Записки научных
семинаров ПОМИ
Том 543, 2025 г.

M. V. Babich, L. A. Bordag, A. M. Khvedelidze, D. M. Mladenov

ON MULTIDIMENSIONAL ANALOGS OF EULER (TAIT-BRYAN) ANGLES AND GRASSMANNIANS

Abstract. We present a generalization of the well known Euler angles method, or more precisely its modification the Tait-Bryan angles method, describing the rotation of an orthogonal frame in $\mathbb{R}^3$. The newly developed system of elementary unitary rotations allows to introduce convenient parameterization in high-dimensional complex-valued unitary spaces. As a by-product we obtain parametrization of the affine parts of Grassmannians and a parameterization of the algebraically-open subsets of conjugation classes of Hermitian matrices by elementary rotations. The parametrization of these objects has many applications especially in quantum information theory.

§1. Introduction

In recent years needs of the quantum information theory in quantitative description of the state space of finite-dimensional quantum systems initiate revival of interest to the issue of parameterizing of certain subsets of the set of the complex positive semidefinite (PSD) matrices. Particularly, a knowledge of algebraic and geometric properties of unitary orbits and the corresponding strata of the space of PSD matrices is crucial for understanding of phenomenon of entanglement. This mathematical issue has a long history and its roots goes back to the eighteenth century. In the simplest case it is related to the description of the orthogonal frames in $\mathbb{R}^3$. The latter one was solved in 1751 by Leonhard Euler who introduced corresponding parameters, in the modern terms the "Euler angles", in the original conventions $z-x'-z''$ rotations or $\alpha-\beta-\gamma$ - angles. In real 3D space the Euler angles characterize an arbitrary rotation as subsequent rotations about different coordinate axes each representing one-parametric subgroup of the SO(3) group. Their sequential execution is able to connect an arbitrary position of the basis with some fixed one. At the end of the 19th century, P. G. Tait and G. H. Bryan modified Euler's method. They choose a different line of nodes and considered rotations about another set

Key words and phrases: Lie groups, unitary group, Grassmanians.

of axes, known as the $z - y' - x''$-convention. The given sequence of rotations determines the system of angles termed nowadays as the *Tait-Bryan angles*. Usually they are widely used in the aerospace and marine dynamics for describing orientation of ships and aircrafts. The present article aims to introduce an analogue of the Tait-Bryan angles in higher dimensional complex spaces baring in mind their further applications in the quantum theory of finite-dimensional systems.

As it was mentioned above PSD matrices parameterization is highly actual for description of composite systems consisting from qubits. In quantum information theory a basic object is an arbitrary 2-level state termed by Schumacher in [1] as *a qubit* emphasizing the differences between the classical bit and its quantum counterpart, a qubit. The unitary symmetry transformations associated to a single qubit is the adjoint $SU(2)$ group action and correspondingly the adjoint $SU(2^n)$ action on n-qubits states. This observation points to the importance of using of a proper parameterization of unitary groups describing composite systems of qubits. With this reason the Euler type parameterization has been used by Tilma, Byrd and Sudarshan in [2, 3] for two qubit density matrices. Latter Cacciatori, Cerchiai and Bertini in [4, 5] used the works [2, 3] as a basis to develop generalization of the Euler angles method to the unitary groups of higher orders. Furthermore, in 2017, the authors of [8] used a generalized Euler angles method to provide an explicit parameterization of the wide class of simple, simply connected compact Lie groups.

The generalized Euler angles parameterizations has a wide range of applications. Below only a few of them where the proper parameterization allowed to overcome computational difficulties are stated. Particularly, in [6] using the Euler angle parameterization of $SU(3)$ group the explicit Hamiltonian reduction from free motion on $SU(3)$ to the motion on the coset space $SU(3)/SU(2) \approx \mathbb{S}^5$ has been performed and proved that the latter is not equivalent to the geodesic motion on $\mathbb{S}^5$ with the standard round metric.

The ideas developed by Tilma and Cacciatori with co-authors in [2, 5] were later used by Stefanov and Mladenov in [9] to describe the pure states of a two-qubit system with similar parameterization for $SU(4)$. The parametrization by means of generalized Euler angles provided in [5] was also used by Slater in [7] to determine the Hilbert–Schmidt separability probabilities for low-dimensional qubit systems. More recently, Abgaryan and Khvedelidze studied an N-level quantum system in [11] and proposed a method to construct all admissible unitary non-equivalent Wigner quasipropability distributions. Aiming to obtain an explicit form of the Wigner functions for single qubit and single qutrit states it was outlined a necessity of using of different sets of the Euler angles for an effective parameterization of the Wigner functions. As another application one can note the problem of an efficient description of density matrices studied in [10] where an explicit parametrization of the density matrix eigenvalues of many level quantum systems in terms the Euler angles has been discussed.

The aforementioned works showcase the difficulties that arise when trying to introduce coordinates on different subsets of the space of Hermitian matrices. These subsets originated from a natural stratification of the state space under the adjoint action of a unitary group. Below proposing a new generalization of the Tait-Bryan angles method to high dimensional spaces we provide also a method for parameterizing of an algebraically open subsets associated to the certain classes of the unitary orbits.

§2. Parameterization of Grassmannian by elementary unitary rotations

Let $V \simeq \mathbb{C}^N, N \geqslant 2$ be a unitary space. Let us call a transformation $\phi \in \mathrm{SU}(V)$ *an elementary unitary rotation* around the $(N-2)$-dimensional subspace $R \subset V$ if it is identical on R.

It is evident that in an orthonormal basis an elementary unitary rotation ϕ has the matrix

$$\begin{pmatrix} \begin{pmatrix} \cos\varphi & -e^{-i\alpha}\sin\varphi \\ e^{i\alpha}\sin\varphi & \cos\varphi \end{pmatrix} & 0 \\ 0 & \mathrm{I} \end{pmatrix}$$

for some $\alpha, \varphi \in \mathbb{R}$, where the last $N-2$ vectors form a basis of R and the first two vectors form a basis of $R^{\perp} \simeq \mathbb{C}^2$.

Consider a space V_{p+q} spanned by two orthogonal subspaces of dimensions p and q:

$$V_{p+q} = V_q \overset{\perp}{\oplus} V_p, \quad q \leqslant p, \quad V_k \simeq \mathbb{C}^k.$$

A Grassmannian that is a manifold of q-dimensional linear subspaces in V_{p+q} we denote by $\mathrm{G}(q, p+q)$. Consider a q-dimensional space $L \in \mathrm{G}(q, p+q)$. Our goal is to construct some *unitized* unitary transformation $\Phi \in \mathrm{SU}(p+q)$ that takes a given q-dimensional L to the (coordinate) subspace V_q.

The transformation Φ will be pq of sequentially performed elementary unitary rotations $\phi_{r,s}$, $0 \leqslant r < q$, $0 \leqslant s < p$ around the subspaces constructed during the process.

Preliminary construction. Let us consider a matrix of such a block structure:

$$\begin{pmatrix} \mathrm{I} & & \\ & 1 & \\ & & \mathrm{I} \\ B' & b' & B \\ 0 & -b & b'' \\ 0 & 0 & 0 \end{pmatrix}.$$

Number of lines in blocks (from top to bottom) $q_1, 1, q_2, p_1, 1, p_2$. Number of columns in blocks (from left to right) $q_1, 1, q_2$.

The following identity can be easily verified:

$$\begin{pmatrix} \mathrm{I} & & & & & \\ & 1 & & & -\bar{b} & \\ & & \mathrm{I} & & & \\ & & & \mathrm{I} & & \\ & b & & & 1 & \\ & & & & & \mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & & \\ & 1 & \\ & & \mathrm{I} \\ B' & b' & B \\ 0 & -b & b'' \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{I} & & \\ & 1 & \\ & & \mathrm{I} \\ B' & \frac{b'}{\beta^2} & \hat{B} \\ 0 & 0 & b'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & & \\ & \beta^2 & -\beta\bar{b}b'' \\ & & \mathrm{I} \end{pmatrix}. \tag{1}$$

Here $b, \bar{b}, \beta \in \mathbb{C}$, $B' \in \mathbb{C}^{p_1 \times q_1}$, $b' \in \mathbb{C}^{p_1 \times 1}$, $B, \hat{B} \in \mathbb{C}^{p_1 \times q_2}$, $b'' \in \mathbb{C}^{1 \times q_2}$, $q = q_1 + q_2 + 1$, $p = p_1 + p_2 + 1$, $\hat{B} := B + \frac{b}{\beta} b' b''$, $\beta := \sqrt{1 + |b|^2}$.

Let us denote

$$\left(\begin{array}{cccccc} \mathrm{I} & & & & & \\ & \frac{1}{\sqrt{1+|b|^2}} & & & & \\ & & \mathrm{I} & & & \\ & & & \mathrm{I} & & \\ & & & & \frac{1}{\sqrt{1+|b|^2}} & \\ & & & & & \mathrm{I} \end{array}\right)\left(\begin{array}{cccccc} \mathrm{I} & & & & & \\ & 1 & & & -\bar{b} & \\ & & \mathrm{I} & & & \\ & & & \mathrm{I} & & \\ & b & & & 1 & \\ & & & & & \mathrm{I} \end{array}\right) =: \phi_{q_1,p_1}. \tag{2}$$

Proposition. *Transformation* $\phi_{q_1,p_1} \in \mathrm{SU}(p+q)$ *is the elementary unitary rotation around coordinate subspace that is the orthogonal complement to the plane spanning the* $(1+q_1)$*-th and* $(q+1+p_1)$*-th basic vectors*:

$$(\vec{e}_{1+q_1}, \vec{e}_{q+1+p_1}) \begin{pmatrix} \cos\varphi & -e^{-i\alpha}\sin\varphi \\ e^{i\alpha}\sin\varphi & \cos\varphi \end{pmatrix}, \quad \varphi := \arctan|b|, \alpha := \arg b.$$

The rotation coincides the assigned direction and the coordinate one in the two-dimensional unitary space.

Note that the rectangle matrix in the right-hand side of (1) has the same structure as the rectangle matrix in the left-hand side, but the $(q+1+p_1)$-th row has one zero more. It means that we can repeat the procedure replacing q_1 with q_1+1 and q_2 with q_2-1. A unitary transformation $\phi_{0,p_1}\ldots\phi_{q-1,p_1}$ gives us one zero line more, that replaces p_2 with p_2+1 and $q_1 = q-1$ transforms to $q_1 = 0$. We can make the next q elementary rotations and vanish the next line. After pq elementary rotations we get the zero block in the right-hand side.

Let us start from the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. The final version of (1) after pq iterations can be rewritten as

$$\Phi\begin{pmatrix} A \\ B \end{pmatrix} = \Phi\begin{pmatrix} \mathrm{I} \\ BA^{-1} \end{pmatrix} A = \begin{pmatrix} \hat{A} \\ 0 \end{pmatrix}, \text{where } \Phi := \underbrace{\phi_{q-1,p-1}\ldots\phi_{0,0}}_{pq \text{ factors}} \tag{3}$$

is a square matrix from $\mathrm{SU}(p+q)$ that is uniquely determined by BA^{-1}. Here $\hat{A}$ is some non-degenerated $q \times q$ matrix. Unessential matrix $\hat{A}$ is the product of A and pq square matrices from the right-hand side of the iterations of (1) for all values q_1 and p_1.

We consider the action of $\mathrm{SU}(p+q)$ on the basis from the right

$$(\vec{e}_1 \ldots \vec{e}_{p+q})^{old}\,\Phi^{-1} = (\vec{e}_1 \ldots \vec{e}_{p+q})^{new}. \tag{4}$$

Our construction gives

$$(\vec{e}_1 \dots \vec{e}_{p+q})^{new}\, \Phi \begin{pmatrix} A \\ B \end{pmatrix} = (\vec{e}_1 \dots \vec{e}_{p+q})^{new} \begin{pmatrix} \hat{A} \\ 0 \end{pmatrix}. \qquad (5)$$

We treat equality (5) as follows. The subspace $L \in \mathrm{G}(q, p+q)$, which is the linear span of the column vectors of the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ in the old basis $(\mathbf{e})^{old}$, is the linear span of the first q vectors of the new basis $(\mathbf{e})^{new}$.

Equality (4) means that the columns Φ^{-1} define the vectors of $(\mathbf{e})^{new}$, and first vectors define point L of the Grassmannian. This columns form an orthonormal basis on L written as the linear combinations of the old basic vectors.

Let us summarize the introduced construction. A point of the Grassmannian $G(q, p+q)$ is given as the subspace spanning the columns of $\begin{pmatrix} A \\ B \end{pmatrix}$. The coordinate functions on the open domain $\det A \neq 0$ of the Grassmannian are the matrix elements of BA^{-1}.

So we construct some SU-transformation Φ that is uniquely defined by the point of the Grassmannian because it is defined by BA^{-1}. The transformation is defined on the algebraically-open domain $\det A \neq 0$. The transformation (matrix Φ^{-1}) moves the coordinate subspace spanning the first n basic vectors to the given point L of the Grassmannian.

It was the case $q \leqslant p$. A case $q > p$ we treat as the construction in question for the orthogonal complement.

We call Φ *an* SU-*coordinate transformation* for the open domain $\det A \neq 0$ of the Grassmannian $\mathrm{G}(q, p+q)$. It gives some standardized sequence of *the elementary* SU(2) *rotations* around coordinate subspaces. These rotations, finely, transform the coordinate subspace to the parametrized point of the Grassmannian $G(q, p+q)$. A coordinate function, in the strict sense of the word, is the complex number b, that is the unique parameter of ϕ_{q_1,p_1} in (2). We collect this b-s for all pq values of the pairs indexes $(q_1, p_1) : 0 \leqslant q_1 < q,\ 0 \leqslant p_1 < p$.

In the simplest case when we have the splitting of $\mathbb{C}^2$ on two orthogonal lines, we get the parametrization of the affine part of the Grassmannian $\mathrm{G}(1,2)$ by the complex variable b. This Grassmannian is just the projective line $\mathbb{C}\mathbf{P}^1$ in this case. Usually it is called *the Riemann sphere*, or the *Bloch sphere*, if the proper parameterization (spherical coordinates) is used.

§3. Analog of Tait-Bryan angles

Let us return to the original problem of parameterizing of certain subsets of the space of the Hermitian matrices.

We define the conjugacy class of the matrix by the set

$$\{(\lambda_k, n_k) : 0 \leqslant k \leqslant m\},$$

where λ_k is the eigenvalue, n_k is its multiplicity. Let this set be ordered according to the multiplicity of the eigenvalues, that is, $i < j \Rightarrow n_i \leqslant n_j$.

We will construct a set SU-coordinate transformations

$$\{\Phi_{ij}, 0 \leqslant i < m, i < j < m\}$$

for each conjugacy class. Consecutive execution of these unitary transformations will transform the set of eigensubspaces of the matrix from the given conjugacy class $\{(\lambda_k, n_k) : 0 \leqslant k \leqslant m\}$ into a set of coordinate subspaces, diagonalize the corresponding Hermitian matrix in some well-defined way.

Let's denote

$$\mathbb{C}^N = (e_0) \overset{\perp}{\oplus} (e_1) \overset{\perp}{\oplus} (e_2) \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} (e_{m-1}) \overset{\perp}{\oplus} (e_m),$$

the partition of the entire space into mutually orthogonal eigenspaces of the Hermitian matrix A. Let the subspaces be ordered by dimension in non-decreasing order (total ordering). Subspaces of the same dimension are in any order.

Let

$$(E_0) \overset{\perp}{\oplus} (E_1) \overset{\perp}{\oplus} (E_2) \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} (E_{m-1}) \overset{\perp}{\oplus} (E_m) = \mathbb{C}^N$$

be a partition of the space into coordinate subspaces of the appropriate dimension: $\dim(E_k) = \dim(e_k) = n_k$.

Our immediate goal is to construct a set of unitary transformations

$$\Phi_{01}\Phi_{02}\ldots\Phi_{0m},$$

whose sequential execution will transform $(e_0)^{\perp}$ into $(E_0)^{\perp}$ and, respectively, (e_0) to (E_0).

Let's first construct Φ_{01}. Let (e_1^1) denotes the subspace that is the intersection of the sum $(e_0) \oplus (e_1) = (e_0, e_1)$ with the coordinate subspace $(E_0)^{\perp}$: $(e_1^1) = (e_0, e_1) \cap (E_0)^{\perp}$. It is easy to see that its dimension $n_1 = n_1 + n_0 + (N - n_0) - N$ coincides with the dimension (e_1).

Thus, the n_0+n_1-dimensional space (e_1,e_0) turns out to be divided into a sum of mutually orthogonal subspaces in two ways:

$$(e_0)\oplus(e_1)=(e_{0;1})\oplus(e_1^1),$$

where $(e_{0;1}):=(e_1^1)^\perp\cap((e_0)\oplus(e_1))\sim(e_0)$ is the orthogonal complement of (e_1^1) in the space (e_0,e_1).

Let us denote by Φ_{01} a unitary transformation $\mathbb{C}^N$, coinciding with the unitary coordinate transformation of the subspace $(e_0)\oplus(e_1)$, which takes (e_0) to $(e_{0;1})$ in (e_0,e_1), and the identity on its orthogonal complement, that is, on $(e_2,\ldots,e_m)$. The image of (e_1) under this transformation will coincide with $(e_1^1)\subset(E_0)^\perp$.

So, we have constructed a unitary Φ_{01}, the result of which is to move the subspace (e_1) into the space (e_1^1) contained in the space $(E_0)^\perp$, and move the space (e_0) into some $(e_{0;1})$, which, by construction, is, of course, orthogonal to (e_1^1), as well as to the remaining (e_k), on which the action of Φ_{01} is trivial.

Let us now construct Φ_{02}. After transforming Φ_{01}, we obtained a new partition of $\mathbb{C}^N$ into the orthogonal sum

$$(e_{0;1})\overset{\perp}{\oplus}(e_1^1)\overset{\perp}{\oplus}(e_2)\overset{\perp}{\oplus}\ldots\overset{\perp}{\oplus}(e_m).$$

It differs from the original partition in that (e_1^1) already lies on $(E_0)^\perp$, and (e_0) has moved inside the space (e_0,e_1), turning into $(e_{0;1})$. All other subspaces (e_k) have not changed yet.

Consider the subspace $(e_{0;1})\oplus(e_2)$, and denote by (e_2^1) its intersection with the coordinate subspace $(E_0)^\perp$, and by $(e_{0;2})$ denote the orthogonal complement to (e_2^1) in the space $(e_{0;1},e_2)$,

$$(e_{0;2}):=(e_2^1)^\perp\cap((e_{0;1})\oplus(e_2))\sim(e_0).$$

Let Φ_{02} be a unitary transformation $\mathbb{C}^N$, identical on $(e_1^1,e_3,\ldots e_m)=((e_{0;1})\oplus(e_2))^\perp$, and which is a unitary coordinate transformation $(e_{0;1})\oplus(e_2)$, taking $(e_{0;1})$ to $(e_{0;2})$ and, accordingly, (e_2) in $(e_2^1)\subset(E_0)^\perp$.

Note that, of course, each of $e_1^1,e_3,\ldots e_m$, is orthogonal to $(e_{0;1})\oplus(e_2)$, because the entire new set $(e_{0;1)},e_1^1,e_2,e_3,\ldots e_m)$ is obtained from the old $(e_0,e_1,e_2,e_3,\ldots e_m)$ by a unitary transformation.

Let's continue constructing Φ_{0k}. The transformation Φ_{0k} sends (e_k) to $(E_0)^\perp$, to the subspace $(e_k^1):=((e_k)\oplus(e_{0;k-1}))\cap(E_0)^\perp$, and the subspace $(e_{0;k-1})$ sends to the subspace $(e_{0;k})$.

Note that $(e_{0;k})$ becomes, in a sense, closer and closer to (E_0), with each step, because the space

$$(e_{0;k})^{\perp} \cap (E_0)^{\perp} = (e_1^1, e_2^1, \dots e_{k-1}^1),$$

orthogonal to both of them, has larger and larger dimensions, with each step. When, after the last step $k = m - 1$, the orthogonal complements to $(e_{0;m})$ and to (E_0) coincide, we obtain the desired unitary transformation

$$\Phi_{0m}\Phi_{0m-1}\dots\Phi_{02}\Phi_{01},$$

which takes (e_0) to $(E_0) = (e_{0;m})$.

Let us now consider the subspace $(E_0)^{\perp}$ as the entire space. We have the same problem, but of a smaller dimension, namely, to combine by a unitary coordinate transformation (identical on (E_0)), the subspaces (e_1^1) and (E_1). Repeating the previous arguments, we obtain a set of unitary coordinate transformations $\Phi_{1m}\Phi_{1m-1}\dots\Phi_{13}\Phi_{12}$ that solves this problem.

Continuing the reduction of dimensionality, we eventually obtain a transformation $\Phi_{m-1,m}$, acting nontrivially in

$$(e_m^{m-1}) \oplus (e_{m-1}^{m-1}) = (E_m) \oplus (E_{m-1})$$

and combining (e_{m-1}) with (E_{m-1}), and, therefore, (e_m) with (E_m), the process will end.

Its result is a set of unitary coordinate transformations

$$\begin{array}{ll} \Phi_{0m}\Phi_{0m-1}\dots\Phi_{02}\Phi_{01} & \lambda_0 \\ \Phi_{1m}\Phi_{1m-1}\dots\Phi_{12} & \lambda_1 \\ \vdots & \\ \Phi_{m-1,m} & \lambda_{m-1} \\ \mathrm{I} & \lambda_m. \end{array} \tag{6}$$

Here the line number "i" corresponds to the eigenvalue λ_i, in the sense that the sequential application of transformations in this line, and those above, transforms the eigensubspace corresponding to λ_i into the coordinate subspace (E_i).

Each Φ_{ij} is represented as a set of successive elementary unitary rotations $\phi_{ij;rs}$. Each of these rotations $\phi_{ij;rs}$ is characterized by a single complex parameter $t_{ij;rs}$.

Theorem 1. *The ordered set of parameters $t_{ij;rs}$, together with the eigenvalue λ_i assigned to each value "i", is a coordinate set of functions on an algebraically open subset of the space of Hermitian matrices with a given set dimensions of the eigenspaces $\{n_0 \leqslant n_1 \leqslant \ldots \leqslant n_m\}$.*

§4. Example. Case of general position in $\mathbb{C}^3$

Let us consider the case of the general position that is an element of SU(3) having three mutually orthogonal subspaces corresponding different eigenvalues $\lambda_0 \neq \lambda_1 \neq \lambda_2 \neq \lambda_0$. The subspaces split $\mathbb{C}^3$ on the orthogonal sum.

In this case all Grassmannians are trivial, they are just the projective lines. The table (6) looks

$$\begin{array}{ll} \Phi_{02}\Phi_{01} & \lambda_0 \\ \Phi_{12} & \lambda_1 \\ \mathrm{I} & \lambda_2. \end{array}$$

The first line in the table consists of two *elementary unitary rotations* Φ_{02}, Φ_{01}. Their combination coincides $(e_0)^\perp$ with $(E_0)^\perp$ and (e_0) with (E_0).

The last rotation Φ_{12} on the plane (E_1, E_2) coincides e_1^1 with E_1 and e_2^1 with E_2, "the game is over".

This procedure gives the classical Tait-Bryan angles in the real case.

4.1. Tait-Bryan angles. The usual notations of "moving" coordinate axes x, y, z and "fixed" X, Y, Z (or x''', y''', z'''). In our notations, they correspond to the lines $(e_0), (e_1), (e_2)$ and $(E_0), (E_1), (E_2)$. Let us carry out the process of successive alignment of the axes described in the article, we will see that it also gives the Tait–Bryan angles (z-y'-x''-convention) in the real case.

The notation "z-y'-x''-convention" means that we perform the sequent rotations, the first around z-axis of the original frame xyz, then around the y-axis of the frame that resulted from this rotation (standard notation $x'y'z'$), and finally around the x-axis of the frame obtained after the first two rotations.

The position of the x-axis after two rotations is final, i.e., $x'' = X = x'''$, since the first two rotations coincide the initial and the final positions of the x-axis. The standard notation for the final position of the reference point is XYZ, or $x'''y'''z'''$, if after each rotation a prime is adding to the axis notation. We note that the latter notation, with three primes, is usually *not used*, people write XYZ.

Thus, the first action is a rotation in the plane

$$(e_0) \oplus (e_1) = (e_0, e_1) = (x, y)$$

around (i.e. being the identity mapping there) the orthogonal complement to it, around the axis $(e_2) = z$ so that during this rotation the axis $(e_1) = y$ goes over to the line of nodes

$$N := (e_0, e_1) \cap (E_0)^{\perp} = (e_0, e_1) \cap (E_1, E_2),$$

"lies on the orthogonal complement to (E_0)". In this case, the new position of e_1 is denoted by $e_1^1 \in (E_1, E_2)$ (it is directed along the line of nodes N), and the new position of e_0 is denoted by $e_{0;1}$. This unitary rotation around (e_2) is denoted by Φ_{01}.

For the second rotation Φ_{02}, we consider the plane spanned by this new $e_{0;1}$ and e_2 orthogonal to it. This plane is obviously the orthogonal complement of the line of nodes $e_1^1 \in (E_1, E_2)$, since these are all new positions of the old (orthogonal) axes after the unitary transformation Φ_{01}.

We make a unitary rotation around the line of nodes (where e_1^1 is already located) as a second action. It place (e_2) on the plane (E_1, E_2). That is, we place the second basis vector (e_2) (the first was (e_1)) on the plane (E_1, E_2), rotating around the line of nodes $N = (e_1^1)$.

The result of rotations Φ_{01} and Φ_{02} is to place the axes (e_1) and (e_2) on the plane (E_1, E_2), and, consequently, to coincide with the axes (e_0) and (E_0).

We make a unitary rotation in the plane (E_1, E_2), combining e_1^1 with E_1, and, therefore, e_2^1 with E_2. These are the "Tait-Bryan angles".

Consider the first action. It is the rotation Φ_{01} (called *a "yaw"*, on the angle usually denoted by ψ) around the "z" axis (this is our (e_2)). This rotation combines the "y" axis with the line of nodes N (classically this is "y'", ours (e_1^1)), that is, the intersection of (x, y) and (Z, Y) (ours $(e_0, e_1) \cap (E_1, E_2)$).

The second action is the rotation Φ_{02} (called *a "pitch"*, on the angle usually denoted by θ) around the line of nodes (the "y'" axis is now directed along it), with its help we place (e_2) (this is z) on the (Z, Y) plane (this is $(E_0)^{\perp}$).

After these two rotations the axis "x" will coincide with the final position $X = x''$ (in our notation the initial (e_0) and the finite (E_0)). The rotation Φ_{12} remains.

The third action is non-trivial in the plane (Y, Z) (in our notation it is $(E_1, E_2) = (E_0)^{\perp}$). It is performed around the axis $x'' = X$ (ours (E_0)) by an angle which is called "*a roll*" (usually it is denoted by ϕ) in order to align the axes with this plane.

§5. Conclusion

The multidimensional version of the Euler-Tait-Bryan angles in the unitary space is presented. The concept of an **elementary unitary rotation** is introduced and a parameterization of the affine part of any Grassmannian by the sequence of the elementary unitary rotations is given. We construct a set of unitary coordinate transformations parameterizing the algebraically-open subsets of the SU(N) conjugation classes of Hermitian matrices. These unitary transformations map the eigenspaces of the representative to the coordinate subspaces and give the set of coordinate functions.

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St. Petersburg Department
of V. A. Steklov Institute of Mathematics,
St. Petersburg, Russia
E-mail: mbabich@pdmi.ras.ru

Поступило 19 сентября 2025 г.

Joint Insatitute for Nuclear Research,
Dubna, 141980, Russia
E-mail: ljudmila@bordag.com

A. Razmadze Mathematical Institute,
Iv. Javakhishvili Tbilisi State University;
Institute of Quantum Physics
and Engineering Technologies,
Georgian Technical University,
Tbilisi, Georgia;
Meshcheryakov Laboratory
of Information Technologies,
Joint Institute for Nuclear Research,
Dubna, 141980, Moscow Region, Russia
E-mail: akhved@jinr.ru

Sofia University “St. Kliment Ohridski”,
1164 Sofia, Bulgaria
E-mail: dimitar.mladenov@phys.uni-sofia.bg