(T)$ is not* 0.

The proof of this proposition is similar to the proofs of similar results for other pairs G, K, for example, see [3, Theorem 2.5]. The important step in the proof is the following

Lemma 2.13. *For any finite sets $X \subset Y \subset \mathbb{N}$ holds*

$$P_Y T(w_{Y,X}) P_Y = P_X.$$

In particular, the orthogonal projection

$$P_{Y,X} \colon H_Y(T) \to H_X(T)$$

is given by $\mathfrak{T}_Y(\epsilon_{Y,X})$.

§3. Spherical representations

3.1.

Definition 3.1. Let K be a subgroup of a group G. We say that (G, K) is a *Gelfand pair* if for any unitary representation T of G and any $g, h \in G$ the operators $PT(g)P$ and $PT(h)P$ commute. Here P is the orthogonal projection onto the space $H(T)^K$ of K-invariant vectors.

Remark 3.2. Note that if (G, K) is a Gelfand pair, it follows that for any irreducible unitary representation T the space $H(T)^K$ is at most one-dimensional.

Definition 3.3. A unitary representation of G is a *spherical representation* of a Gelfand pair (G, K) if it possesses a cyclic K-invariant vector ξ (it means that the orbit of ξ under the action of G is total in $H(T)$). We will always assume that $\|\xi\| = 1$.

Such a vector will be called a *spherical vector*, and the corresponding matrix element $\varphi(g) := \langle T(g)\xi; \xi\rangle$ will be called a *spherical function* of T. It is well-known that any irreducible spherical representation if uniquely determined by its spherical function (see, for example, [6, Chapter 8])

Proposition 3.4 (cf. [1, Proposition 3.6]). *For any partition $\alpha \in \mathcal{P}$ the pair $(S(\infty); K^\alpha)$ is a Gelfand pair.*

Let T be a spherical representation of the pair $(S(\infty), K^\alpha)$. Then the corresponding representation $\mathfrak{T}$ of the semigroup $\mathfrak{G}^\alpha$ is one-dimensional and the spherical function $\varphi(g)$ of T is given by the character of the representation $\mathfrak{T}$ of the semigroup $\mathfrak{G}^\alpha$.

Proof. Recall that $\mathfrak{G}^\alpha$ is a commutative semigroup. So for any irreducible representation T of $S(\infty)$ the operators $PT(g)P = \mathfrak{T}(\theta^\alpha(g))$ commute. The second part follows immediately from the fact that ξ is K-invariant. □

The total classification of spherical representations of the pair $(S(\infty), K^\alpha)$ was first obtained by Nessonov in [1]. In [2] Neretin gives another proof, using the semigroup approach. Strictly speaking, Neretin proves this result only in the case of partitions into finitely many parts (i.e., $|I| = m < \infty$), but in fact, all of his arguments work for the infinite case.

For the sake of brevity we will call unitary representation T of $S(\infty)$ that is spherical to a pair $(S(\infty), K^\alpha)$ an α-spherical representation. Now we will show how to construct irreducible α-spherical representations of $S(\infty)$.

3.2. Let us recall the definition of the countable tensor product of Hilbert spaces.

Suppose we have a collection of Hilbert spaces $(V)_{k\in\mathbb{N}}$ and let us fix some unit vectors $\xi_k \in V_k$.

For any $n \leqslant m$ we define an inclusion

$$\bigotimes_{i=k}^{n} V_k \to \bigotimes_{k=1}^{m} V_k$$

by

$$v_1 \otimes \cdots \otimes v_n \mapsto v_1 \otimes \cdots \otimes v_n \otimes \xi_{n+1} \cdots \otimes \xi_m.$$

The inductive limit of this direct system is an inner product space and we will denote its completion by $\bigotimes_{k\in\mathbb{N}}(V_k, \xi_k)$.

When all V_k are copies of the same Hilbert space V we will write $V^{\otimes\infty}(\xi)$, where ξ denotes the vector $\xi = \xi_1 \otimes \xi_2 \otimes \dots$

Let us choose in each V_k some orthonormal basis $\xi_k = e_0^{(k)}, e_1^{(k)}, e_2^{(k)}, \dots$ Then $\bigotimes_{k\in\mathbb{N}}(V_k, \xi_k)$ has an orthonormal basis e_f

$$e_f = e_{f(1)}^{(1)} \otimes e_{f(2)}^{(2)} \otimes \cdots,$$

with almost all values $f(k)$ equal to 0.

3.3. Now we will construct some representations.

Fix a partition $\alpha \in \mathcal{P}^I$.

Let V be a Hilbert space, and $(v_i)_{i\in I}$ be some unit vectors generating V and such that v_i and v_j are not collinear for $i \neq j$. We take a countable number of copies of V and choose $\xi_k = v_i$ when k lies in α_i.

The infinite symmetric group $S(\infty)$ acts on the space $V^{\otimes\infty}(\xi)$ by permuting the terms.

Let H be the closure of the cyclic span of ξ under this action, i.e., $H = \overline{\operatorname{span}\{T(g)\xi \mid g \in S(\infty)\}}$, and let S be the restriction of the representation above to H. The following theorem holds.

Theorem 3.5. 1. *This representation is irreducible. It is β-spherical for any $\beta \succcurlyeq \alpha$.*

2. (*cf.* [1, Theorem 3.4]) *Its spherical function is*

$$\varphi(g) = \prod_{i\neq j\in I} \langle v_j, v_i\rangle^{d_{ij}^{\alpha}(g)}$$

(*here we assume that* $0^0 = 1$).

It follows that S is uniquely determined by a partition α and an $I \times I$ Gram matrix $G = (\langle v_j, v_i\rangle)_{i,j\in I}$. We will denote this representation by $S(\alpha, G)$.

3. (*cf.* [1, Proposition 4.10]) *Two representations $S(\alpha, G)$ and $S(\alpha, G')$ are equivalent if and only if there is a complex diagonal matrix $D = (d_i)_{i\in I}$ with $|d_i| = 1$, such that $G' = DGD^{-1}$. In other words, if and only if the corresponding systems of vectors (v_i), (v_i') can be obtained from each other by composition of an isometry and a diagonal map $v_i \mapsto d_i v_i$, $|d_i| = 1$.*

   This gives us an equivalence relation on Gram matrices. We will denote by $\mathcal{G}^I$ the set of some representatives of these equivalence classes.
4. (*cf.* [1, Theorem 3.7], [2, Theorem 8.1]) *All β-spherical representations can be obtained in this way. I.e., if S is some β-spherical representation, then there is a partition $\alpha \in \mathcal{P}^I$ such that $\alpha \preccurlyeq \beta$ and a Gram matrix $G \in \mathcal{G}^I$ such that S is equivalent to $S(\alpha, G)$.*

Proof. We will not prove the last statement.

1. The space of K_n^α-invariant vectors in $V^{\otimes\infty}(\xi)$ coincides with $V^{\otimes n}$. It follows that ξ is the only (up to multiplication by scalar) K^α-invariant vector in $V^{\otimes\infty}(\xi)$, hence S is irreducible. Indeed, if $H(S) = H \oplus H'$ and P is the orthogonal projection to H, then $P(\xi)$ is K^α-invariant and, therefore, either $H = H(S)$ or $H' = H(S)$. It is also clear that ξ is K_β-invariant for any partition β finer then α.
2. The spherical function is

$$\varphi(g) = \langle T(g)\xi, \xi\rangle = \prod_k \langle \xi_{g^{-1}(k)}, \xi_k\rangle = \prod_{i\neq j} \langle v_j, v_i\rangle^{d_{ij}^\alpha(g)}.$$

3. In [2, Lemma 8.2] Neretin shows that the semigroup $\mathfrak{G}^\alpha$ is generated by cycles $\mathfrak{s}[i_1, \dots, i_p]$ that are defined as follows: if we write $\mathfrak{G}^\alpha$ as a semigroup of matrices, then any element from $\mathfrak{G}^\alpha$ can be written as a linear combination (with coefficients in positive integers) of elementary matrices E_{ij} for $i \neq j$ (with ∞ on the diagonal). Then for any $p \geqslant 2$ and any pairwise distinct $i_1, \dots, i_p \in I$ we define cycles as elements

$$\mathfrak{s}[i_1 \dots, i_p] = E_{i_1 i_2} + E_{i_2 i_3} + \dots + E_{i_p i_1}.$$

   Any element $\mathfrak{g} \in \mathfrak{G}^\alpha$ can be represented as a finite product of cycles.

   Now, two irreducible spherical representations S and S' are equivalent if and only if their corresponding spherical functions φ and φ' are the same. Or, equivalently, if the characters χ, χ' of the semigroup $\mathfrak{G}^\alpha$

are the same. Now, it is enough to compute the value of χ and χ' on cycles $\mathfrak{s}[i_1, \dots, i_p]$. We have the following formula

$$\chi(\mathfrak{s}[i_1, \dots, i_p]) = g_{i_1 i_2} \cdots g_{i_p i_1}.$$

So we are left to prove the following.

Lemma 3.6. *Let G, G' be two Gram matrices indexed by the set I such that $g_{ii} = g'_{ii} = 1$. Then $G' = DGD^{-1}$ for some diagonal complex matrix D, $|d_i| = 1$, if and only if for any $p \geqslant 2$ and any pairwise distinct $i_1, \dots, i_p \in I$ the following holds*:

$$g_{i_1 i_2} \cdots g_{i_p i_1} = g'_{i_1 i_2} \cdots g'_{i_p i_1}. \tag{3.1}$$

Proof. Note that the condition $G' = DGD^{-1}$ means that $g'_{ij} = g_{ij}\frac{d_i}{d_j}$ for any $i, j \in I$.

The "only if" part is trivial. Now suppose that (3.1) is satisfied. In particular, $|g_{ij}|^2 = |g'_{ij}|^2$ and $g_{ij} = 0$ if and only if $g'_{ij} = 0$.

Let Γ be undirected graph with vertices indexed by I and such that two vertices i, j are connected if and only if $g_{ij} \neq 0$. Let C denote the set of all connected components c and let us choose in every component c a spanning tree Γ_c with a fixed root i_c. For any $i \in c$ we set the value d_i by induction on the length of the path from i_c to i. We set $d_{i_c} = 1$ and for any path $i_c, \dots, j, i$ we set $d_i = d_j \frac{g'_{ij}}{g_{ij}}$.

We defined all d_i for $i \in c$, and if i and j are connected by an edge in the tree we have the required property. Now suppose that i and j are connected in Γ, but not in Γ_c. Then if we add this edge to Γ_c we will obtain a cycle $j = i_1, i_2, \dots, i_p = i$. Applying (3.1) we obtain

$$\frac{g'_{ij}}{g_{ij}} = \frac{g_{i_1 i_2} \cdots g_{i_{p-1} i_p}}{g'_{i_1 i_2} \cdots g'_{i_{p-1} i_p}} = \frac{d_{i_2} \cdots d_{i_p}}{d_{i_1} \cdots d_{i_{p-1}}} = \frac{d_i}{d_j}. \qquad \square$$

$\square$

Remark 3.7. In the construction of $S(\alpha, G)$ we demanded that v_i, v_j must not be collinear for $i \neq j$. Suppose that $\tilde{S}$ is a representation constructed from partition β and vectors v_i in the manner described above, but without the non-collinearity condition on v_i. Then $\tilde{S}$ is equivalent to $S(\alpha, G)$, where $\alpha \preccurlyeq \beta$ is obtained from β by uniting all α_i, α_j such that $\mathbb{C}v_i = \mathbb{C}v_j$, and G is the corresponding Gram matrix.

We could describe β-spherical representations either as $S(\alpha, G)$ for $\alpha \preccurlyeq \beta$ with non-collinear condition or as representations $S(\tilde{G})$ depending only on Gram matrix $\tilde{G}$ without the condition. For several reasons, the first description is more convenient to us.

Example 3.8. When β is a partition of $\mathbb{N}$ onto two sets β_1, β_2 any β-spherical representation is uniquely determined by a real number $a \in [0,1]$. Indeed, this representation is defined by the value $g_{12} = \langle v_2, v_1 \rangle$. But since we may multiply v_2 by any scalar $|d| = 1$, we may assume that g_{12} is real and positive. The value $g_{12} = 1$ corresponds to the trivial representation, $g_{12} = 0$ to the representation induced from the trivial representation of K^β.

§4. Admissible representations

In this section we construct some representations $T(\alpha, n, \lambda, G)$ and later prove that any β-admissible representation can be obtained in this way.

Let us fix some finite number n. The irreducible representation of $S(n)$ corresponding to a Young diagram λ will be denoted by R^λ.

The pair $(S_n(\infty), K_n^\beta)$ is a Gelfand pair and their spherical representations are determined by some partition $\alpha^{(n)} \in \mathcal{P}^I([\bar{n}])$ such that $\alpha^{(n)} \succcurlyeq \beta^{(n)}$ and $G \in \mathcal{G}^I$. We will denote this representation by $S(\alpha^{(n)}, G)$.

We can construct the induced representation

$$T(\alpha, n, \lambda, G) = \mathrm{Ind}_{S(n)S_n(\infty)}^{S(\infty)} R^\lambda \otimes S(\alpha^{(n)}, G).$$

Theorem 4.1. *The representation $T(\alpha, n, \lambda, G)$ is irreducible and α-admissible.*

Proof. Denote the subgroup $S(n)S_n(\infty)$ by L and the representations $S(\alpha^{(n)}, G)$ and $T(\alpha, n, \lambda, G)$ by S and T respectively.

The space $H(T)$ consists of functions $f\colon S(\infty) \to H(R^\lambda) \otimes H(S)$ satisfying $f(xhg) = R^\lambda(x)S(h)f(g)$ for all $x \in S(n)$, $h \in S_n(\infty)$ and $g \in S(\infty)$, and such that f lies in $\ell^2(L\backslash S(\infty))$.

The last condition means the following: $\|f(g)\|$ is constant along right cosets of L, so we can consider the function $\bar{f}\colon L\backslash S(\infty) \to \mathbb{C}$ defined by $\bar{f}(Lg) = \|f(g)\|$ and we want

$$\sum_\sigma \|\bar{f}(\sigma)\|^2 < \infty,$$

where σ runs along all right cosets $L\backslash S(\infty)$.

The space of K_n^α-invariant functions in $H(T)$ is exactly the space of functions with support in L. Indeed, let f be K_n^α-invariant. Then $\bar{f}$ is constant along K_n^α-orbits, so if an orbit gK_n^α intersects with infinitely many right cosets Lx, then $f(g) = 0$.

The right coset Lx consists of all y such that $x^{-1}([n]) = y^{-1}([n])$, while an orbit gK_n^α consists of all y such that $y(k) = g(k), k \in [n]$ and $y(\alpha_i) = g(\alpha_i)$ for any $i \in I$.

So we need to show that if g does not lie in L, its orbit gK_n^α intersects with infinitely many cosets Lx. Note that gK_n^α intersects with Lx if and only if $x^{-1}([n])$ lies the set $K_n^\alpha g^{-1}([n]) = \{xg^{-1}([n]) \mid x \in K_n^\alpha\}$. But when $g \notin L$ there exists a point $a \in g^{-1}([n]) \setminus [n]$, therefore the set $K_n^\alpha g^{-1}([n])$ is infinite and gK_n^α intersects with infinitely many Lx.

So, if f is K_n^α-invariant, $f(g) = 0$ for any $g \notin L$.

All such functions f are determined by their value at identity $f(1)$, and, moreover, $f(1)$ must lie in $(H(R^\lambda \otimes S))^{K_n^\alpha} = H(R^\lambda) \otimes \xi$. So, the subspace of K_n^α-invariant functions is $S(n)$-irreducible and it is cyclic, so T itself is irreducible.

Since T is irreducible with $H_n^\alpha(T) \neq 0$, it is α-admissible. □

Note that if β is a partition such that $[\beta] \succcurlyeq [\alpha]$, then $T(\alpha, X, \lambda, G)$ is clearly β-admissible.

Theorem 4.2 (cf. [1, Theorem 5.9]). *Let T be some irreducible β-admissible representation of $S(\infty)$. Then there exists a finite number n, a Young diagram λ of the size n, a partition $\alpha \in \mathcal{P}^I$ such that $[\alpha] \preccurlyeq [\beta]$ and a Gram matrix $G \in \mathcal{G}^I$ such that T is equivalent to $T(\alpha, n, \lambda, G)$.*

Remark 4.3. On the first glance, it seems possible to reformulate the statement of this theorem in the following way: if T is an irreducible β-admissible representation of $S(\infty)$, then there exists a finite set X, an irreducible representation R of $S(X)$ and a $\beta^{(X)}$-spherical representation S of $S_X(\infty)$ such that

$$T = \operatorname{Ind}_{S(X)S(\infty)[X]}^{S(\infty)}(R \otimes S).$$

But this reformulation will not be correct, as is shown in the example below.

Example 4.4. Let β be the partition of $\mathbb{N}$ onto the sets of even and odd numbers, i.e., $I = \{1, 2\}$, $\beta_1 = \{1, 3, 5, \ldots\}$, $\beta_2 = \{2, 4, 6 \ldots\}$. Now, let us change β a bit. Write

$$\beta_1' = \beta_1 \setminus \{1\} = \{3, 5, 7, \ldots\},$$
$$\beta_2' = \beta_2 \cup \{1\} = \{1, 2, 4, 6, \ldots\}.$$

Any β'-spherical representation is uniquely determined by a real number $a \in [0,1]$ (see Example 3.8). The representation $S(\beta', a)$ is β-spherical when $a \neq 0$ (more on equivalence between $S(\alpha, G)$ and $S(\alpha', G')$ for different α, α' will be given in Section 5).

Now take $T = S(\beta', 0)$. It is β-spherical, but is not equivalent to $\mathrm{Ind}_{S(X)S_X(\infty)}^{S(\infty)}(R \otimes S)$ for any choice of X, R, S, because the latter representation does not have any non-zero $K^{\beta'}$-invariant vectors.

Definition 4.5 ([1, Definition 5.2])**.** The *depth* of an irreducible β-admissible representation T is the smallest number n such that there exists a partition β' equivalent to β such that $H_n^{\beta'}(T)$ is not zero.

Proposition 4.6. *The depth of* $T(\alpha, n, \lambda, G)$ *is* n.

Proof. The proof repeats the argument of Theorem 4.1.

If $f\colon S(\infty) \to H(R^\lambda) \otimes H(S^{(\alpha)}, G)$ is K_k^β-invariant for some $\beta \sim \alpha$, then $f(g) = 0$ for any g such that orbit gK_k^β intersects with infinitely many cosets Lx. If $k < n$ it happens for any $g \in S(\infty)$. Indeed, recall that the orbit gK_k^β intersects with the coset Lx if and only if $x^{-1}([n])$ lies in $K_k^\beta g^{-1}([n])$. But when $k < n$ the set $K_k^\beta g^{-1}([n])$ is infinite for any $g \in S(\infty)$, so the only K_k^β-invariant vector f is 0. □

Recall that for any two finite sets $X \subset Y$ we introduced an idempotent $\epsilon_{Y,X}^\beta$ such that the orthogonal projection $H_Y^\beta \to H_X^\beta$ is given by $\mathfrak{T}_Y(\epsilon_{Y,X}^\beta)$.

Lemma 4.7. *Let* ϵ *be some idempotent in* $\mathfrak{G}^\beta(Y)$*, i.e., an element*

$$\epsilon = \begin{bmatrix} 1_X & (B_j)^t \\ (B_i) & (0) \end{bmatrix}$$

for some $X \subset Y$ *and* $B_i \subset Y \setminus X$, $i \in I$*. Then* $\mathfrak{T}_Y^\beta(\epsilon)$ *gives the orthogonal projection from the space* H_Y^β *to the space* H_X^α *for any partition* α *satisfying:*

$$\alpha_i^{(X)} = \beta_i^{(Y)} \cup B_i.$$

Proof. Since $H_Y^\alpha = H_Y^\beta$, this projection is given by

$$\mathfrak{T}_Y^\alpha(\epsilon_{Y,X}^\alpha) = P_Y^\alpha T(w_{Y,X}^\alpha) P_Y^\alpha = P_Y^\beta T(w_{Y,X}^\alpha) P_Y^\beta = \mathfrak{T}_Y^\beta(\epsilon). \qquad \square$$

Proof of the theorem. Suppose that n is the depth of T and $H(T)$ possesses non-zero $K_n^{\beta'}$-invariant vectors for some $\beta' \sim \beta$.

If $n = 0$, then T is β'-spherical, and everything is proven. So now suppose $n > 0$.

If $X \subsetneq [n]$ and $\epsilon \in \mathfrak{G}^{\beta'}(n)$ is an idempotent of the form

$$\epsilon = \begin{bmatrix} 1_X & (B_j)^t \\ (B_i) & (0) \end{bmatrix},$$

$\mathfrak{T}_n(\epsilon)$ is a projection on the space $H_X^{\tilde{\beta}}$, where $\tilde{\beta}^{(X)} = \beta'^{(n)} \sqcup B_i$. Let $|X| = k < n$ and g be any permutation from $S(\infty)$ that maps X to $[k]$. Then $T(g^{-1})H_X^{\tilde{\beta}} = H_k^{g(\tilde{\beta})} = 0$ since $g(\tilde{\beta}) \sim \beta$.

So $\mathfrak{T}_n(\epsilon)$ acts on $H_n^{\beta'}$ by 0 for any idempotent $1 \neq \epsilon \in \mathfrak{G}^{\beta'}(n)$. Let $\mathfrak{I}$ be the two-sided ideal generated by all such ϵ. We claim that $\mathfrak{I}$ consists of all $\mathfrak{g}$

$$\mathfrak{g} = \begin{bmatrix} A & (C_j) \\ (B_i) & (d_{ij}) \end{bmatrix}$$

such that $\mathrm{dom}(A) \neq [n]$.

Indeed, suppose that $\mathfrak{g}$ be as above. Then take $X = \mathrm{dom}(A)$ and

$$\epsilon = \begin{bmatrix} 1_X & (B_j)^t \\ (B_i) & (0) \end{bmatrix},$$

where B_i are the same as in $\mathfrak{g}$. Then

$$\mathfrak{g} \cdot \epsilon = \begin{bmatrix} A \cdot 1_X & (AB_j^t + C_j) \\ (B_i \cdot 1_X + B_i) & (B_i B_j^t + d_{ij}) \end{bmatrix} = \begin{bmatrix} A & (C_j) \\ (B_i) & (d_{ij}) \end{bmatrix} = \mathfrak{g} \tag{4.1}$$

lies in the ideal $\mathfrak{I}$.

So the representation $\mathfrak{T}_n^{\beta'}$ is uniquely determined by its restriction to the subset $\mathfrak{G}_0 = \mathfrak{G}^{\beta'}(n) \setminus \mathfrak{I}$ that consists of all elements

$$\mathfrak{g} = \begin{bmatrix} A & (0) \\ (0) & (d_{ij}) \end{bmatrix},$$

where A is a permutation $A \in S(n)$. The set $\mathfrak{G}_0$ is a sub-semigroup of $\mathfrak{G}^{\beta'}(n)$ and is isomorphic to the direct product of $S(n)$ and $\mathfrak{G}^{\beta'^{(n)}}$. Its irreducible representations are exactly $R^\lambda \otimes \chi$, where χ is some character of $\mathfrak{G}^{\beta'^{(n)}}$ and uniquely corresponds to some $\beta'^{(n)}$-spherical representation of $S_n(\infty)$. $\square$

Note that the proof of this theorem gives one an algorithm on how to find all the parameters α, n, λ, G given an irreducible β-admissible representation T.

§5. Unitary equivalence for β-admissible representations

In this section we find conditions for when two representations $T_1 = T(\alpha, n, \lambda, G)$ and $T_2 = T(\beta, m, \mu, G')$ are unitary equivalent.

Proposition 5.1. *Representation $T = T(\alpha, n, \lambda, G)$ is β-admissible if and only if $[\beta] \succcurlyeq [\alpha]$.*

Proof. Suppose that $\alpha = (\alpha_i)_{i\in I}$, $\beta = (\beta_j)_{j\in J}$ and $[\beta] \not\succcurlyeq [\alpha]$. It means that for any N there exist $j \in J$ such that $\beta_j^{(N)}$ intersects with at least two different $\alpha_k^{(N)}, \alpha_l^{(N)}$.

Firstly, assume $n = 0$ and $T = S(\alpha, G)$ is a spherical representation with spherical vector $\xi = \xi_1 \otimes \cdots$.

Let $\eta \in H(T)$ be unit K_n^β-invariant vector. Since ξ is cyclic, for any $\varepsilon > 0$ there exists

$$\eta_\varepsilon = \sum_{i=1}^{m} c_i T(g_i)\xi, \quad \|\eta - \eta_\varepsilon\| < \varepsilon/2$$

and one can easily check that

$$\|T(k)\eta_\varepsilon - \eta_\varepsilon\| < \varepsilon$$

holds. For any $k \in K_n^\beta$

$$\begin{aligned}\|T(k)\eta_\varepsilon\|^2 = \|\eta_\varepsilon\|^2 &= \sum_{i,j=1}^{m} c_i\bar{c_j}(T(g_i)\xi, T(g_j)\xi)\\ &= \sum c_i\bar{c_j}\varphi(g_j^{-1}g_i) > 1 - \varepsilon/2\end{aligned}$$

since η is unit. In the similar manner,

$$(T(k)\eta_\varepsilon, \eta_\varepsilon) = \sum c_i\bar{c_j}\varphi(g_j^{-1}kg_i).$$

It is true for all $k \in K_n^\beta$. We may take $N > n$ such that all g_i lie in $S(N)$. Then for any $k \in K_N^\beta$

$$\begin{aligned}\varepsilon > \|T(k)\eta_\varepsilon - \eta_\varepsilon\| &= \sum_{i,j=1}^{m} (c_i\bar{c_j}\varphi(g_j^{-1}g_i))(2 - \varphi(k) - \bar{\varphi(k)})\\ &> (1 - \varepsilon/2)(2 - \varphi(k) - \bar{\varphi(k)}).\end{aligned}$$

Now, since $[\beta] \not\succcurlyeq [\alpha]$ we have an infinite collection of pairwise different numbers $x_1, y_1, x_2, y_2, \ldots > N$ such that for any s elements x_s, y_s lie in the same part β_j, but in different parts α_k, α_l. If we now take

$k_r = (x_1, y_2) \cdots (x_r, y_r) \in K_N^\beta$, the sequence $\varphi(k_r)$ converges to 0, so for r large enough

$$(2 - \varphi(k_r) - \varphi(\bar{k}_r)) > \frac{\varepsilon}{2 - \varepsilon}$$

and we have a contradiction.

Now suppose $T = T(\alpha, n, \lambda, G)$ and $f \neq 0$ is K_N^β-invariant. We may assume that $N > n$. Using the same argument we used in the proof of Theorem 4.2 and Proposition 4.6 we obtain that the support of f consists only of those g such that $g^{-1}([n]) \subset [N]$. Moreover, f is determined by its values on the elements g_X, where $X \subset [N]$ is any subset of size n and $g_X \in S(N)$ are some permutations mapping X onto $[n]$. Since any $k \in K_N^\beta$ commutes with all g_X and lies in $S_n(\infty)S_n(\infty)$, all af the values g_X must lie in $\in H(R^\lambda) \otimes H(S(\alpha^{(n)}, G))^{K_N^\beta}$, so for f to be non-zero, $S(\alpha^{(n)}, G)$ must be $\beta^{(n)}$-admissible, i.e., $[\beta] \preccurlyeq [\alpha]$. □

It follows that if two representations $T_1 = T(\alpha, n, \lambda, G)$ and $T_2 = T(\beta, m, \mu, G')$ are equivalent, then $\alpha \sim \beta$ and $m = n$, since the depth of α-admissible representation does not depend on the choice of representative α in the equivalence class.

Proposition 5.2. *Let $\alpha \in \mathcal{P}^I$ be some partition of $\mathbb{N}$, $G = (g_{ij}) \in \mathcal{G}^I$ be some Gram matrix, and $S = S(\alpha, G)$ be the corresponding spherical representation. Recall that it is a closed subspace of the Hilbert space $V^{\otimes\infty}(\xi)$. Now suppose that $\beta \sim \alpha$ is obtained from α by moving some number k from α_i to α_j. In other words, there are $i \neq j \in I$ and $k \in \alpha_i$ such that $\beta_i = \alpha_i \setminus \{k\}$, $\beta_j = \alpha_j \cup \{k\}$ and $\beta_{i'} = \alpha_{i'}$ for $i' \neq i, j$.*

*Then S has a non-zero K^β-invariant vector η if and only if $g_{ij} \neq 0$. In this case we say that β is obtained from α by a G-*permissible movement.

Proof. Note that for any $\beta \sim \alpha$ the space $V^{\otimes}(\xi)$ has exactly one (up to multiplication by scalar) K^β-invariant vector η, namely $\eta = \eta_1 \otimes \ldots$, where $\eta_k = v_i$ when $k \in \beta_i$.

Now, if $g_{ij} = 0$, then for any $x \in S(\infty)$ vector η is orthogonal to $T(x)\xi$, so lies in the orthogonal complement to $H(S)$ in $V^{\otimes\infty}(\xi)$. It follows that $H(S)$ does not have any non-zero K^β-invariant vectors. In particular, $S(\alpha, G)$ is not equivalent to $S(\beta, H)$ for any choice of H.

Now suppose $g_{ji} \neq 0$. Let $\alpha_j = (a_j^1 < a_j^2 < \ldots)$ and take

$$\eta_n = \frac{1}{n g_{ji}} \sum_{s=1}^{n} T(k, a_j^s)\xi.$$

Then

$$(\eta_n, \eta_n) = \frac{1}{n^2|g_{ji}|^2} \sum_{s\neq t=1}^{n} \langle T(k, a_j^s, a_j^t)\xi, \xi\rangle + \frac{n}{n^2|g_{ji}|^2}$$
$$= \frac{n(n-1)|g_{ji}|^2}{n^2|g_{ji}|^2} + \frac{n}{n^2|g_{ji}|^2};$$
$$(\eta_n, \eta) = \frac{1}{ng_{ji}} \sum_{s=1}^{n} (T(k, a_j^s)\xi, \eta) = \frac{1}{ng_{ji}} ng_{ji} = 1;$$
$$\|\eta - \eta_n\|^2 = 1 + \frac{n(n-1)}{n^2} + \frac{1}{n|g_{ji}|^2} - 2 \to 0,$$

So η lies in $H(S)$ and S is equivalent to $S(\beta, G)$ for the same Gram matrix G. □

Example 5.3. Suppose that $|I| = 2$, α is a partition onto even and odd numbers and $\beta \sim \alpha$. Recall that α-spherical representations are parametrized by $g \in [0, 1]$. From the proposition it follows that $S(\alpha, g)$ is equivalent to $S(\beta, g)$ whenever $g \neq 0$. Representations $S(\alpha, 0)$ and $S(\beta, 0)$ are equivalent if and only if β is obtained from α by some finite permutation of $\mathbb{N}$.

Definition 5.4. Let α be some partition of $\mathbb{N}$ and suppose that $\beta \sim \alpha$ can be obtained from α by some finite permutation and by composition of finite number of G-permissible movements. We will say that α and β are *G-close*.

Definition 5.5. For a partition $\alpha \in \mathcal{P}^I$ and a Gram matrix $G \in \mathcal{G}^I$ we define a graph $\Gamma(G)$ as in the proof of the third part of Theorem 3.5. I.e, its vertices are indexed by I, and two vertices $i \neq j$ are connected by an edge if $g_{ij} \neq 0$. Then connected components c of $\Gamma(G)$ define a splitting of I into subsets $I = \bigsqcup_{c\in c} I_c$.

For any $N \geqslant 0$, $\alpha \in \mathcal{P}^I$ and $G \in \mathcal{G}^I$, we define a collection of numbers $l(\alpha, G)^{(N)} = \left(l(\alpha, G)_c^{(N)}\right)_{c\in C}$, where $l(\alpha, G)_c^{(N)}$ denotes the number of points $k \in [N]$ such that $k \in \alpha_i$ and $i \in I_c$.

Then two partitions α, β are G-close if and only if for all N large enough we have $l(\alpha, G)^{(N)} = l(\beta, G)^{(N)}$.

Proposition 5.6 (cf. [1, Theorem 3.7])**.** *Let $\alpha \in \mathcal{P}^I$, $G \in \mathcal{G}^I$ and $S(\alpha, G)$ be the corresponding representation of $S(\infty)$ lying in the space $V^{\otimes\infty}(\xi)$.*

Let C be the set of connected components of the graph $\Gamma(G)$ and for any $c \in C$ we set V_c to be the closed subspace of V generated by all v_i for $i \in c$. Then V splits into orthogonal partition

$$V = \bigoplus_{c \in C} V_c$$

and, in turn, G splits into direct sum of Gram matrices $G_c \in \mathcal{G}^c$.

Now let β be the partition obtained from α by uniting all parts lying in the same component. In other words, $\beta \in \mathcal{P}^C$ and

$$\beta_c = \bigsqcup_{i \in c} \alpha_i.$$

Then the representation $S(\alpha, G)$ is equivalent to the induced representation

$$\operatorname{Ind}_{K^\beta}^{S(\infty)} \bigotimes_{c \in C} S(\alpha|_{\beta_c}, G_c),$$

where $\alpha|_{\beta_c} \in \mathcal{P}^c(\beta_c)$ is the partition $\beta_c = \bigsqcup_{i \in c} \alpha_i$ of the set β_c, and the representation $\bigotimes S(\alpha|_{\beta_c}, G_c)$ is defined as follows. Its space is

$$\bigotimes_{c \in C} (H(S(\alpha|_{\beta_c}, G_c)), \xi^c),$$

where ξ^c is the spherical vector of the corresponding representation. The action of $K^\beta = \sum_{c \in C} S(\beta_c)$ is defined naturally.

Proof. Proposition 5.2 gives us a description of the space H of $S(\alpha, G)$. For any $\alpha' \sim \alpha$ the corresponding $K^{\alpha'}$-invariant vector ξ' lies in H when α' is G-close to α and is orthogonal to H otherwise. It follows that H has a dense subspace

$$\bigcup_{n \in \mathbb{N}} V_{c_1} \otimes V_{c_2} \otimes \cdots \otimes V_{c_n} \otimes \xi_{n+1} \otimes \cdots ,$$

where for any connected component c there are exactly $l(\alpha, G)_c^{(n)}$ indexes equal to c among $c_1, \ldots, c_n$.

Now, for any $k \in \mathbb{N}$ write $c(k)$ for the connected component such that $k \in \alpha_i \subset \beta_{c(k)}$. Then the space H' that is the closure of

$$\bigcup_{n \in \mathbb{N}} V_{c(1)} \otimes V_{c(2)} \otimes \cdots \otimes V_{c(n)} \otimes \xi_{n+1} \otimes \cdots$$

is K^β-invariant and equivalent to the representation $\bigotimes_{c \in C} S(\alpha|_{\beta_c}, G_c)$.

From this the equivalence between $S(\alpha, G)$ and $\mathrm{Ind} \bigotimes S(\alpha|_{\beta_c}, G_c)$, becomes clear. □

Theorem 5.7. *Two representations $S(\alpha, G)$ and $S(\beta, G')$ are equivalent if and only if all of the following conditions are satisfied*

(i) *$\alpha \sim \beta$. We will assume that α and β are indexed by the same set I.*
(ii) *$G' = DGD^{-1}$ for some diagonal matrix D with $|d_i| = 1$. In particular, they define the same subsets $I_c \subset I$ and the same notion of G-closeness.*
(iii) *α and β are G-close.*

Proof. The "if" part of the proposition is clear. Now suppose that $S(\alpha, G)$ and $S(\beta, G')$ are equivalent. It immediately follows that α and β must be equivalent. Suppose that N is such that $\alpha^{(N)} = \beta^{(N)}$. Then for any $x \in S_N(\infty)$ the spherical functions $\varphi^{\beta}(x) = \varphi^{\alpha}(x)$, so $G = DGD^{-1}$.

Now, from the previous arguments we see that the subspaces $H(S(\alpha, G))$ and $H(S(\beta, G))$ of $V^{\otimes\infty}(\xi)$ coincide whenever α and β are G-close and orthogonal otherwise. In the latter case the representations are not equivalent since in $V^{\otimes\infty}(\xi)$ there is only one (up to multiplication) K^{β}-invariant vector. □

We are ready to prove the following

Theorem 5.8 (cf. [1, Theorem 6.11]). *Two representations*

$$T_1 = T(\alpha, n, \lambda, G) \quad \text{and} \quad T_2 = T(\beta, m, \mu, G')$$

are equivalent if and only if

(i) $\alpha \sim \beta$;
(ii) $n = m$;
(iii) $\lambda = \mu$;
(iv) $G' = DGD^{-1}$;
(v) *partitions $\alpha^{(n)}$ and $\beta^{(n)}$ of $[\bar{n}]$ are G-close.*

Proof. We have already seen that if T_1 and T_2 are equivalent, then $\alpha \sim \beta$ and $n = m$. The space $H(T_1)$ must contain K_n^{β}-invariant vectors f uniquely determined by $f(1) \in H(R^{\lambda}) \otimes H(S(\alpha^{(n),G})^{K_n^{\beta}})$, so $\alpha^{(n)}$ and $\beta^{(n)}$ must be G-close and $G' = DGD^{-1}$ must hold. □

§6. Representations admissible with respect to two partitions

The classification results also give us the conditions under which an irreducible unitary representation T is α- and β-admissible for different partitions α, β. We will denote by $\mathrm{IrrAdm}(\alpha)$ the set of equivalence classes of irreducible α-admissible representations.

If T is α-admissible, then $T = T(\gamma, n, \lambda, G)$ for some $[\gamma] \preccurlyeq [\alpha]$. It is also β-admissible if and only if $[\gamma] \preccurlyeq [\beta]$.

Definition 6.1. Let $\alpha, \beta \in \mathcal{P}$. We say that γ is the *infimum* of α and β if $\gamma \preccurlyeq \alpha, \beta$ and $\gamma' \preccurlyeq \gamma$ for any other $\gamma' \preccurlyeq \alpha, \beta$. We will denote this infimum by $\alpha \wedge \beta$.

Proposition 6.2. *The infimum $\alpha \wedge \beta$ always exists.*

Proof. Let $\alpha \in \mathcal{P}^I$, $\beta \in \mathcal{P}^J$ and Γ be the graph with vertices indexed by I, such that vertices i, i' are connected by an edge if there exists $j \in J$ such that β_j intersects with both $\alpha_i, \alpha_{i'}$. Let C be the set of connected components of Γ, and put for any $c \in C$

$$\gamma_c = \bigsqcup_{i \in c} \alpha_i.$$

It is clear that $\gamma = \alpha \wedge \beta$. □

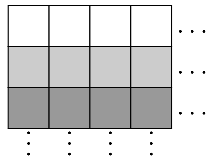
Definition 6.3. We will say that an equivalence class $[\gamma]$ is the *infimum* of $[\alpha]$ and $[\beta]$ if $[\gamma] \preccurlyeq [\alpha], [\beta]$ and $[\gamma'] \preccurlyeq [\gamma]$ for any other $[\gamma'] \preccurlyeq [\alpha], [\beta]$.

Unfortunately, the infimum of equivalence classes of partition may not exist. When it does we will denote it by $[\alpha] \wedge [\beta]$.

Example 6.4. Here we will provide an example of two equivalence classes $[\alpha], [\beta]$ such that the infimum does not exist. More explicitly, we will construct a series $[\gamma(n)]$ of equivalence classes of partitions satisfying the following conditions:

(1) $[\gamma(0)] \prec [\gamma(1)] \prec \cdots \preccurlyeq [\alpha], [\beta]$;
(2) for any other $[\gamma] \preccurlyeq [\alpha], [\beta]$ there exists n such that $[\gamma] \prec [\gamma(n)]$.

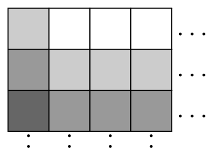
Let $\alpha \in \mathcal{P}^I$ be a partition with countable index set $I = \{1, 2, \ldots\}$, suppose that $\alpha_i = (a_i^1 < a_i^2 < \cdots)$, and suppose also that I is ordered in such a way that $a_1^1 < a_2^1 < \ldots$. Note that this defines the bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ that sends a_i^j to the pair (i, j). It will be convenient for us to visualise the partition α as follows (boxes of the same colour belong to the same part α_i, the numbers a_i^j grow from left to right in every row and from top to bottom in the first column).

$\alpha =$ 

Now define $\beta = (\beta_i)_{i \in I}$ in the following way

$$\beta_1 = \alpha_1 \setminus \{a_1^1\}, \quad \beta_i = \big(\alpha_i \setminus \{a_i^1\}\big) \cup \{a_{i-1}^1\},\ i \neq 1.$$

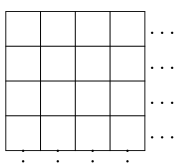
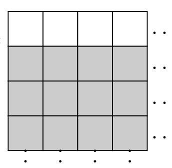
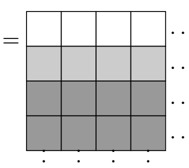
We can visualise β in the same manner.

$\beta =$ 

For any $n \geqslant 0$ let us denote by $\gamma(n) \in \mathcal{P}^{[n+1]}$ the partition of $\mathbb{N}$ into $n+1$ parts defined as follows:

$$\gamma(n)_i = \alpha_i,\ i \leqslant n, \quad \gamma(n)_{n+1} = \mathbb{N} \setminus \bigcup_{i=1}^{n} \alpha_i.$$

For example, for small n we have

$\gamma(0) =$  , $\gamma(1) =$  , $\gamma(2) =$  .

One can note that $\gamma(n)^{(a_n^1)} = \alpha^{(a_n^1)} \wedge \beta^{(a_n^1)}$.

Indeed, it is clear that the finest partition that is coarser then both α and β is the trivial partition $\gamma(0)$. Once you remove the first a_n^1 elements (i.e., the first n elements in the first column and some finite number of elements in other columns) the sets $\alpha_i^{(a_n^1)}$ and $\beta_i^{(a_n^1)}$ coincide for $i = 1, \ldots, n$, and we have $\alpha^{(a_n^1)} \wedge \beta^{(a_n^1)} = \gamma(n)^{(a_n^1)}$.

So we constructed an increasing sequence of equivalence classes that are all coarser then $[\alpha]$ and $[\beta]$:

$$[\gamma(1)] \prec [\gamma(2)] \prec \cdots \prec [\alpha], [\beta].$$

Now suppose that $[\gamma] \preccurlyeq [\alpha], [\beta]$ is any equivalence class that is coarser then both $[\alpha], [\beta]$. Then there exists a number N such that $\gamma^{(N)} \preccurlyeq \alpha^{(N)}, \beta^{(N)}$.

But the infimum $\alpha^{(N)} \wedge \beta^{(N)}$ is $\gamma(n)^{(N)}$ for n satisfying $a_n^1 \leqslant N < a_{n+1}^1$, so we have $[\gamma] \preccurlyeq [\gamma(n)] \prec [\gamma(n+1)]$.

The argument used in this example can be applied in general situation.

Let α, β be any partitions, and suppose that $\{[\gamma(n)]\}_{n\in\mathbb{N}}$ is the sequence of equivalent classes of partitions defined by $\gamma(n)^{(n)} = \alpha^{(n)} \wedge \beta^{(n)}$. Then $\{[\gamma(n)]\}_{n\in\mathbb{N}}$ satisfies the conditions from the Example 6.4:

(1) $[\gamma(0)] \preccurlyeq [\gamma(1)] \preccurlyeq \cdots \preccurlyeq [\alpha], [\beta]$;
(2) for any other $[\gamma] \preccurlyeq [\alpha], [\beta]$ there exists n such that $[\gamma] \preccurlyeq [\gamma(n)]$.

It follows that the infimum exists if and only if the sequence $[\gamma(n)]$ stabilises. For example, it happens when both α, β are partitions of $\mathbb{N}$ into finitely many parts. Indeed, in this case we may take N large enough so that $\alpha_i \cap \beta_j \subset [N]$ for any α_i and β_j such that their intersection is finite. Then $[\gamma(n)] = [\gamma(n+1)]$ for all $n > N$.

Proposition 6.5. *Let α, β be two partitions and $\gamma(n)$ be as above. Then*

$$\bigcup_{n\in\mathbb{N}} \mathrm{IrrAdm}(\gamma(n))$$

is the set of all irreducible representations that are admissible with respect to both α and β.

In particular, when the infimum $[\gamma]=[\alpha]\wedge[\beta]$ exists, the set of all irreducible representations admissible with respect to both α and β is $\mathrm{IrrAdm}(\gamma)$.

The proof follows directly from the Proposition 5.1 and the reasonings above.

Note that this set can be described in other terms.

Let $K_n := K_n^{\gamma(n)}$. These groups form a decreasing sequence

$$K_0 \supset K_1 \supset \cdots$$

that satisfy the conditions $1 - 3$ from Definition 1.2. So, we may define the $\{K_n\}$-topology in the usual way. The set of equivalence classes of irreducible representations continuous with respect to this topology is exactly $\bigcup_{n\in\mathbb{N}} \mathrm{IrrAdm}(\gamma(n))$.

It follows from the fact that $K_N^{\gamma(n)}$ is sandwiched between K_n and K_N (depending on what is greater, n or N).

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