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AN EXACT CALCULATION OF ORIENTATION OF A FREELY ROTATING RIGID BODY

. We present an algorithmically implementable solution to the problem of determining the exact orientation of a freely rotating rigid body as an explicit function of time.

Introduction. Although the problem of determining the torque free motion of a rigid body was declared to be solved by Jacobi [14], his solution remained barely accessible to his followers who too often (instead) advertised the construction of Poinsot of the polehode, rolling without slipping on the herpolehode. But the Poinsot construction is neither a complete solution [20] nor is it a (partial) solution to this wily problem. Maxwell had alerted us [16] to the problem of "freely rotating body", being "so intricate" that many "have come to erroneous conclusions on this subject". Since then, obliviously to his admonition, more victims have fallen pray to the problem's multiple traps, even upon attempting to solve its simpler special versions, such as the Feynman's wobbling plate [12] (which erroneous solution led to a Nobel prize in physics [13]). Viktor Savinykh was so stunned (if not horrified) by the Dzhanibekov's discovery (on June 25, 1985) of the mysteriously sudden flipping of the wingnut (on board of the space station "Salute 7") that he remained speechless about it (ever since). Frequent spectacles which have pretended to merely further explain a "solved problem" are becoming waningly less convincing in our age, where an efficient algorithmic implementation has become a desired standard of solution. Unraveling the algebraic structure of the solution, as Galois dramatically taught us in his last letter, is the one and only key to its understanding, without which no efficient nor error free algorithms can be implemented. With this paper, we disclose the interwoven structure, which we presented in [4], of the solution of the problem of determining the exact orientation of a freely rotating rigid body as an explicit function

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of time, and we bring to attention the cunning "opposite-sign-interlaces" which must be carefully addressed to avoid errors, otherwise inevitable.

Recent research, conducted by E.A. Mityushov and further advanced by N. P. Kopytov [15], focused on comparing processed experimental data, gathered from IMU,[1] monitoring a torque free rigid body motion, with its computer modeling, has exposed the need for efficient algorithms for deducing the rigid body orientation from angular velocity measurements, as well as, algorithms for efficiently foreseeing and visualizing the rotation for a given angular momentum and energy. Via relying on exact (not merely approximate) solutions, such algorithms must depart from common, prone to cumulative errors, numerical methods towards algorithmic procedures, based on an understanding of the algebraic structure of torque free rigid body motion.

The rotation matrix as a function of the rotating coordinates of the angular velocity. Let $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ denote the unit vectors, directed along the principal axes of inertia of a rigid body, corresponding to the principal momenta A, B and C. The three coordinate functions p, q and r of the angular velocity $\boldsymbol{\omega} = p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$ of such freely rotating rigid body are then determined for a given (conserved) angular momentum $\boldsymbol{m}$ and (twice the) energy $h := \boldsymbol{\omega} \cdot \boldsymbol{m} = Ap^2 + Bq^2 + Cr^2$.

Each row of the (orthonormal) matrix

$$S := \begin{pmatrix} \alpha_0 \cos\psi - \alpha_1 \sin\psi & \alpha_0 \sin\psi + \alpha_1 \cos\psi & Ap/m \\ \beta_0 \cos\psi - \beta_1 \sin\psi & \beta_0 \sin\psi + \beta_1 \cos\psi & Bq/m \\ \gamma_0 \cos\psi - \gamma_1 \sin\psi & \gamma_0 \sin\psi + \gamma_1 \cos\psi & Cr/m \end{pmatrix},$$

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} := \frac{1}{\sqrt{\omega^2 - h^2/m^2}} \begin{pmatrix} (1 - Ah/m^2)\, p \\ (1 - Bh/m^2)\, q \\ (1 - Ch/m^2)\, r \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} := \frac{1}{\sqrt{m^2\omega^2 - h^2}} \begin{pmatrix} (B - C)\, qr \\ (C - A)\, rp \\ (A - B)\, pq \end{pmatrix},$$

$$\psi = \psi(t) := \frac{h\,t}{m} + \left(\frac{h}{m} - \frac{m}{A}\right)\left(\frac{h}{m} - \frac{m}{B}\right)\left(\frac{h}{m} - \frac{m}{C}\right)\int_0^t \frac{dt}{\omega^2 - h^2/m^2},$$ [2]

[1] IMU is the standard abbreviation for "Inertial Measurement Unit" which is an electronic device for triaxial measurement of angular speed and (linear) acceleration.

[2] We have used the non-boldface letters ω and m to denote the moduli of the angular velocity and the angular momentum, respectively, so $\omega^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega} = p^2 + q^2 + r^2$ and $m^2 = \boldsymbol{m} \cdot \boldsymbol{m} = A^2p^2 + B^2q^2 + C^2r^2$.

which we calculated in [4],[3] provides the three coordinates of a corresponding rotating unit vector with respect to an inertial (that is, a non-rotating)[4] frame which directing unit vectors shall be denoted by $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z} := \boldsymbol{m}/m$.[5]

Accordingly, each column of the matrix S provides the three coordinates of a corresponding non-rotating unit vector with respect to the rotating

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[3]The (three) differential identities

$$\begin{pmatrix} A\dot{p}/m \\ B\dot{q}/m \\ C\dot{r}/m \end{pmatrix} = \sqrt{\omega^2 - \frac{h^2}{m^2}} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}, \quad \begin{pmatrix} \dot{\alpha_0} \\ \dot{\beta_0} \\ \dot{\gamma_0} \end{pmatrix} = \left(\dot{\psi} - \frac{h}{m} \right) \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix},$$

$$\begin{pmatrix} \dot{\alpha_1} \\ \dot{\beta_1} \\ \dot{\gamma_1} \end{pmatrix} = \left(\frac{h}{m} - \dot{\psi} \right) \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} - \sqrt{\omega^2 - \frac{h^2}{m^2}} \begin{pmatrix} Ap/m \\ Bq/m \\ Cr/m \end{pmatrix},$$

where the dot denotes differentiation with respect to the (implicit) time variable (which we shall denote by t), together with the (three) "cross product" identities

$$\boldsymbol{\omega} \times \left(\frac{Ap}{m} \boldsymbol{i} + \frac{Bq}{m} \boldsymbol{j} + \frac{Cr}{m} \boldsymbol{k} \right) = -\sqrt{\omega^2 - \frac{h^2}{m^2}} \, (\alpha_1 \boldsymbol{i} + \beta_1 \boldsymbol{j} + \gamma_1 \boldsymbol{k}),$$

$$\boldsymbol{\omega} \times (\alpha_0 \boldsymbol{i} + \beta_0 \boldsymbol{j} + \gamma_0 \boldsymbol{k}) = \frac{h}{m} (\alpha_1 \boldsymbol{i} + \beta_1 \boldsymbol{j} + \gamma_1 \boldsymbol{k}),$$

$$\boldsymbol{\omega} \times (\alpha_1 \boldsymbol{i} + \beta_1 \boldsymbol{j} + \gamma_1 \boldsymbol{k}) = \sqrt{\omega^2 - \frac{h^2}{m^2}} \left(\frac{Ap}{m} \boldsymbol{i} + \frac{Bq}{m} \boldsymbol{j} + \frac{Cr}{m} \boldsymbol{k} \right) - \frac{h}{m} (\alpha_0 \boldsymbol{i} + \beta_0 \boldsymbol{j} + \gamma_0 \boldsymbol{k}),$$

imply that the differential matricial identity $\dot{S} = -WS$, $W := \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$ holds, as required. Alternatively, it might be expressed as the identity

$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i} = \left(\left(\sqrt{\omega^2 - \frac{h^2}{m^2}} \, \frac{Ap}{m} - \frac{h\alpha_0}{m} \right) \sin\psi - \frac{h\alpha_1}{m} \cos\psi \right) \boldsymbol{x}$$

$$- \left(\left(\sqrt{\omega^2 - \frac{h^2}{m^2}} \, \frac{Ap}{m} - \frac{h\alpha_0}{m} \right) \cos\psi + \frac{h\alpha_1}{m} \sin\psi \right) \boldsymbol{y} + \sqrt{\omega^2 - \frac{h^2}{m^2}} \, \alpha_1 \boldsymbol{z}$$

which is invariant under simultaneous cyclic permutations of the three ordered triples $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, $(p,\, q,\, r)$ and $(A,\, B,\, C)$, inducing cyclic permutations of the two ordered triples $(\alpha_0,\, \beta_0,\, \gamma_0)$ and $(\alpha_1,\, \beta_1,\, \gamma_1)$.

[4]The terms "rotating vector" and "rotating coordinates" are borrowed from my short article [8].

[5]We have tacitly excluded the case where $\boldsymbol{\omega}$ and $\boldsymbol{m}$ are colinear, that is, the case of the vanishing difference $\omega^2 - h^2/m^2$, including the critical motion of the Dzhanibekov top which was addressed in [4].

frame. Thus, the matrix S might formally be construed via the relation

$$\begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} = S \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}.$$

We might also readily express the angular velocity $\boldsymbol{\omega}$, with respect to the non-rotating frame, thereby tracing a *herpolehode* as

$$\boldsymbol{\omega} = \sqrt{\omega^2 - h^2/m^2}\,(\cos\psi\,\boldsymbol{x} + \sin\psi\,\boldsymbol{y}) + h/m\,\boldsymbol{z}.$$

Note that the condition $(Ah-m^2)(Bh-m^2)(Ch-m^2) = 0$ implies that ψ is linear (as a function of time) although it does not imply the constancy of the angular speed ω, whereas the condition $(A-B)(B-C)(C-A) = 0$ does imply the constancy of ω, along with the linearity of ψ.

Introduce the MacCullagh ellipsoid of inertia:

$$\frac{\zeta^2}{A} + \frac{\eta^2}{B} + \frac{\xi^2}{C} = \frac{h}{m^2},\ \zeta := \frac{Ap}{m},\ \eta := \frac{Bq}{m},\ \xi := \frac{Cr}{m}$$

and observe that the values Ah/m^2, Bh/m^2 and Ch/m^2 (which appear in the rows of the matrix S) are the squares of the (corresponding) lengths of its principal axes which are presumed to coincide with the principal axes of inertia. The curve traced by the tip of the (fixed) unit vector $\boldsymbol{z}$ on the (rotating) MacCullagh ellipsoid might be referred to as the *polehode*.[6]

The rotating coordinates of the angular velocity as explicit functions of time. Assuming that the principal momenta of inertia are strictly ascendingly ordered $A < B < C$ and the condition $(Ah - m^2)(Bh - m^2)(Ch - m^2) \neq 0$ is met, we might, guided by [4], calculate provisory coordinates ω_1, ω_2 and ω_3, of $\boldsymbol{\omega}$ as explicit functions of time:

$$\omega_1 = \omega_1(t) = f(t, A, B, C),\ \omega_2 = \omega_2(t) = f(t, B, C, A),$$
$$\omega_3 = \omega_3(t) = f(t, C, A, B),$$

[6]One ought not confuse the MacCullagh ellipsoid with the Poinsot ellipsoid, where the polehode for the latter ellipsoid traces the tip of the angular velocity (not the angular momentum) in the rotating frame. In order to emphasize the difference, the adjective "kinetic" is added, in [20], to the term "polehode" in our case where the tip is that of the angular momentum (not the angular velocity). We must also emphasize that the construction of MacCullagh is superior to the construction of Poinsot which was rightfully noted, in [20], to be "incomplete". And indeed, the construction of Poinsot is "blind" to the motion of the Dzhanibekov top, as discussed in [4].

$$f(t,A,B,C) := \sqrt{\frac{BC}{(A-B)(A-C)}}\; i\,\mathcal{S}\!\left(t, \sqrt{\frac{(A-B)(m^2-Ch)}{ABC}}, \sqrt{\frac{(A-C)(m^2-Bh)}{ABC}}\right),^{7}$$

$$i := \sqrt{-1},\;\; \mathcal{S}(t,\mu,\nu) := \sqrt{\mu\nu}\;\mathcal{S}\!\left(\sqrt{\mu\nu}\;\left(t+\frac{\pi}{2M(\mu,\nu)}\right), -\frac{\mu}{\nu}\right),$$

where $\mathcal{S}(\cdot,\cdot)$ is the Galois alternative elliptic function, as defined in [1, 6],[8] and $M(\cdot,\cdot)$ is the arithmetic-geometric mean (of its two variables).

Insofar, however, the calculated provisory coordinates ω_1, ω_2 and ω_3 are complex-valued for real-valued argument t. In fact, all three coordinate functions (above presented) share a simple pole at zero,[9] so we shall proceed by shifting their (shared) argument t by an imaginary value, thereby ensuring their simultaneous real-valuedness upon restricting their (shared) time domain to a "shifted" real line.[10]

In order to determine the desired shift of the time domain we calculate the three half-periods $T_1 := T(A,B,C)$, $T_2 := T(B,C,A)$ and

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[7]We shall consistently choose the branch of the square root which values lie in the first quadrant of the complex plane. No ambiguities will arise since the arguments of the double-valued square root function, throughout this paper, are restrcited to lie in the upper half-plane.

[8]Note that the quadrivariate function f is invariant under transposition of its last two arguments since the trivariate function $\mathcal{S}$ is, that is,

$$f(t,A,B,C) = f(t,A,C,B),\;\; \mathcal{S}(t,\mu,\nu) = \mathcal{S}(t,\nu,\mu).$$

The said invariance is a consequence of the identity for the alternative elliptic function:

$$\mathcal{S}(t,1/k) = \mathcal{S}(t,k),$$

which, in particular, is evident from the relation:

$$\mathcal{S}(t,k) = \sqrt{k}\;\mathrm{sn}(t/\sqrt{k},k),$$

where $\mathrm{sn}(\cdot,k)$ is the Jacobi elliptic sine function which elliptic modulus is k. Note, however, that while the function $\mathrm{sn}(\cdot,k)$ is invariant under flipping the sign of the elliptic modulus k, $\mathcal{S}(\cdot,k)$ is not. Instead, it satisfies the identity

$$\mathcal{S}(t,-k) = -i\,\mathcal{S}(it,k)$$

which corresponds to the identity $\mathcal{R}(t,-k) = -\mathcal{R}(it,k)$, satisfied by the Galois essential elliptic function $\mathcal{R}(\cdot,\cdot)$, via which the the alternative elliptic function $\mathcal{S}(\cdot,\cdot)$ might alternatively (no pun intended) be defined, as discussed in [2, 5, 6].

[9]The poles are inherited from the (simple) pole of the function $\mathcal{S}(t,\mu,\nu)$, viewed for fixed values of μ and ν, as a function of its first argument, with residue -1 at $t=0$.

[10]By a "shifted" real line we mean a line, in the complex plane, parallel to the real line.

$T_3 := T(C, A, B)$,[11] where

$$T(A,B,C) := \frac{\pi\sqrt{ABC}}{2M\left(\sqrt{(A-B)(m^2-Ch)}, \sqrt{(A-C)(m^2-Bh)}\right)},$$

and observe that

$$\omega_1(T_1) = \omega_2(T_2) = \omega_3(T_3) = 0,$$

$$\omega_2(T_1)^2 = \frac{m^2-Ch}{B(B-C)},\ \omega_3(T_2)^2 = \frac{m^2-Ah}{C(C-A)},\ \omega_1(T_3)^2 = \frac{m^2-Bh}{A(A-B)},$$

$$\omega_3(T_1)^2 = \frac{m^2-Bh}{C(C-B)},\ \omega_1(T_2)^2 = \frac{m^2-Ch}{A(A-C)},\ \omega_2(T_3)^2 = \frac{m^2-Ah}{B(B-A)}.$$

So, having assumed B to be the intermediate principal moment of inertia, the square ω_2^2 is guaranteed to be nonnegative at all three argument values T_1, T_2 and T_3, whereas the squares ω_3^2 and ω_1^2 are guaranteed to be simultaneously nonnegative (only) at T_2. With the principal momenta (strictly) ascendingly ordered, the square ω_3^2 (ω_1^2) is additionaly guaranteed to be nonnegative at T_1 (T_3) if and only if $Bh \leq (\geq)\ m^2$. With the latter condition satisfied, the sign of coordinate function ω_3 (ω_1) is preserved and, if the strengthened condition $Bh < (>)\ m^2$ is satisfied, its half-period T_3 (T_1) is positive and finite.[12] Whatever the case, the values $\omega_1(t \pm T_2)$, $\omega_2(t \pm T_2)$ and $\omega_3(t \pm T_2)$ are guaranteed to be real for real-valued argument t.[13] From now on, without loss of generality, we might assume that $p(t) = \omega_1(t + T_2)$, $q(t) = \omega_2(t + T_2)$ and $r(t) = \omega_3(t + T_2)$.

The time-dependent projections of the angular momentum on the directed Galois axes. The Euler equations, satisfied by the functions p, q and r, are equivalent to the three identities:

$$\dot{\mathcal{S}}_1(t) = \mathcal{S}_2(t)\,\mathcal{S}_3(t),\ \dot{\mathcal{S}}_2(t) = \mathcal{S}_3(t)\,\mathcal{S}_1(t),\ \dot{\mathcal{S}}_3(t) = \mathcal{S}_1(t)\,\mathcal{S}_2(t), \qquad (1)$$

where we have denoted, for brevity, by $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ the three elliptic functions which were explicitly presented as being proportional to ω_1, ω_2

[11] $T(A, B, C)$ is a half-period (only) for $f(\cdot, A, B, C)$. It is a quarter-period for both $f(\cdot, B, C, A)$ and $f(\cdot, C, A, B)$.

[12] The duality of the Dzhanibekov top, corresponding to the critical case of motion $Bh = m^2$, was discussed in [3, 4].

[13] Whether we add or subtract T_2 is further determined by the chosen orientation of the coordinate system (which we imposed before determining the signs of p, q and r).

and ω_3, respectively. All three identities follow from a single identity which is the first of the following pair of identities:

$$\begin{aligned}\dot{\mathcal{S}}(t,\mu,\nu) &= \mathcal{S}(t,\sqrt{\mu^2-\nu^2},i\,\nu)\,\mathcal{S}(t,\sqrt{\nu^2-\mu^2},i\,\mu)\\ &= \mathcal{S}(t,\mu,\nu)\,\mathcal{S}(t,i(\mu+\nu),i(\mu-\nu)),^{14}\end{aligned} \tag{2}$$

The equivalence of identities (1) to Euler equations is based on the following two equalities

$$(A-B)(m^2-Ch)+(B-C)(m^2-Ah)+(C-A)(m^2-Bh)=0,$$

$$A\sqrt{\frac{BC}{(A-B)(A-C)}}\;i=(C-B)\sqrt{\frac{CA}{(B-C)(B-A)}}\sqrt{\frac{AB}{(C-A)(C-B)}}$$

with the latter equality still preserved under cyclic permutations of the strictly ascendingly ordered principal momenta of inertia.[15]

The second of the pair of identities (2) might be rewritten as

$$\dot{\mathcal{S}}(t,\mu,\nu)=\mathcal{S}(t,\mu,\nu)\,\mathcal{T}(t,\mu,\nu),\;\;\mathcal{T}(t,\mu,\nu):=\mathcal{S}(t,i(\mu-\nu),i(\mu+\nu)),$$

alluding us, in particular, to introduce a function $\mathcal{T}_2$ as the logarithmic derivative of $\mathcal{S}_2$, that is,

$$\mathcal{T}_2(t):=\frac{\dot{\mathcal{S}}_2(t)}{\mathcal{S}_2(t)}$$

$$=\mathcal{S}\left(t,\frac{\sqrt{(C-B)(m^2-Ah)}-\sqrt{(A-B)(m^2-Ch)}}{\sqrt{ABC}},\frac{\sqrt{(C-B)(m^2-Ah)}+\sqrt{(A-B)(m^2-Ch)}}{\sqrt{ABC}}\right)$$

Thus and since $\mathcal{T}_2$ shares with $\mathcal{S}_2$ the same simple pole with the same residue (-1) at zero, $\mathcal{S}_2$ must, as discussed in [6], be the derivative of the function $t\mapsto\ln(\mathcal{T}_2(t/2))$, that is,

$$\mathcal{S}_2(t)=\frac{\dot{\mathcal{T}}_2(t/2)}{2\mathcal{T}_2(t/2)}.$$

On the other hand, $\mathcal{S}_2$ is the logarithmic derivative of (the sum and the difference) $\mathcal{S}_3\pm\mathcal{S}_1$, and the residue of the the sum $\mathcal{S}_3+\mathcal{S}_1$ at zero (-2)

[15]The equality, however, is not preserved if C and A are transposed (so would be the ordering of the principal momenta reversed). It is the reason for our assumption that a strict ordering (which cannot be arbitrarily reversed) was imposed upon the principal momenta of inertia. Reversing their order requires taking (instead) the branch of the square root function in the fourth quadrant (the complex conjugate of the first) after restricting its domain to the lower half-plane. In particular, we would have to reset $\sqrt{-1}$ to $-i$.

matches the residue of the function $t \mapsto \mathcal{T}_2(t/2)$. In fact, four functions disguise one and the same function, up to a shift of its argument:

$$\mathcal{T}_2(t/2) = \mathcal{S}_3(t) + \mathcal{S}_1(t),\ \mathcal{T}_2(t/2 + T_2) = -\mathcal{S}_3(t) - \mathcal{S}_1(t),$$
$$\mathcal{T}_2(t/2 + T_3) = \mathcal{S}_3(t) - \mathcal{S}_1(t),\ \mathcal{T}_2(t/2 + T_1) = -\mathcal{S}_3(t) + \mathcal{S}_1(t),$$

and if we go on to introduce the function

$$g(t, B, C, A) := \frac{CA\,i}{C-A}\mathcal{T}_2\left(\frac{t+T_2}{2}\right) = \frac{CA\,i}{C-A}\left(\mathcal{S}_3(t+T_2) + \mathcal{S}_1(t+T_2)\right)$$
$$= g_3\,C\,r(t) + g_1\,A\,p(t),\ g_3 := \sqrt{\frac{A(B-C)}{B(A-C)}},\ g_1 := \sqrt{\frac{C(A-B)}{B(A-C)}},$$

then it turns out to coincide, up to a shift of its argument, with the four(!) projections of the angular momentum on the (directed) Galois axes, as announced at the PCA conference on April 17, 2024 [9]:

$$g(t, B, C, A) = (g_3\,\boldsymbol{k} + g_1\,\boldsymbol{i}) \cdot \boldsymbol{m},\ g(t + 2\,T_2, B, C, A) = -(g_3\,\boldsymbol{k} + g_1\,\boldsymbol{i}) \cdot \boldsymbol{m},$$
$$g(t+2\,T_3, B, C, A) = (g_3\,\boldsymbol{k} - g_1\,\boldsymbol{i}) \cdot \boldsymbol{m},\ g(t+2\,T_1, B, C, A) = (-g_3\,\boldsymbol{k} + g_1\,\boldsymbol{i}) \cdot \boldsymbol{m}.^{16}$$

Conclusion. Two cases of rigid body motion, separated by critical (dual) motion (for which $m^2 = Bh$) must be distinguished, along with determining whether to add the imaginary time shift T_2 to the arguments of the provisory coordinate functions or subtract it, in accordance with the imposed orientation of the coordinate system. These two interrelated algorithmic steps are crucial for ensuring a correct implementation of the theoretical solution into practice. With this correct implementation, the critical motion is then visualized as (genuinely) dual motion, where two distinct choices of imposing the orientation would correspond to two (not one!) distinct solutions of critical motion, corresponding to one and the same angular momentum pseudovector, as was demonstrated in [17]. The computation of the precession angle ψ as an incomplete elliptic integral of the third type must be preceded by evaluating a complete elliptic integral of the third type, including the cases of its degeneration to complete elliptic integrals of second and first types, as discussed in detail in [7].

[16] The three non-trivial shifts $2\,T_2$, $2\,T_3$ and $2\,T_1$ of the argument of the function $g(\cdot, B, C, A)$ are, of course, its three half-periods. They correspond to the three non-trivial elements of the Klein four-group. An non-trivial action of this group is, in particular, demonstrated by the constancy of the product $g(t, B, C, A)\,g(t + 2\,T_1, B, C, A)) = (h - m^2/B)\,CA/(C - A)$, as pointed out in [10, 11].

A dedication and an acknowledgment. The author would like to dedicate this article to Anatoly Vershik (December 28, 1933 – February 14, 2024), who will be remembered for his exceptionally noble soul.

A first algorithmic implementation of the exact calculation of orientation of a freely rotating rigid body, which is presented in this article, was carried out by Egor Moschjonok in [18, 19].

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