Записки научных
семинаров ПОМИ
Том 543, 2025 г.

I. Baskov

THE SPLITTING OF THE DE RHAM COHOMOLOGY OF SOFT FUNCTION ALGEBRAS IS MULTIPLICATIVE

Abstract. Let A be a real soft function algebra. In [1] we have obtained a canonical splitting $\mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \cong \mathrm{H}^*(X,\mathbb{R}) \oplus$ (something) via the canonical maps $\Lambda_A : \mathrm{H}^*(X,\mathbb{R}) \to \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}})$ and $\Psi_A : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathrm{H}^*(X,\mathbb{R})$. In this paper we prove that these maps are multiplicative.

§1. Introduction

All algebras are assumed to be commutative. To an algebra A over a field k one associates a dg-algebra $\Omega^\bullet_{A|k}$ with $\Omega^0_{A|k} = A$, called the de Rham dg-algebra, in a standard way (see [4, §3]).

In this paper we extend the results of the paper [1]. There, for a soft sheaf of k-algebras $\mathcal{F}$ on a compact Hausdorff space X we have constructed a linear map

$$\Lambda_{\mathcal{F}} : \mathbb{H}^*(X, \underline{k}_X[0]) \to \mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k}).$$

Here the domain is the cohomology of X with coefficients in the constant sheaf $\underline{k}_X$. This map is natural with respect to morphisms of k-ringed spaces. For an arbitrary space X and a subalgebra $A \hookrightarrow C(X)$ of the $\mathbb{R}$-algebra of real-valued continuous functions on X we have constructed a linear map

$$\Psi_A : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]).$$

This map is natural with respect to continuous maps of spaces covered by a homomorphism of algebras. We have proved that for a compact Hausdorff space X and a soft subsheaf $\mathcal{F}$ of C_X, the sheaf of $\mathbb{R}$-algebras of real-valued continuous functions, the composition $\Psi_{\mathcal{F}(X)} \circ \Lambda_{\mathcal{F}}$ is the identity map. Thus, the groups $\mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$ canonically split off of $\mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)})$.

Key words and phrases: de Rham cohomology of algebras, soft function algebra, canonical splitting.

This research was partially financed by the Russian Science Foundation via agreement No. 24-21-00119.

But there is still a question of whether the maps $\Lambda_{\mathcal{F}}$ and Ψ_A are multiplicative with respect to the cup product on cohomology. The answer is yes and is given by Theorem 1 and Theorem 11.

Acknowledgments. I would like to thank Dr. Semën Podkorytov for his immense patience, along with many fruitful discussions and his invaluable help in drafting this paper. I am grateful to the St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences for their financial assistance.

§2. Cup product in hypercohomology

2.1. Hypercohomology groups. By a complex we mean a non-negative cochain complex. By default, a complex consists of k-vector spaces. For a sheaf $\mathcal{F}$ of vector spaces over a field k we denote by $\mathcal{F}[0]$ the complex of sheaves with $\mathcal{F}$ in degree 0 and other terms zero. We denote by $\underline{k}_X$ the constant sheaf on a space X associated with the field k.

For any complex $\mathcal{F}^\bullet$ there is a complex $\mathcal{I}^\bullet$ of injective sheaves and a quasi-isomorphism $i : \mathcal{F}^\bullet \to \mathcal{I}^\bullet$, see [7, Proposition 8.4]. We call $\mathcal{I}^\bullet$ an *injective resolution* of $\mathcal{F}^\bullet$.

One defines the *hypercohomology* of a complex of sheaves $\mathcal{F}^\bullet$ as

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) := \mathrm{H}^*(\mathcal{I}^\bullet(X))$$

for some injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. We refer the reader to [8, Definition 10.2] for the precise definition of the hypercohomology groups.

There is a canonical homomorphism

$$\Upsilon : \mathrm{H}^*(\mathcal{F}^\bullet(X)) \to \mathbb{H}^*(X, \mathcal{F}^\bullet),$$

natural with respect to morphisms of complexes of sheaves.

Take two complexes of sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ on topological spaces X and Y, respectively. Suppose $\phi : Y \to X$ is a continuous map and $\gamma : \mathcal{F}^\bullet \to \phi_*\mathcal{G}^\bullet$ is a morphism of complexes of sheaves. Then there is the induced map on the hypercohomology $\mathbb{H}^*(\phi, \gamma) : \mathbb{H}^*(X, \mathcal{F}^\bullet) \to \mathbb{H}^*(Y, \mathcal{G}^\bullet)$.

2.2. External cup product. For two complexes of sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ on a space X one defines their tensor product over k, $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$, as a complex of sheaves on X, see [3, Chapter II, 6.1]. We construct a map

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{G}^\bullet) \to \mathbb{H}^*(X, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet),$$

called the *external cup product*. Choose injective resolutions

$$i_1 : \mathcal{F}^\bullet \to \mathcal{I}^\bullet, \quad i_2 : \mathcal{G}^\bullet \to \mathcal{J}^\bullet, \quad j : \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet \to \mathcal{K}^\bullet.$$

The map of complexes

$$i_1 \otimes i_2 : \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet \to \mathcal{I}^\bullet \otimes \mathcal{J}^\bullet$$

is a quasi-isomorphism, as the tensor product of sheaves over a field is an exact functor. There exists a unique up to homotopy morphism of complexes $\beta : \mathcal{I}^\bullet \otimes \mathcal{J}^\bullet \to \mathcal{K}^\bullet$ making the following diagram commute up to homotopy

$$\begin{array}{ccc} \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet & \xrightarrow{i_1 \otimes i_2} & \mathcal{I}^\bullet \otimes \mathcal{J}^\bullet \\ \Big\downarrow j & \swarrow_{\beta} & \\ \mathcal{K}^\bullet, & & \end{array}$$

see [6, Lemma 13.18.6 and Lemma 13.18.7].

Taking the global sections over X one obtains the natural map

$$\beta(X) : (\mathcal{I}^\bullet \otimes \mathcal{J}^\bullet)(X) \to \mathcal{K}^\bullet(X).$$

On the other hand, one has the natural map

$$\mathcal{I}^\bullet(X) \otimes \mathcal{J}^\bullet(X) \to (\mathcal{I}^\bullet \otimes \mathcal{J}^\bullet)(X).$$

Composing these maps and taking the cohomology leads to the natural map

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{G}^\bullet) \to \mathbb{H}^*(X, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet).$$

This product is natural in $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. Moreover, it is natural in triples $(X, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$.

The external cup product is compatible with Υ, that is, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}^*(\mathcal{F}^\bullet(X)) \otimes \mathrm{H}^*(\mathcal{G}^\bullet(X)) & \longrightarrow & \mathrm{H}^*((\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)(X)) \\ \Big\downarrow \Upsilon \otimes \Upsilon & & \Big\downarrow \Upsilon \\ \mathbb{H}^*(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{G}^\bullet) & \xrightarrow{\cup} & \mathbb{H}^*(X, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet), \end{array} \tag{1}$$

where the top map is induced by the natural map of complexes

$$\mathcal{F}^\bullet(X) \otimes \mathcal{G}^\bullet(X) \to (\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)(X).$$

2.3. Internal cup product. Let $\mathcal{F}^\bullet$ be a sheaf complex with multiplication, that is, a complex of sheaves $\mathcal{F}^\bullet$ comes with a morphism of complexes

$$\mathcal{F}^\bullet \otimes \mathcal{F}^\bullet \xrightarrow{\times} \mathcal{F}^\bullet,$$

called the multiplication. Composing with the external cup product, this multiplication gives rise to the product

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{F}^\bullet) \xrightarrow{\cup} \mathbb{H}^*(X, \mathcal{F}^\bullet),$$

which we call the (*internal*) *cup product.*

The cup product on hypercohomology satisfies the following properties.

(1) The product on the hypercohomology

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{F}^\bullet) \xrightarrow{\cup} \mathbb{H}^*(X, \mathcal{F}^\bullet)$$

is natural in the following sense. Let $\gamma : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of sheaf complexes with multiplication, that is, the diagram

$$\begin{array}{ccc} \mathcal{F}^\bullet \otimes \mathcal{F}^\bullet & \xrightarrow{\times} & \mathcal{F}^\bullet \\ {\scriptstyle \gamma\otimes\gamma}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathcal{G}^\bullet \otimes \mathcal{G}^\bullet & \xrightarrow{\times} & \mathcal{G}^\bullet \end{array}$$

is commutative.

Then we have the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^*(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{F}^\bullet) & \xrightarrow{\cup} & \mathbb{H}^*(X, \mathcal{F}^\bullet) \\ {\scriptstyle \mathbb{H}^*(\gamma)\otimes\mathbb{H}^*(\gamma)}\downarrow & & \downarrow{\scriptstyle \mathbb{H}^*(\gamma)} \\ \mathbb{H}^*(X, \mathcal{G}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{G}^\bullet) & \xrightarrow{\cup} & \mathbb{H}^*(X, \mathcal{G}^\bullet). \end{array}$$

(2) Take a space X with a sheaf complex with multiplication $\mathcal{F}^\bullet$ and a space Y with a sheaf complex with multiplication $\mathcal{G}^\bullet$. Let $\phi : Y \to X$ be a continuous map and $\gamma : \mathcal{F}^\bullet \to \phi_* \mathcal{G}^\bullet$ be a morphism of sheaf complexes with multiplication. Then we have the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^*(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^*(X, \mathcal{F}^\bullet) & \xrightarrow{\cup} & \mathbb{H}^*(X, \mathcal{F}^\bullet) \\ {\scriptstyle \mathbb{H}^*(\phi,\gamma)\otimes\mathbb{H}^*(\phi,\gamma)}\downarrow & & \downarrow{\scriptstyle \mathbb{H}^*(\phi,\gamma)} \\ \mathbb{H}^*(Y, \mathcal{G}^\bullet) \otimes \mathbb{H}^*(Y, \mathcal{G}^\bullet) & \xrightarrow{\cup} & \mathbb{H}^*(Y, \mathcal{G}^\bullet). \end{array}$$

(3) The product on hypercohomology is compatible with Υ.
Explicitly, the multiplication on $\mathcal{F}^\bullet$ induces the product

$$\mathrm{H}^*(\mathcal{F}^\bullet(X)) \otimes \mathrm{H}^*(\mathcal{F}^\bullet(X)) \to \mathrm{H}^*(\mathcal{F}^\bullet(X)).$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}^*(\mathcal{F}^\bullet(X)) \otimes \mathrm{H}^*(\mathcal{F}^\bullet(X)) & \longrightarrow & \mathrm{H}^*(\mathcal{F}^\bullet(X)) \\ \big\downarrow{\scriptstyle \Upsilon\otimes\Upsilon} & & \big\downarrow{\scriptstyle \Upsilon} \\ \mathbb{H}^*(X,\mathcal{F}^\bullet) \otimes \mathbb{H}^*(X,\mathcal{F}^\bullet) & \xrightarrow{\ \cup\ } & \mathbb{H}^*(X,\mathcal{F}^\bullet). \end{array}$$

The sheaf complex $\underline{k}_X[0]$ carries multiplication

$$\underline{k}_X[0] \otimes \underline{k}_X[0] \to \underline{k}_X[0].$$

The resulting cup product

$$\mathbb{H}^*(X, \underline{k}_X[0]) \otimes \mathbb{H}^*(X, \underline{k}_X[0]) \xrightarrow{\cup} \mathbb{H}^*(X, \underline{k}_X[0])$$

coincides with the usual cup product on the sheaf cohomology, see [7, §5.3.2].

§3. Algebraic de Rham forms

For any commutative algebra with unity A over a field k we denote by $\Omega^\bullet_{A|k}$ the dg-algebra of algebraic de Rham forms, see [4, §3].

Let X be a compact Hausdorff space. Consider a soft sheaf of algebras $\mathcal{F}$ on X. In [1, Section 2.4] we consider the complex of presheaves $\Omega^\bullet_{\mathcal{F}|k}$ on X with $\Omega^\bullet_{\mathcal{F}|k}(U) = \Omega^\bullet_{\mathcal{F}(U)|k}$. For $n \geqslant 0$ we denote by ${}^+\Omega^n_{\mathcal{F}|k}$ the associated sheaf of $\Omega^n_{\mathcal{F}|k}$. The sheaves ${}^+\Omega^n_{\mathcal{F}|k}$ form a complex of sheaves ${}^+\Omega^\bullet_{\mathcal{F}|k}$. We denote by

$$\mathrm{sh} : \Omega^\bullet_{\mathcal{F}(X)|k} = \Omega^\bullet_{\mathcal{F}|k}(X) \to {}^+\Omega^\bullet_{\mathcal{F}|k}(X)$$

the sheafification map.

We define a morphism of sheaf complexes with multiplication

$$\epsilon : \underline{k}_X[0] \to {}^+\Omega^\bullet_{\mathcal{F}|k},$$

called the coaugmentation, by $\epsilon(1) := 1$.

The wedge product

$$\Omega^\bullet_{A|k} \otimes \Omega^\bullet_{A|k} \xrightarrow{\wedge} \Omega^\bullet_{A|k}$$

for any k-algebra A gives rise to a product

$$\Omega^\bullet_{\mathcal{F}|k} \otimes \Omega^\bullet_{\mathcal{F}|k} \xrightarrow{\wedge} \Omega^\bullet_{\mathcal{F}|k}$$

for the complex of presheaves $\Omega^\bullet_{\mathcal{F}|k}$. Passing to the associated sheaves gives rise to a product

$$^+\Omega^\bullet_{\mathcal{F}|k} \otimes {}^+\Omega^\bullet_{\mathcal{F}|k} \xrightarrow{\wedge} {}^+\Omega^\bullet_{\mathcal{F}|k}.$$

Taking the global sections yields the commutative diagram

$$\begin{array}{ccc} \Omega^\bullet_{\mathcal{F}(X)|k} \otimes \Omega^\bullet_{\mathcal{F}(X)|k} & \xrightarrow{\quad\wedge\quad} & \Omega^\bullet_{\mathcal{F}(X)|k} \\ {\scriptstyle \mathrm{sh}\otimes\mathrm{sh}}\downarrow & & \downarrow{\scriptstyle \mathrm{sh}} \\ {}^+\Omega^\bullet_{\mathcal{F}|k}(X) \otimes {}^+\Omega^\bullet_{\mathcal{F}|k}(X) & \xrightarrow{\quad\wedge\quad} & {}^+\Omega^\bullet_{\mathcal{F}|k}(X). \end{array} \tag{2}$$

Now, we have the cup product

$$\mathbb{H}^*(X, {}^+\Omega^\bullet_{\mathcal{F}|k}) \otimes \mathbb{H}^*(X, {}^+\Omega^\bullet_{\mathcal{F}|k}) \xrightarrow{\cup} \mathbb{H}^*(X, {}^+\Omega^\bullet_{\mathcal{F}|k}).$$

§4. The map $\Lambda_{\mathcal{F}}$ is multiplicative

In this section we briefly recall the construction of the natural map

$$\Lambda_{\mathcal{F}} : \mathbb{H}^*(X, \underline{k}_X[0]) \to \mathbb{H}^*(X, {}^+\Omega^\bullet_{\mathcal{F}|k})$$

constructed in [1, Chapter 3]. Then we prove that the map $\Lambda_{\mathcal{F}}$ is multiplicative.

Definition. *For a soft sheaf of algebras $\mathcal{F}$ on a compact Hausdorff space X we define the map $\Lambda_{\mathcal{F}}$ by the following diagram:*

$$\begin{array}{ccc} \mathbb{H}^*(X, \underline{k}_X[0]) & \xrightarrow{\mathbb{H}^*(\epsilon)} & \mathbb{H}^*(X, {}^+\Omega^\bullet_{\mathcal{F}|k}) \\ & & \cong\uparrow{\scriptstyle \Upsilon} \\ & \searrow^{\Lambda_{\mathcal{F}}} & \mathrm{H}^*({}^+\Omega^\bullet_{\mathcal{F}|k}(X)) \\ & & \cong\uparrow{\scriptstyle \mathrm{H}^*(\mathrm{sh})} \\ & & \mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k}). \end{array}$$

The maps Υ and $\mathrm{H}^*(\mathrm{sh})$ are isomorphisms, see [1, Section 3.2].

The map $\mathrm{H}^*(\mathrm{sh})$ is a multiplicative by Diagram 2. *The map Υ is multiplicative by the property* 3 *of the internal cup product. The map $\mathbb{H}^*(\epsilon)$ is multiplicative by the property* 1 *of the internal cup product.*

Hence, we obtain the following theorem:

Theorem 1. *The map*

$$\Lambda_{\mathcal{F}} : \mathbb{H}^*(X, \underline{k}_X[0]) \to \mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k})$$

is multiplicative.

§5. Simplicial techniques

We call a fibrant pointed simplicial set (K, o) *connected* if for each $v \in K_0$ there exists a simplicial map $\Delta[1] \to K$ such that $0 \mapsto o$ and $1 \mapsto v$ (here $0, 1 \in \Delta[1]_0$ are the ends). We call (K, o) *contractible* if there exists a basepoint preserving homotopy $\Delta[1] \times K \to K$ that maps the bottom base to o and is the identity on the top base.

A simplicial vector space B is called a simplicial module over a simplicial algebra A if there is a simplicial linear map

$$A \otimes B \xrightarrow{\times} B$$

that makes B_p into an A_p-module for each $p \geqslant 0$, cf. [10, Chapter 2.6].

Lemma 2. *Let B be a simplicial module over a connected simplicial algebra A. Then B is contractible.*

Proof. As A is connected there exists a simplicial map

$$\gamma : \Delta[1] \to A$$

connecting 0 and 1. Then the composition

$$\Delta[1] \times B \xrightarrow{\gamma \times \mathrm{id}} A \times B \xrightarrow{\times} B$$

is the required homotopy. □

We call a simplicial complex of vector spaces $A^\bullet$ *degree-wise contractible* if A^q is contractible for each $q \geqslant 0$. Note that in [2] the degree-wise contractible simplicial complexes are called "extendable complexes". We call a simplicial map of simplicial complexes $A^\bullet \to B^\bullet$ a *dimension-wise quasi-isomorphism* if it is a quasi-isomorphism in each simplicial dimension.

Corollary 3. *Let $A^\bullet$ be a simplicial graded algebra such that the simplicial algebra A^0 is connected. Then $A^\bullet$ is degree-wise contractible.*

Corollary 4. *Let $A^\bullet$ and $B^\bullet$ be simplicial graded algebras and $A^\bullet$ be degree-wise contractible. Then $A^\bullet \otimes B^\bullet$ is degree-wise contractible.*

Proof. The simplicial vector space $(A^\bullet \otimes B^\bullet)^q$ is a simplicial module over the simplicial algebra A^0, which is connected as it is contractible. □

For any simplicial complex $A^\bullet$ and a simplicial set K one defines a complex $A^\bullet(K)$ in a usual way: for $q \geqslant 0$ the vector space $A^q(K)$ consists of all simplicial maps $K \to A^q$, cf. [2, Chapter 10, (b)]. If $A^\bullet$ is a simplicial dg-algebra, then $A^\bullet(K)$ is a dg-algebra.

Lemma 5. *Let $\theta : A^\bullet \to B^\bullet$ be a dimension-wise quasi-isomorphism of degree-wise contractible simplicial complexes. Then for any simplicial set K, the induced morphism of complexes $\theta(K) : A^\bullet(K) \to B^\bullet(K)$ is a quasi-isomorphism.*

Proof. See [2, Proposition 10.5]. □

§6. The map Ψ is multiplicative

In the paper [1] we have contructed the map

$$\Psi : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$$

for any topological space X and any subalgebra A of the algebra of real-valued continuous functions on X. In this section we prove that this map is multiplicative.

6.1. Simplicial dg-algebra $\mathbb{R}[0]_{\mathrm{const}}$. We denote by $\mathbb{R}[0]_{\mathrm{const}}$ the simplicial dg-algebra that equals $\mathbb{R}[0]$ in each dimension. We have

$$\mathbb{R}[0]_{\mathrm{const}} \otimes \mathbb{R}[0]_{\mathrm{const}} = \mathbb{R}[0]_{\mathrm{const}}.$$

6.2. Simplicial dg-algebra $\mathrm{C}^\bullet$. We denote by $\mathrm{C}^\bullet$ the simplicial dg-algebra of (unnormalized) simplicial cochains on the combinatorial simplices $\Delta[n]$, cf. [2, Chapter 10, (d)]. The multiplication is the usual Alexander–Whitney cup product. There is a unique morphism of simplicial dg-algebras $\epsilon : \mathbb{R}[0]_{\mathrm{const}} \to \mathrm{C}^\bullet$, called the coaugmentation.

Lemma 6. *The simplicial dg-algebra $\mathrm{C}^\bullet$ is degree-wise contractible. Moreover the coaugmentation map $\epsilon : \mathbb{R}[0]_{\mathrm{const}} \to \mathrm{C}^\bullet$ is a dimension-wise quasi-isomorphism.*

Cf. [2, Lemma 10.12] for the normalized simplicial cochains.

6.3. Simplicial dg-algebra $\Omega^\bullet_\flat$. In [1, Chapter 5] we have introduced the simplicial dg-algebra of flat cochains $\Omega^\bullet_\flat$. For each $n \geqslant 0$, the dg-algebra $\Omega^\bullet_\flat(\Delta^n)$ consists of affine cochains on Δ^n bounded with respect to the flat seminorm.

Consider the simplicial algebra Lip, where, for each $n \geqslant 0$, the algebra $\mathrm{Lip}(\Delta^n)$ consists of all Lipschitz functions on Δ^n. By [1, 5.2(2)] there is a morphism of simplicial algebras

$$\zeta : \mathrm{Lip} \to \Omega^0_\flat.$$

Lemma 7. *The simplicial dg-algebra $\Omega^\bullet_\flat$ is degree-wise contractible.*

Proof. By construction of $\Omega^\bullet_\flat$, for each $q \geqslant 0$, $\Omega^q_\flat$ is a simplicial module over $\Omega^0_\flat$ and, hence, over Lip. By Lemma 2 it suffices to prove that the simplicial algebra Lip is connected, which is easy to see. □

There is a unique morphism of simplicial dg-algebras $\epsilon : \mathbb{R}[0]_{\mathrm{const}} \to \Omega^\bullet_\flat(\Delta^n)$, called the coaugmentation.

Lemma 8. *The map $\epsilon : \mathbb{R}[0]_{\mathrm{const}} \to \Omega^\bullet_\flat$ is a dimension-wise quasi-isomorphism.*

Proof. See the construction of $\Omega^\bullet_\flat(\Delta^n)$ and [9, Chapter VII, Theorem 12A]. □

6.4. Spectrum of a finitely generated algebra. For a set $Z \subset \mathbb{R}^n$ one defines the simplicial set $\mathrm{Sing}_{\,\mathrm{Lip}}(Z)$ of Lipschitz singular simplices.

We denote by $\mathrm{S}^\bullet_{\mathrm{Lip}}(Z)$ the complex of Lipschitz singular cochains on Z. This complex can be equipped with the Alexander–Whitney cup product, which is non-commutative. We have

$$\mathrm{C}^\bullet(\mathrm{Sing}_{\,\mathrm{Lip}}(Z)) = \mathrm{S}^\bullet_{\mathrm{Lip}}(Z),$$

where the product on the left, that comes from the product on $\mathrm{C}^\bullet$, coincides with the Alexander–Whitney cup product on the right.

For a finitely generated $\mathbb{R}$-algebra B one constructs the *real maximal spectrum* $\mathrm{spec}_{\mathbb{R}} B$. It is an algebraic set. For a Lipschitz singular simplex $\sigma : \Delta^n \to \mathrm{spec}_{\mathbb{R}} B$ one introduces the dg-algebra morphism

$$\mu(\sigma) : \Omega^\bullet_{B|\mathbb{R}} \to \Omega^\bullet_\flat(\Delta^n),$$

see [1, Section 6.1].

Lemma 9. *Take $h : [m] \to [n]$ a morphism in the category $\mathbf{\Delta}$. The following diagram is commutative:*

$$\begin{array}{ccc} \Omega^\bullet_{B|\mathbb{R}} & & \\ {\scriptstyle \mu(\sigma)}\downarrow & \searrow^{\mu(h^*\sigma)} & \\ \Omega^\bullet_\flat(\Delta^n) & \xrightarrow{h^*} & \Omega^\bullet_\flat(\Delta^{n-1}). \end{array}$$

Proof. The proof is analogous to the proof of [1, Lemma 21]. □

The lemma allows us to construct a dg-algebra morphism

$$\hat{\mu} : \Omega^\bullet_{B|\mathbb{R}} \to \Omega^\bullet_\flat(\mathrm{Sing}_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B))$$

as

$$\hat{\mu}(\omega)(\sigma) = \mu(\sigma)(\omega) \tag{3}$$

for any $\omega \in \Omega^\bullet_{B|\mathbb{R}}$ and any $\sigma \in \mathrm{Sing}_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B)_n$.

Any simplex $\beta \in \Delta[n]_m$ gives rise to a affine singular simplex $\Delta^m \to \Delta^n$. This yields the restriction morphism of simplicial complexes

$$\tau : \Omega^\bullet_\flat \to \mathrm{C}^\bullet,$$

which is not multiplicative.

We have a map of complexes

$$\xi_B : \Omega^\bullet_{B|\mathbb{R}} \to \mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B)$$

defined as

$$\xi_B(\omega)(\sigma) = \tau(\mu(\sigma)(\omega))(\mathrm{id}_{[n]}), \tag{4}$$

where $\mathrm{id}_{[n]} \in \Delta[n]_n$, see [1, Section 6.2].

The following diagram is commutative by Formula 3 and Formula 4:

$$\begin{array}{ccc} \Omega^\bullet_\flat(\mathrm{Sing}_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B)) & \xrightarrow{\tau(\mathrm{Sing}_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B))} & \mathrm{C}^\bullet(\mathrm{Sing}_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B)) \\ \uparrow{\scriptstyle \hat{\mu}} & & \| \\ \Omega^\bullet_{B|\mathbb{R}} & \xrightarrow[\xi_B]{} & \mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B). \end{array} \tag{5}$$

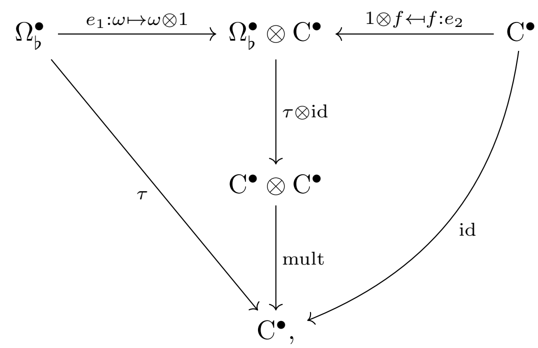
Lemma 10. *The map*

$$\mathrm{H}^*(\xi_B) : \mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \to \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_\mathbb{R} B))$$

is multiplicative.

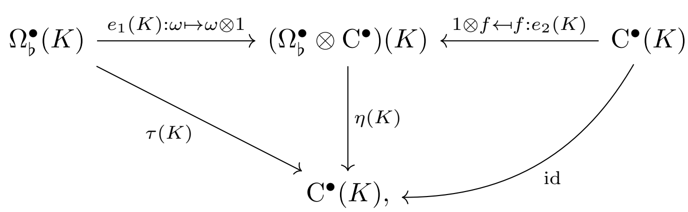
The proof follows [2, Theorem 10.9].

Proof. Consider the following commutative diagram of simplicial dg-algebras and simplicial cochains maps:



where mult is the multiplication of $\mathrm{C}^\bullet$. Taking tensor product of coaugmentations on $\Omega^\bullet_\flat$ and $\mathrm{C}^\bullet$ we obtain coaugmentations on $\Omega^\bullet_\flat \otimes \mathrm{C}^\bullet$ and $\mathrm{C}^\bullet \otimes \mathrm{C}^\bullet$. By Lemma 6, Lemma 7 and Corollary 4 all the simplicial dg-algebras in this diagram are degree-wise contractible. By Lemma 6, Lemma 8 and the fact that all the cochain maps in the diagram preserve coaugmentation, all cochain maps are dimension-wise quasi-isomorphism. Notice, that the maps e_1 and e_2 are in fact morphisms of simplicial dg-algebras. Let $\eta : \Omega^\bullet_\flat \otimes \mathrm{C}^\bullet \to \mathrm{C}^\bullet$ be the composition of the vertical arrows.

Substituting a simplicial set K gives rise to the commutative diagram of dg-algebras and cochains maps



where all of the arrows are quasi-isomorphisms by Lemma 5. The maps $e_1(K)$ and $e_2(K)$ are morphisms of dg-algebras.

Take the cohomology. The map $\mathrm{H}^*(e_2(K))$ is an isomorphism. Since $\mathrm{H}^*(\eta(K)) \circ \mathrm{H}^*(e_2(K))$ is the identity, we have $\mathrm{H}^*(\eta(K)) = (\mathrm{H}^*(e_2(K)))^{-1}$. We have

$$\mathrm{H}^*(\tau(K)) = \mathrm{H}^*(\eta(K)) \circ \mathrm{H}^*(e_1(K)) = (\mathrm{H}^*(e_2(K)))^{-1} \circ \mathrm{H}^*(e_1(K)),$$

hence the map $\mathrm{H}^*(\tau(K))$ is multiplicative.

Substitute $K = \mathrm{Sing}_{\,\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B)$ and together with Diagram 5 one concludes that the map

$$\mathrm{H}^*(\xi_B) : \mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \to \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B))$$

is multiplicative. □

We denote by ${}^+\mathrm{S}^\bullet_{\mathrm{Lip},\mathrm{spec}_{\mathbb{R}} B}$ the complex of sheaves of Lipschitz singular cochains on $\mathrm{spec}_{\mathbb{R}} B$. It can be equipped with the Alexander–Whitney cup product, which is non-commutative.

6.5. The algebra of continuous functions.

Theorem 11. *Let X be a topological space, $A \subset C(X)$ be a subalgebra of the algebra of real-valued continuous functions on X. Then the map*

$$\Psi_A : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$$

is multiplicative.

Proof. Take a finitely generated subalgebra $B \subset A$. One has a continuous function $\Gamma_B : X \to \mathrm{spec}_{\mathbb{R}} B$. There is a morphism of dg-algebras

$$\mathrm{sh} : \mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B) \to {}^+\mathrm{S}^\bullet_{\mathrm{Lip},\mathrm{spec}_{\mathbb{R}} B}(\mathrm{spec}_{\mathbb{R}} B)$$

natural in subalgebra B.

There is also a morphism of sheaf complexes with multiplication

$$\epsilon : \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B}[0] \to {}^+\mathrm{S}^\bullet_{\mathrm{Lip},\mathrm{spec}_{\mathbb{R}} B}$$

which is a quasi-isomorphism, see [1, Lemma 3].

Consider the following maps.

(1) The map

$$\mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \xrightarrow{\mathrm{H}^*(\xi_B)} \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B))$$

is multiplicative by Lemma 10.

(2) The map

$$\mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B)) \xrightarrow{\mathrm{H}^*(\mathrm{sh})} \mathrm{H}^*({}^+\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B))$$

is a clearly multiplicative.

(3) The map

$$\mathrm{H}^*({}^+\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B)) \xrightarrow{\Upsilon} \mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, {}^+\mathrm{S}^\bullet_{\mathrm{Lip},\mathrm{spec}_{\mathbb{R}} B})$$

is multiplicative by the property 3 of the internal cup product.

(4) The map

$$\mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, {}^{+}\mathrm{S}^{\bullet}_{\mathrm{Lip},\mathrm{spec}_{\mathbb{R}} B}) \xrightarrow{(\mathbb{H}^*(\epsilon))^{-1}} \mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B}[0])$$

is multiplicative by the property 1 of the internal cup product.

(5) The map

$$\mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B}[0]) \xrightarrow{\Gamma^*} \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$$

is multiplicative by the property 2 of the internal cup product.

Hence the composition of the maps (1)–(5)

$$\mathrm{H}^*(\Omega^{\bullet}_{B|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$$

is multiplicative.

The map Ψ_A is constructed by passing to the colimit of the above map over all finitely generated subalgebras $B \subset A$, hence is multiplicative. □

References

1. I Baskov, *The de Rham cohomology of soft function algebras.* — New York J. Math., **29** (2023) 1302–1340.
2. Y. Felix,, S. Halperin, J.-C. Thomas, *Rational homotopy theory.* Springer, New York (2001).
3. R. Godement, *Topologie algébrique et théorie des faisceaux.* Hermann, Paris (1958).
4. E. Kunz, *Kähler differentials.* Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig (1986).
5. S. Mac Lane, *Homology.* Springer, Cham (1963).
6. Stacks, Tag 013G. `https://stacks.math.columbia.edu/tag/013G.`
7. Cl. Voisin, *Hodge Theory and Complex Algebraic Geometry* I. Cambridge University Press (2002).
8. T. Wedhorn, *Manifolds, sheaves, and cohomology.* Springer Spektrum, Wiesbaden (2016).
9. H. Whitney, *Geometric integration theory.* Princeton University Press, Princeton, NJ (1957).
10. D. Quillen, *Homotopical algebra.* Lecture Notes in Mathematics, Springer, Cham (1967).

St. Petersburg Department
of Steklov Mathematical Institute
of Russian Academy of Sciences
E-mail: `baskovigor@pdmi.ras.ru`

Поступило 18 сентября 2025 г.