Записки научных
семинаров ПОМИ
Том 539, 2024 г.

S. Repin, T. Samrowski

ESTIMATES OF ERRORS GENERATED BY UNCERTAIN DATA IN A COUPLED PIESO-ELECTRIC PROBLEM

Abstract. The paper is concerned with a coupled piezo-electric problem with incompletely known coefficients of the elasticity tensor and two other tensors that define electric properties of the media. Due to this uncertainty, the problem possesses a set (cloud) of equally probable solutions instead of the unique solution. Quantitative characteristics of this set are derived by a posteriori estimates of the functional type. They give an upper bound of the cloud diameter and lower bound of maximal diameter of the ball inscribed. The estimates are fully computable. They are based on solving algebraic optimisation problems of low dimensionality related to the sets containing possible coefficients. In the case of isotropic elasticity with the Poisson's ratio close to 0.5, it is shown that even tiny values of uncertainty in the coefficient may generate very large errors in the solution.

§1. Introduction

Mathematical models of real life problems are never known exactly. There always exists a certain lack of knowledge on geometry, physical constants, and other parameters. Depending on a particular problem, this fact may or may not be ignored. A special branch of error analysis called *uncertainty quantification* studies effects caused by incomplete knowledge on the problem data (e.g., see [1, 23]).

In this paper, we consider boundary-value problems for a coupled system of partial differential equations whose coefficients contain indeterminacy. Our approach to analysis of uncertainty errors is based on using the machinery of functional type a posteriori estimates. It was suggested in [16], where these mathematical tools were used to deduce guaranteed and fully computable bounds of uncertainty errors for a boundary value problem of elliptic type. The case of elliptic type system was investigated in [10]. Here we study a more complicated pieso-electric problem. It belongs to

Key words and phrases: errors generated by uncertain data, a posteriori error estimates of the functional type, coupled pieso-electric problem.

the class of so-called multiphysics problems that involve several equations of different types associated with different processes or phenomena. Piezoelectricity is an electromechanical energy transduction mechanism so that the corresponding model combines elasticity equations with the equations describing density of electric charge.

The first (linear) mathematical model of this kind was derived by W. Voigt [27]. Various advanced models of an elastic medium with polarization we presented later in [12, 13, 25], and [26]. Effects evoked by thermal and magnetic fields are considered in [5, 6] and [11]. In [2], the authors investigated a linear model, excluding hysteresis effects, to describe the interaction between the elastic and electrical fields within a three-dimensional piezoelectric matrix containing metallic inclusions.

Typical approaches to solving multiphysics problems numerically involve sequentially solving the selected equation using previously computed numerical solutions of other equations. Hence the errors affect each other and enhances the overall error. This principal difficulty concerns numerical errors as well as those caused by data uncertainty. To overcome it we need fully computable and guaranteed a posteriori error estimates. For some coupled problems in continuum mechanics these estimates have been derived in [8,16,18,20,21]. Their derivation is based on the theory of a posteriori error estimates of the functional type introduced in [14–16] and other publications cited therein. Unlike a posteriori estimates of other types, estimates of they do not rely on specific information about the approximate solution and do not contain mesh-dependent constants. Thus, they provide computable measures of accuracy for a broad range of problems (see [9] for practical applications to finite element and other numerical methods and [19] for analysis of modeling errors).

In this paper, we use them to study effects caused by incomplete knowledge of the coefficients in the coupled mathematical model of pieso-electric crystal. Classical and generalised formulations of the mathematical model are discussed in Section 2, where we also introduce notation and functional spaces. A posteriori estimates of the functional type are presented in Section 3 and two–sided error bounds of errors generated by uncertain data are deduced in Section 4 (estimates (4.22), (4.31), and and (4.34)). In addition, we discuss a particular case, where the elastic part is defined by the isotropic relation with the Poisson's coefficient close to 0.5. It is shown that here we have effects similar to those earlier noticed in [10]. If the elasticity equation is supplied with purely Dirishlét boundary conditions, then the ratio of the solution cloud diameter to maximal variation of the coefficient may grow catastrophically when approaching to 0.5. In this case, even tiny uncertainty in this coefficient may generate very large uncertainty in the solution so that any quantitative analysis of the problem may become senseless. To avoid this situation the Dirichlét boundary conditions must be compatible with a divergence free field.

§2. Mathematical model

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d \in \{2,3\}$) with Lipschitz continuous boundary Γ. The model of a pieso-electric body operates with the vector valued function $\mathbf{u} : \Omega \to \mathbb{R}^d$ (elastic displacement) and scalar field $\varphi : \Omega \to \mathbb{R}$ (electric potential). Within the framework of linear elasticity model, the strain tensor $\boldsymbol{\varepsilon}$ is defined as the symmetric part of the displacement gradient, i.e., $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$. The system of equations

$$\operatorname{Div} \boldsymbol{\sigma}(\mathbf{u}, \varphi) + \mathbf{f} = 0, \tag{2.1}$$

$$\operatorname{div} \mathbf{p}(\mathbf{u}, \varphi) + g = 0 \tag{2.2}$$

describes deformation of the body occupying Ω. It contains the body force vector $\mathbf{f}$ and the scalar field of the electric charge density g. Here and later on Div and div denote the divergence operators for the tensor and vector valued functions, respectively, i.e., $\operatorname{Div} \boldsymbol{\tau} = \nabla \cdot \boldsymbol{\tau} = \tau_{ij,j}$ and $\operatorname{div} \mathbf{q} = \nabla \cdot \mathbf{q} = q_{i,i}$. In these relations, and later on the Einstein summation convention of summation over the repeated indices is adopted.

The stress tensor $\boldsymbol{\sigma}$ and the dielectric flux $\mathbf{p}$ are coupled via the linear piezoelectric material law:

$$\boldsymbol{\sigma}(\mathbf{u}, \varphi) = \mathbb{L}\boldsymbol{\varepsilon}(\mathbf{u}) \; + \; \mathbb{B} \cdot \nabla\varphi, \tag{2.3}$$

$$\mathbf{p}(\mathbf{u}, \varphi) = \mathbb{K} \cdot \nabla\varphi \; - \; \mathbb{B}^T : \boldsymbol{\varepsilon}(\mathbf{u}). \tag{2.4}$$

In (2.3), $\mathbb{L} = \{L_{ijkl}\}$ is the forth-order tensor of elastic moduli, which satisfies the condition

$$\gamma_1^2(\mathbb{L})\,|\boldsymbol{\varepsilon}|^2 \leqslant \mathbb{L}\,\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \leqslant \gamma_2^2(\mathbb{L})\,|\boldsymbol{\varepsilon}|^2, \quad \forall \boldsymbol{\varepsilon} \in \mathbb{M}^{d\times d}_{\mathrm{sym}}, \tag{2.5}$$

where $\mathbb{M}^{d\times d}_{\mathrm{sym}}$ is the space of symmetric real valued $d \times d$ tensors. We assume that $L_{ijkl} \in L_\infty(\Omega)$ and possess natural symmetry properties:

$$\mathbb{L}_{ijkm} = \mathbb{L}_{jikm} = \mathbb{L}_{kmij} \in L^\infty(\Omega), \quad i,j,k,m = 1,\dots,d. \tag{2.6}$$

Constitutive relations (2.3)–(2.4) also contain the (third-order) piezoelectric tensor

$$\mathbb{B} = \{B_{ijs}\},\ B_{ijs} \in L^\infty(\Omega)$$

and the (second-order) dielectric material tensor $\mathbb{K} = \{K_{ij}\} \in \mathbb{M}^{d\times d}_{\rm sym}$. It satisfies the conditions

$$K_{ij} = K_{ji} \in L^\infty(\Omega), \quad \gamma_1^2(\mathbb{K})\,|\boldsymbol{\zeta}|^2 \leqslant \mathbb{K}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} \leqslant \gamma_2^2(\mathbb{K})\,|\boldsymbol{\zeta}|^2, \quad \forall \boldsymbol{\zeta} \in \mathbb{R}^d. \tag{2.7}$$

The system (2.1)–(2.4) is supplied with the boundary conditions

$$\mathbf{u} = \mathbf{u}_0 \text{ on } \Gamma^1_D, \quad \varphi = \varphi_0 \text{ on } \Gamma^2_D \tag{2.8}$$

stated for the case where the Dirichlet boundary conditions $\mathbf{u}_0$ for the elastic component of the solution is defined on Γ^1_D and $\varphi = \varphi_0$ for the electric component is defined on Γ^2_D. It is assumed that these parts of Γ have positive surface measures. We note that, in general, Γ^1_D and Γ^2_D are two different parts of the boundary Γ. On the remaining parts $\Gamma^1_N := \Gamma\backslash\Gamma^1_D$ and $\Gamma^2_N := \Gamma \setminus \Gamma^2_D$ we impose the homogeneous Neumann conditions

$$\boldsymbol{\sigma}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma^1_N, \quad \mathbf{p}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma^2_N, \tag{2.9}$$

where $\mathbf{n}$ denotes the unit outward normal to Γ. We introduce Hilbert spaces

$$H(\Omega,\, \mathrm{Div}) := \{\boldsymbol{\tau} \in \mathrm{L}^2(\Omega,\, \mathrm{M}^{\mathrm{d}\times\mathrm{d}}_{\mathrm{s}}),\ \mathrm{Div}\,\boldsymbol{\tau} \in \mathrm{L}^2(\Omega,\, \mathbb{R}^{\mathrm{d}})\}$$

and

$$H(\Omega,\, \mathrm{div}) := \{\mathbf{q} \in \mathrm{L}^2(\Omega,\, \mathbb{R}^{\mathrm{d}}),\ \mathrm{div}\,\mathbf{q} \in \mathrm{L}^2(\Omega)\}$$

supplied with the norms

$$\|\mathbf{u}\|_{\mathrm{div}} := \left(\int\limits_\Omega (|u|^2 + |\mathrm{div} u|^2)\right)^{1/2} \text{ and } \|\boldsymbol{\sigma}\|_{\mathrm{Div}} := \left(\int\limits_\Omega (|\boldsymbol{\sigma}|^2 + |\mathrm{Div}\boldsymbol{\sigma}|^2)\right)^{1/2},$$

respectively.

$H^1(\Omega)$ and $H^1(\Omega, \mathbb{R}^d)$ denote the Sobolev spaces of scalar and vector valued functions defined in Ω which are square summable with the first derivatives and

$$V_0 := \left\{\mathbf{v} \in V := H^1(\Omega;\mathbb{R}^3) \,\middle|\, \mathbf{v}|_{\Gamma^1_D} = 0\right\},$$
$$M_0 := \left\{\psi \in M := H^1(\Omega) \,\middle|\, \psi|_{\Gamma^2_D} = 0\right\}.$$

Then

$$V_0 + \mathbf{u_0} := \{\mathbf{v} \in V \ \mid\ \mathbf{v} = \mathbf{w} + \mathbf{u_0},\ \mathbf{w} \in V_0\},$$

and

$$M_0 + \varphi_0 := \{\psi \in M \ | \ \psi = \eta + \varphi_0, \ \eta \in M_0\}$$

are the sets of functions satisfying the prescribed boundary conditions.

Let $\mathbf{f} \in L_2(\Omega, \mathbb{R}^d)$, $g \in L_2(\Omega)$,

$$F(\mathbf{w}) := \int\limits_\Omega \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x}, \qquad \text{and} \qquad G(\eta) \ := \int\limits_\Omega g \, \eta \, \mathrm{d}\mathbf{x}.$$

The generalized solution of the problem (2.1)-(2.4) is defined as the pair of functions $\mathbf{u} \in V_0 + \mathbf{u_0}$ and $\varphi \in M_0 + \varphi_0$ satisfying the system of integral identities

$$\int\limits_\Omega \boldsymbol{\sigma}(\mathbf{u}, \mathbf{w}) : \varepsilon(\mathbf{w}) \, dx = F\,(\mathbf{w}), \quad \forall \mathbf{w} \in V_0, \tag{2.10}$$

$$\int\limits_\Omega \mathbf{p}(\eta, \mathbf{u}) \cdot \nabla \eta \, dx = G\,(\eta), \quad \forall \eta \in M_0, \tag{2.11}$$

where $\boldsymbol{\sigma}$ and $\mathbf{p}$ are defined by the relations (2.3) and (2.4). Well–posdness of the problem can be established by known methods (e.g., see [24]).

For our purposes, it is convenient to rewrite (2.10)-(2.11) as a saddle point problem:
find $(\mathbf{u}, \varphi) \in (V_0 + \mathbf{u}_0) \times (M_0 + \varphi_0)$ such that

$$L(\mathbf{u}, \, \psi) \leqslant L(\mathbf{u}, \, \varphi) \leqslant L(\mathbf{v}, \varphi) \qquad \forall \psi \in M_0 + \varphi_0, \ \forall \mathbf{v} \in V_0 + \mathbf{u}_0, \tag{2.12}$$

where the Lagrangian is defined by the relation

$$L(\mathbf{v}, \psi) := \int\limits_\Omega \left(\frac{1}{2} \mathbb{L} \boldsymbol{\varepsilon}(\mathbf{v}) \, : \, \boldsymbol{\varepsilon}(\mathbf{v}) + \nabla \psi \cdot \mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{v}) - \frac{1}{2} \, \mathbb{K} \, \nabla \psi \cdot \nabla \psi \right) dx + G(\psi) - F(\mathbf{v}).$$

It is easy to see that:

$$\begin{array}{rcl} L(\mathbf{v}, \cdot) & : & V_0 + \mathbf{u}_0 \to \mathbb{R} - \text{convex and continuous}, \\ L(\cdot, \psi) & : & M_0 + \varphi_0 \to \mathbb{R} - \text{concave and continuous}, \\ L(\mathbf{v}, 0) & \to & +\infty \ \text{ for } \|\mathbf{v}\|_V \to +\infty, \\ L(0, \psi) & \to & -\infty \ \text{ for } \|\psi\|_M \to +\infty. \end{array}$$

Hence existence of $(\mathbf{u}, \varphi)$ follows from known results in convex analysis (e.g., see [4]).

Consider the right hand side of the saddle point relation:

$$L(\mathbf{u},\varphi) \leqslant L(\mathbf{v},\varphi) \qquad \forall \mathbf{v} \in V_0 + \mathbf{u}_0.$$

We represent it in the variational form

$$J_\varphi(\mathbf{u}) \leqslant J_\varphi(\mathbf{v}) \qquad \forall \mathbf{v} \in V_0 + \mathbf{u}_0,$$

where

$$J_\varphi(\mathbf{v}) := \int\limits_\Omega \left(\frac{1}{2}\,\mathbb{L}\,\boldsymbol{\varepsilon}(\mathbf{v}) \,:\, \boldsymbol{\varepsilon}(\mathbf{v}) + \nabla\varphi \cdot \mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{v}) \right) dx - F(\mathbf{v}).$$

The minimizer $\mathbf{u}$ satisfies the relation

$$\int\limits_\Omega \left(\mathbb{L}\,\boldsymbol{\varepsilon}(\mathbf{u}) \,:\, \boldsymbol{\varepsilon}(\mathbf{w}) + \nabla\varphi \cdot \mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{w}) \right) dx - F(\mathbf{w}) = 0,\ \forall \mathbf{w} \in V_0. \tag{2.13}$$

which coincides with (2.10).

The left hand side of (2.12) reads

$$I_\mathbf{u}(\varphi) \leqslant I_\mathbf{u}(\psi) \qquad \forall \psi \in M_0 + u_0,$$

where

$$I_\mathbf{u}(\psi) := \int\limits_\Omega \left(\frac{1}{2}\,\mathbb{K}\,\nabla\psi \cdot \nabla\psi - \nabla\psi \cdot \mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{u}) \right) dx - G(\psi).$$

Here, the minimizer φ satisfies the integral relation

$$\int\limits_\Omega \left(\mathbb{K}\,\nabla\varphi \cdot \nabla\eta - \nabla\eta \cdot \mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{u}) \right) dx - G(\eta) = 0 \qquad \forall \eta \in M_0, \tag{2.14}$$

which coincides with (2.11).

§3. Two–sided error bounds

Let $\mathbf{v} \in V_0 + \mathbf{u}_0$ and $\psi \in M_0 + \varphi_0$ be viewed as approximations of $\mathbf{u}$ and φ, respectively. Then $\mathbf{u} - \mathbf{v}$ and $\phi - \psi$ are the corresponding errors. Computable bounds of these errors follow from (2.13) and (2.14). Notice that the norm

$$|[\mathbf{u},\varphi]|^2 := \|\boldsymbol{\varepsilon}(\mathbf{u})\|^2_{\mathbb{L}} + \|\nabla\varphi\|^2_{\mathbb{K}} \tag{3.1}$$

is the natural energy norm associated with the problem. It is convenient to use it as a measure of errors.

Let us set $\mathbf{w} = \mathbf{u} - \mathbf{v}$ in (2.13) and $\eta = \varphi - \psi$ in (2.14). Then, these relations can be represented in the following equivalent forms

$$\int\limits_\Omega (\mathbb{L}\,\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}) + \nabla(\varphi-\psi)\cdot\mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}))\,dx - F(\mathbf{u}-\mathbf{v})$$
$$+ \int\limits_\Omega (\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}) + \nabla\psi\cdot\mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}))\;dx = 0 \quad (3.2)$$

and

$$\int\limits_\Omega (\mathbb{K}\nabla(\varphi-\psi)\cdot\nabla(\varphi-\psi) - \nabla(\varphi-\psi)\cdot\mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}))\;dx - G(\varphi-\psi)$$
$$+ \int\limits_\Omega (\mathbb{K}\,\nabla\psi\cdot\nabla(\varphi-\psi) - \nabla(\varphi-\psi)\cdot\mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{v}))\,dx = 0. \quad (3.3)$$

By adding (3.2) and (3.3) we obtain

$$|[\mathbf{u}-\mathbf{v},\varphi-\psi|]^2 \;=\; \int\limits_\Omega (\,\mathbb{L}\,\boldsymbol{\varepsilon}(\mathbf{v}) + \nabla\psi\cdot\mathbb{B}) : \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v})\,dx$$
$$+ \int\limits_\Omega (\mathbb{K}\,\nabla\psi\cdot\nabla(\varphi-\psi) - \nabla(\varphi-\psi)\cdot\mathbb{B} : \boldsymbol{\varepsilon}(\mathbf{v}))\;dx$$
$$- \int\limits_\Omega (\mathbf{f}\cdot(\mathbf{u}-\mathbf{v}) + g(\varphi-\psi))\,dx. \quad (3.4)$$

To rearrange the right hand side of (3.4), we introduce two functions $\boldsymbol{\tau} \in H^+(\Omega, \mathrm{Div})$ and $q \in H^+(\Omega, \mathrm{div})$. For them we have integral identities

$$\int\limits_\Omega (\boldsymbol{\tau} : \boldsymbol{\varepsilon}(\mathbf{w}) + \mathrm{Div}\,\boldsymbol{\tau}\cdot\mathrm{w})\,\mathrm{dx} = 0 \qquad \forall w \in V_0, \quad (3.5)$$
$$\int\limits_\Omega (\mathbf{q}\cdot\psi + \mathrm{div}\;\mathbf{q}\;\psi)\mathrm{dx} = 0 \qquad \forall\psi \in M_0. \quad (3.6)$$

By (3.5) and (3.6) we represent (3.4) in the form

$$|[\mathbf{u}-\mathbf{v},\varphi-\psi|]^2 \;=\; \int\limits_\Omega (\,\mathbb{L}\,\boldsymbol{\varepsilon}(\mathbf{v})\,+\nabla\psi\cdot\mathbb{B}-\boldsymbol{\tau}):\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v})\,dx$$
$$+\int\limits_\Omega \nabla(\varphi-\psi)\cdot(\mathbb{K}\,\nabla\psi-\mathbb{B}:\boldsymbol{\varepsilon}(\mathbf{v})-\mathbf{q})\;dx$$
$$-\int\limits_\Omega(\mathrm{Div}\,\boldsymbol{\tau}+\mathbf{f})\cdot(\mathbf{u}-\mathbf{v})\,dx-\int\limits_\Omega(\mathrm{div}\,\mathbf{q}+g)\,(\varphi-\psi)\,dx. \quad (3.7)$$

This identity implies guaranteed error bounds. Consider the quantities

$$\mathcal{M}_1(\mathbf{v},\psi,\boldsymbol{\tau}) := \|\boldsymbol{\tau}-\mathbb{L}\,\varepsilon(\mathbf{v})-\mathbb{B}\cdot\nabla\psi\|_{\mathbb{L}^{-1}}+\mu_F(\mathbb{L},\Omega,\Gamma_D^1)\|\mathbf{f}+\mathrm{Div}\boldsymbol{\tau}\| \quad (3.8)$$

and

$$\mathcal{M}_2(\mathbf{v},\psi,\mathbf{q}) := \|\mathbf{q}-\mathbb{K}\nabla\psi+\mathbb{B}^T:\,\varepsilon(\mathbf{v})\|_{\mathbb{K}^{-1}}+\mu_F(K,\Omega,\Gamma_D^2)\|g+\mathrm{div}\mathbf{q}\|, \quad (3.9)$$

where μ_F are constants in the Friedrichs type inequalities

$$\|\mathbf{w}\| \leqslant \mu_F(\mathbb{L},\Omega,\Gamma_D^1)\,\|\varepsilon(\mathbf{w})\|_{\mathbb{L}} \quad \forall\mathbf{w}\in V_0,$$

and

$$\|\eta\| \leqslant \mu_F(\mathbb{K},\Omega,\Gamma_D^2)\,\|\nabla\eta\|_{\mathbb{K}} \quad \forall\varphi\in M_0.$$

Here and later on $\|\cdot\|$ stands for the L^2 norm of a vector or scalar valued function. The quantities $\mathcal{M}_1$ and $\mathcal{M}_2$ contain only known functions (approximations $\mathbf{v}$ and ψ, and the functions $\boldsymbol{\tau}\in H^+(\Omega,\mathrm{Div})$ and $q\in H^+(\Omega,\mathrm{div})$ that can be viewed as approximations of the elastic stress and of the dielectric flux, respectively).
Theorem below shows that these two quantities majorate the error norm.

Theorem 3.1. i) *For any $\mathbf{v}\in V_0+\mathbf{u_0}$ and $\psi\in M_0$ combined error norm is bounded from above by the estimate*

$$|[\mathbf{u}-\mathbf{v},\,\varphi-\psi|]^2 \leqslant \mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau})+\mathcal{M}_2^2(\mathbf{v},\psi,\mathbf{q}), \quad (3.10)$$

where $\boldsymbol{\tau}$ and q are arbitrary functions in the spaces $H^+(\Omega,\mathrm{Div})$ and $H^+(\Omega,\mathrm{div})$, respectively.

ii) *The right-hand side of (3.10) vanishes if and only if*

$$\mathbf{v}=\mathbf{u},\ \psi=\varphi,\ \boldsymbol{\tau}=\mathbb{L}\,\varepsilon(\mathbf{u})+\mathbb{B}\cdot\nabla\varphi,\ \textit{and}\ \mathbf{q}=\mathbb{K}\,\nabla\varphi+\mathbb{B}^T:\varepsilon(\mathbf{u}).$$

Proof of Theorem 3.1 follows from the identity (3.7). The reader find it and comments on applications to error estimation for finite element approximations in [8].

Remark 3.2. It is easy to see that

$$\mu_F(\mathbb{L},\Omega,\Gamma^1_D)\leqslant c_2(\mathbb{L})C_K\mu_F(\Omega,\Gamma^1_D),\quad \mu_F(\mathbb{K},\Omega,\Gamma^2_D)\leqslant\gamma_2(\mathbb{K})\,\mu_F(\Omega,\Gamma^2_D),$$

where C_K is the constant in the Korn's inequality and $\mu_F(\Omega,\Gamma^1_D)$ and $\mu_F(\Omega,\Gamma^2_D)$ are constants in the inequalities

$$\|\mathbf{w}\|\leqslant\mu_F(\Omega,\Gamma^1_D)\|\nabla\mathbf{w}\|,\ \ \forall\mathbf{w}\in V_0,\quad \|\eta\|\leqslant\mu_F(\Omega,\Gamma^2_D)\|\nabla\eta\|,\ \ \forall\varphi\in M_0.$$

In general, finding majorants of $\mu_F(\Omega,\Gamma^1_D)$ and $\mu_F(\Omega,\Gamma^2_D)$. may be a difficult task. However, there are known methods that can be used to compute majorants of such constants (e.g., see [3], [7], [17], [22]).

Above presented error majorants are helpful for estimation the accuracy of numerical approximation. In what follows, we are focused on a different task: *evaluation of modeling errors generated by uncertain data*. For this purpose, we also need error minorants.

By 2.13), we find that

$$\begin{aligned}
&J_\varphi(\mathbf{v})-J_\varphi(\mathbf{u})\\
&=\int\limits_\Omega\left(\frac12\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v})-\frac12\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})+\nabla\varphi\cdot\mathbb{B}:\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\right)dx-F(\mathbf{v}-\mathbf{u})\\
&=\int\limits_\Omega\left(\frac12\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}):\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})+\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{u}):\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})+\nabla\varphi\cdot\mathbb{B}:\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\right)dx-F(\mathbf{v}-\mathbf{u})\\
&=\frac12\|\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\|^2_{\mathbb{L}}.
\end{aligned}\tag{3.11}$$

Since

$$\frac12\|\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\|^2_{\mathbb{L}}\geqslant J_\varphi(\mathbf{v})-J_\varphi(\mathbf{v}+\mathbf{w})\quad\forall\mathbf{w}\in V_0,$$

(3.11) implies the lower bound

$$\begin{aligned}
&\frac12\|\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\|^2_{\mathbb{L}}\\
&\geqslant\sup_{\mathbf{w}\in V_0}\left\{F(\mathbf{w})-\int\limits_\Omega\Big((\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{v})+\nabla\varphi\cdot\mathbb{B}):\boldsymbol{\varepsilon}(\mathbf{w})+\frac12\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{w}):\boldsymbol{\varepsilon}(\mathbf{w})\Big)dx\right\}
\end{aligned}\tag{3.12}$$

In (3.12), the function φ is generally unknown. Therefore, if we wish to have a fully computable lower bound, then a certain approximation ψ, should be used instead of φ. The respective difference enters the estimate and, as a result, the lower bound comes in the form

$$\frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\|^2_{\mathbb{L}} \geqslant \sup_{\mathbf{w}\in V_0}\Big\{F(\mathbf{w}) - \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{w})\|^2_{\mathbb{L}} - \int\limits_{\Omega}(\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{v}) + \nabla\psi\cdot\mathbb{B}) : \boldsymbol{\varepsilon}(\mathbf{w})\,dx - \|\nabla(\psi-\varphi)\|_{\Omega}\|\mathbb{B}:\boldsymbol{\varepsilon}(\mathbf{w})\|_{\Omega}\Big\} \quad (3.13)$$

Here the norm $\|\psi - \varphi\|$ can be estimated by Theorem 1. To find a suitable function $\mathbf{w}$ it is convenient to maximize the first three terms in the right hand side of (3.13) (what amounts solving a quadratic minimization problem). A practical way to apply this estimate is to define a suitable $\mathbf{w}$ by maximization of the first three terms in the right hand side and use it to define the quantity $\|\mathbb{B}:\boldsymbol{\varepsilon}(\mathbf{w})\|_{\Omega}$ in the last term.

Similar arguments lead to a lower bound for another component of the error

$$\frac{1}{2}\|\nabla(\psi-\varphi)\|^2_{\mathbb{K}} \geqslant \sup_{\eta}\Big\{G(\eta) - \frac{1}{2}\|\nabla\eta\|^2_{\mathbb{K}} - \int\limits_{\Omega}(\mathbb{K}\nabla\eta\cdot\nabla\psi - \nabla\eta\cdot\mathbb{B}:\boldsymbol{\varepsilon}(\mathbf{u}))dx\Big\}. \quad (3.14)$$

Summation of (3.13) and (3.14) yields a lower bound of the combined energy norm (3.1), which is used in the next section.

§4. Errors generated by uncertain data

4.1. Problem with uncertain data. Now we consider the case, where $\mathbb{L}$, $\mathbb{B}$, and $\mathbb{K}$ are not known exactly. This situation is typical for applications, where physical parameters of the system are usually known with certain accuracy only. Hence instead of concrete $\mathbb{L}$, $\mathbb{K}$, and $\mathbb{B}$ we can only say that these matrixes belong to some sets. A natural way to define such sets is to introduce "mean" values $\overline{\mathbb{L}}$, $\overline{\mathbb{K}}$, $\overline{\mathbb{B}}$ and possible variations around them. We assume that

$$\begin{aligned} \mathbb{L} &= \overline{\mathbb{L}} + \delta\mathbb{L}; \quad \mathbb{L}\in\mathcal{L}, \\ \mathbb{K} &= \overline{\mathbb{K}} + \delta\mathbb{K}; \quad \mathbb{K}\in\mathcal{K}, \\ \mathbb{B} &= \overline{\mathbb{B}} + \delta\mathbb{B}; \quad \mathbb{B}\in\mathcal{B}, \end{aligned} \quad (4.1)$$

where $\mathcal{L}$, $\mathcal{K}$, and $\mathcal{B}$ are the sets that contain equally probable matrices of the coefficients. In the simplest case,

$$\mathcal{L} := \left\{ L_{ijkm} : |\mathbb{L}| = \sum_{i,j,k,m} L^2_{ijkm} \leqslant \epsilon^2_L \right\},$$

$$\mathcal{K} := \left\{ K_{ij} \ : |\mathbb{K}| \ = \ \sum_{i,j} K^2_{ij} \leqslant \epsilon^2_K \right\},$$

$$\mathcal{B} := \left\{ B_{ijk} \ : |\mathbb{B}| \ = \ \sum_{i,j,k} B^2_{ijk} \leqslant \epsilon^2_B \right\}.$$

Here possible coefficients of the matrices are contained in balls, whose radii are ϵ_L, ϵ_K, and ϵ_B. In general, the sets $\mathcal{L}$, $\mathcal{K}$, and $\mathcal{B}$ may have more complicated structures defined by a certain amount of parameters, but this will not require essential changes of the method presented below. By $\delta\mathcal{L}$, $\delta\mathcal{K}$, and $\delta\mathcal{B}$ we define the sets of admissible variations around $\overline{\mathbb{L}}$, $\overline{\mathbb{K}}$, and $\overline{\mathbb{B}}$, respectively, i.e.,

$$\mathcal{L} = \overline{\mathbb{L}} + \delta\mathcal{L}, \quad \mathcal{K} = \overline{\mathbb{K}} + \delta\mathcal{K}, \quad \mathcal{B} = \overline{\mathbb{B}} + \delta\mathcal{B}.$$

Let $\gamma_1(\overline{\mathbb{K}})$ and $\gamma_2(\overline{\mathbb{K}})$ be two positive constants in the inequality

$$\gamma_1^2(\overline{\mathbb{K}})\,|\boldsymbol{\zeta}|^2 \leqslant \ \overline{\mathbb{K}}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} \ \leqslant \gamma_2^2(\overline{\mathbb{K}})\,|\boldsymbol{\zeta}|^2 \qquad\qquad \forall \boldsymbol{\zeta} \in \mathbb{R}^d,$$

which is analogous to (2.7). Then for all $\boldsymbol{\zeta} \in \mathbb{R}^d$, we have

$$\begin{aligned}\beta_{1,K}\overline{\mathbb{K}}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} &\leqslant (\gamma_1^2(\overline{\mathbb{K}}) - \epsilon_K)\,|\boldsymbol{\zeta}|^2 \leqslant \mathbb{K}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} = \overline{\mathbb{K}}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} + \delta\mathbb{K}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} \\ &\leqslant \gamma_2^2(\overline{\mathbb{K}})\,|\boldsymbol{\zeta}|^2 + |\delta\mathbb{K}\boldsymbol{\zeta}|\,|\boldsymbol{\zeta}| \leqslant (\gamma_2^2(\overline{\mathbb{K}}) + \epsilon_K)\,|\boldsymbol{\zeta}|^2 \leqslant \beta_{2,K}\overline{\mathbb{K}}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}, \quad (4.2)\end{aligned}$$

where

$$\beta_{1,K} := 1 - \frac{\epsilon_K}{\gamma_2^2(\overline{\mathbb{K}})} \quad \text{and} \quad \beta_{2,K} := \frac{\gamma_2^2(\overline{\mathbb{K}}) + \epsilon_K}{\gamma_1^2(\overline{\mathbb{K}})}.$$

We assume that variations around the "mean" matrix $\mathbb{K}$ are limited, i.e.,

$$\gamma_2^2(\overline{\mathbb{K}}) > \epsilon_K,$$

so that $\beta_{1,K} > 0$ and all matrices in $\mathcal{K}$ are positive definite. From (4.2) it follows that

$$\beta_{1,K}\|\psi\|^2_{\overline{\mathbb{K}}} \ \leqslant \|\psi\|^2_{\mathbb{K}} \leqslant \ \beta_{2,K}\|\psi\|^2_{\overline{\mathbb{K}}} \quad \forall \psi \in M_0. \qquad (4.3)$$

Analogously, let $\gamma_1(\overline{\mathbb{L}})$ and $\gamma_2(\overline{\mathbb{L}})$ be positive constants in the inequality

$$\gamma_1^2(\overline{\mathbb{L}})\,|\boldsymbol{\tau}|^2 \leqslant \overline{\mathbb{L}}\boldsymbol{\tau} : \boldsymbol{\tau} \leqslant \gamma_2^2(\overline{\mathbb{L}})\,|\boldsymbol{\tau}|^2 \qquad \forall \boldsymbol{\tau} \in \mathbb{M}_s^{d\times d}.$$

Then for all $\boldsymbol{\tau} \in \mathbb{R}^d$, we have

$$\begin{aligned} \beta_{1,L}\overline{\mathbb{L}}\boldsymbol{\tau}\cdot\boldsymbol{\tau} &\leqslant (\gamma_1^2(\overline{\mathbb{L}}) - \epsilon_L)\,|\boldsymbol{\tau}|^2 \leqslant \mathbb{L}\boldsymbol{\tau} : \boldsymbol{\tau} = \overline{\mathbb{L}}\boldsymbol{\tau}\cdot\boldsymbol{\tau} + \delta\mathbb{L}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \\ &\leqslant \gamma_2^2(\overline{\mathbb{L}})\,|\boldsymbol{\tau}|^2 + |\delta\mathbb{L}\boldsymbol{\tau}|\,|\boldsymbol{\tau}| \leqslant (\gamma_2^2(\overline{\mathbb{L}}) + \epsilon_L)\,|\boldsymbol{\tau}|^2 \leqslant \beta_{2,L}\overline{\mathbb{L}}\boldsymbol{\tau}\cdot\boldsymbol{\tau}, \end{aligned} \tag{4.4}$$

where

$$\beta_{1,L} := 1 - \frac{\epsilon_L}{\gamma_2^2(\overline{\mathbb{L}})} \quad \text{and} \quad \beta_{2,L} := \frac{\gamma_2^2(\overline{\mathbb{L}}) + \epsilon_L}{\gamma_1^2(\overline{\mathbb{L}})}$$

and it is assumed that $\gamma_2^2(\mathbb{L}) > \epsilon_L$. Hence

$$\beta_{1,L}\|\varepsilon(\mathbf{v})\|^2_{\overline{\mathbb{L}}} \leqslant \|\varepsilon(\mathbf{v})\|^2_{\mathbb{L}} \leqslant \beta_{2,L}\|\varepsilon(\mathbf{v})\|^2_{\overline{\mathbb{L}}} \qquad \forall \mathbf{v} \in V_0. \tag{4.5}$$

Let $\mathcal{D}$ denote the set of all admissible $\mathbb{L}$, $\mathbb{K}$, and $\mathbb{B}$ satisfying (4.1). This set of data generates the corresponding sets of all admissible solutions

$$\Upsilon := \{u \in V_0 \ | \ u \text{ is solution of (2.1)–(2.9) for some } (\mathbb{L},\mathbb{K},\mathbb{B}) \in \mathcal{D}\}\,,$$
$$\varPhi := \{\varphi \in M_0 \ | \ \varphi \text{ is solution of (2.1)–(2.9) for some } (\mathbb{L},\mathbb{K},\mathbb{B}) \in \mathcal{D}\}\,.$$

Thus, instead of a single solution we have these sets of equally possible solutions, which can be called "solution clouds". In Fig. 1 these sets are coloured blue and $\mathcal{D} := \mathcal{L} \cup \mathcal{K} \cup \mathcal{B}$.

The quantities

$$R_\Upsilon := \sup_{\mathbf{u}\in\Upsilon} \|\mathbf{u} - \overline{\mathbf{u}}\|_V \quad \text{and} \quad R_\varPhi := \sup_{\varphi\in\varPhi} \|\varphi - \overline{\varphi}\|_M \tag{4.6}$$

are important characteristics of Υ and $\varPhi$. They show sizes of Υ and $\varPhi$ in terms of the energy norms and, therefore, give a presentation on the accuracy limits induced by uncertain data. In view of (2.5), (2.7), and Korn's inequality they are equivalent to

$$R_{\Upsilon,L} := \sup_{\mathbf{u}\in\Upsilon} \|\varepsilon(\mathbf{u} - \overline{\mathbf{u}})\|_{\mathbb{L}} \quad \text{and} \quad R_{\varPhi,K} := \sup_{\varphi\in\varPhi} \|\nabla(\varphi - \overline{\varphi})\|_{\mathbb{K}}. \tag{4.7}$$

The quantities defined by (4.6) and (4.7) define (in terms of different norms) sizes of balls containing Υ and $\varPhi$.

Computations with the accuracy $e_{\mathbf{u}}$ for approximations of $\mathbf{u}$ and e_ϕ for approximations of φ are sensefull if numerical errors are much larger than the errors caused by data uncertainty, i.e., if $e_{\mathbf{u}} > R_{\Upsilon,L}$ and $e_\phi > R_{\varPhi,K}$. Therefore, our first goal is to deduce estimates of $R_{\Upsilon,L}$ and $R_{\varPhi,K}$.

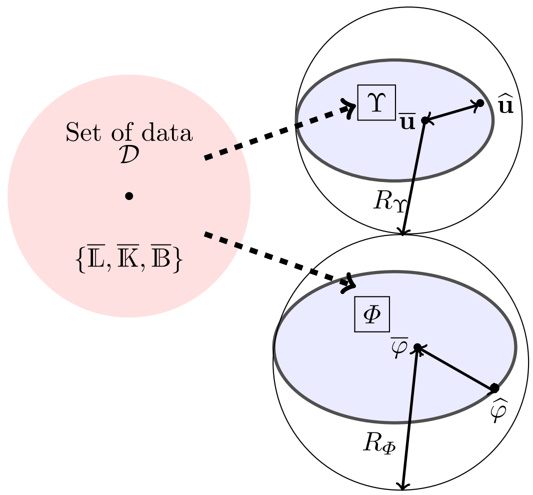

Figure 1. The sets $\mathcal{D}$, Υ, and Φ.

In principle, there is a straightforward way to get a presentation on these quantities. It consists of solving sufficiently large amount of problems related to different data and comparing the respective solutions. However, this way is very expensive and not reliable. In what follows, we suggest another method, where fully guaranteed estimates of $R_{\Upsilon,L}$ and $R_{\Phi,K}$ are obtained by solving algebraic problems associated with the sets of parameters $\mathcal{L}$, $\mathcal{K}$, and $\mathcal{B}$. The crucial step on this way is done by means of a posteriori estimates of the functional type discussed in the preceding section.

4.2. Majorants of $R_{\Upsilon,L}$ and $R_{\Phi,K}$. Let $\overline{\mathbf{u}}$ and $\overline{\varphi}$ denote the solution of the system (2.1)-(2.4) in Ω with the matrices $\overline{\mathbb{L}}, \overline{\mathbb{B}}$ and $\overline{\mathbb{K}}$, i.e.,

$$\overline{\boldsymbol{\sigma}} = \overline{\mathbb{L}}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) + \nabla\overline{\varphi} \cdot \overline{\mathbb{B}}, \tag{4.8}$$

$$\overline{\mathbf{p}} = \overline{\mathbb{K}}\,\nabla\overline{\varphi} - \overline{\mathbb{B}}^T : \boldsymbol{\varepsilon}(\overline{\mathbf{u}}). \tag{4.9}$$

$$\operatorname{Div}\overline{\boldsymbol{\sigma}} + \mathbf{f} = 0, \tag{4.10}$$

$$\operatorname{div}\overline{\mathbf{p}} + g = 0. \tag{4.11}$$

All the problems are supplied with the same boundary conditions

$$\overline{\mathbf{u}} = \mathbf{u}_0 \text{ on } \Gamma^1_D, \quad \overline{\varphi} = \varphi_0 \text{ on } \Gamma^2_D,$$
$$\overline{\boldsymbol{\sigma}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma^1_N, \quad \overline{\mathbf{p}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma^2_N.$$

Let $\mathbf{u}$ and φ be defined by (2.1), (2.2), (2.3, and (2.4). We compare these functions with $\overline{\mathbf{u}}$ and $\overline{\varphi}$. Due to (4.10) and (4.11) the last terms of (3.8) and (3.9) vanish and, therefore,

$$\mathcal{M}_1(\overline{\mathbf{u}}, \overline{\varphi}, \overline{\boldsymbol{\sigma}}) = \|\overline{\boldsymbol{\sigma}} - \mathbb{L}\,\varepsilon(\overline{\mathbf{u}}) - \mathbb{B} \cdot \nabla\overline{\varphi}\|_{\mathbb{L}^{-1}} = \|\mathrm{T}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{L}, \delta\mathbb{B})\|_{\mathbb{L}^{-1}}$$

and

$$\mathcal{M}_2(\overline{\mathbf{u}}, \overline{\varphi}, \overline{\mathbf{p}}) = \|\overline{\mathbf{p}} - \mathbb{K}\nabla\overline{\varphi} + \mathbb{B}^T : \varepsilon(\overline{\mathbf{u}})\|_{\mathbb{K}^{-1}} = \|\mathrm{S}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{K}, \delta\mathbb{B})\|_{\mathbb{K}^{-1}},$$

$$\mathrm{T}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{L}, \delta\mathbb{B}) := \delta\mathbb{L}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) + \nabla\overline{\varphi} \cdot \delta\mathbb{B}, \tag{4.12}$$
$$\mathrm{S}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{K}, \delta\mathbb{B}) := \delta\mathbb{K}\,\nabla\overline{\varphi} - \delta\mathbb{B} : \boldsymbol{\varepsilon}(\overline{\mathbf{u}}). \tag{4.13}$$

and by (3.10) we obtain

$$\|\boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})\|^2_{\mathbb{L}} + \|\nabla(\varphi-\overline{\varphi})\|^2_{\mathbb{K}} \leqslant \|\mathrm{T}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{L}, \delta\mathbb{B})\|_{\mathbb{L}^{-1}} + \|\mathrm{S}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{K}, \delta\mathbb{B})\|_{\mathbb{K}^{-1}}, \tag{4.14}$$

In view of (4.3) and (4.5), we have

$$\beta_{1,L}\|\boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})\|^2_{\overline{\mathbb{L}}} + \beta_{1,K}\|\nabla(\varphi-\overline{\varphi})\|^2_{\overline{\mathbb{K}}} \leqslant \|\boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})\|^2_{\mathbb{L}} + \|\nabla(\varphi - \overline{\varphi})\|^2_{\mathbb{K}}. \tag{4.15}$$

From (4.2) and (4.4) it follows that

$$\|\boldsymbol{\zeta}\|^2_{\mathbb{K}^{-1}} \leqslant \frac{1}{\beta_{1,K}}\|\boldsymbol{\zeta}\|^2_{\overline{\mathbb{K}}^{-1}} \quad \text{and} \quad \|\boldsymbol{\tau}\|^2_{\mathbb{L}^{-1}} \leqslant \frac{1}{\beta_{1,L}}\|\boldsymbol{\tau}\|^2_{\overline{\mathbb{L}}^{-1}}.$$

Hence from (4.14) and (4.15) it follows that

$$\begin{aligned}\beta_{1,L}&\|\boldsymbol{\varepsilon}(\mathbf{u} - \overline{\mathbf{u}})\|^2_{\overline{\mathbb{L}}} + \beta_{1,K}\|\nabla(\varphi - \overline{\varphi})\|^2_{\overline{\mathbb{K}}} \\ &\leqslant \frac{1}{\beta_{1,L}}\|\mathrm{T}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{L}, \delta\mathbb{B})\|_{\overline{\mathbb{L}}^{-1}} + \frac{1}{\beta_{1,K}}\|\mathrm{S}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{K}, \delta\mathbb{B})\|_{\overline{\mathbb{K}}^{-1}}.\end{aligned} \tag{4.16}$$

The left hand side of (4.16) depends on the solutions $\overline{u}$ and $\overline{\varphi}$ of the "central" problem (4.8)–(4.11) and variations $\delta\mathbb{L}$, $\delta\mathbb{K}$, and $\delta\mathbb{B}$. It gives an upper bound of the modeling error generated by exact solutions of two different problems. Indeed, taking supremum in the left hand side of (4.16) with respect to $u \in \Upsilon$ and $\varphi \in \mathit{\Phi}$ is equivalent to taking supremum over the sets of admissible parameters in the right one. Thus, we arrive at the principal estimate

$$\begin{aligned}&\beta_{1,L} R_{\Upsilon,L} + \beta_{1,K} R_{\Phi,K}\\ &\quad\leqslant \sup_{\substack{\delta\mathbb{L}\in\delta\mathcal{L},\\ \delta\mathbb{K}\in\delta\mathcal{K},\\ \delta\mathbb{B}\in\delta\mathcal{B}}} \Big\{ \frac{1}{\beta_{1,L}} \|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|^2_{\mathbb{L}^{-1}} + \frac{1}{\beta_{1,K}} \|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|^2_{\mathbb{K}^{-1}} \Big\}.\end{aligned} \tag{4.17}$$

Notice that (4.17) gives an upper bound for the weighted sum of $R_{\Upsilon,L}$ and $R_{\Upsilon,K}$ *by analysing the set of possible data $\mathcal{D}$ instead of the sets of possible solutions Υ and Φ.*

To get estimates for $R_{\Upsilon,L}$ and $R_{\Upsilon,K}$ separately, we need to apply a bit different estimation method. Consider the identity (3.7), where $\mathbf{v} = \overline{\mathbf{u}}$, $\boldsymbol{\tau} = \overline{\boldsymbol{\sigma}}$, $\psi = \overline{\varphi}$, and $\mathbf{q} = \overline{\mathbf{p}}$. We have

$$\begin{aligned}&\|\boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})\|^2_{\mathbb{L}} + \|\nabla(\varphi-\overline{\varphi})\|^2_{\mathbb{K}}\\ &= \int_\Omega \Big(\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B}) : \boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})dx + \mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\cdot\nabla(\varphi-\overline{\varphi})\Big)dx,\end{aligned} \tag{4.18}$$

Integrals in (4.18) are easy to estimate:

$$\begin{aligned}\int_\Omega \mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B}) : \boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})dx &\leqslant \|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|_{\overline{\mathbb{L}}^{-1}} \|\boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})\|_{\overline{\mathbb{L}}}\\ &\leqslant \frac{1}{2\alpha_1}\|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|^2_{\overline{\mathbb{L}}^{-1}} + \frac{\alpha_1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})\|^2_{\overline{\mathbb{L}}}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\int_\Omega \mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B}) \cdot \nabla(\varphi-\overline{\varphi})dx &\leqslant \|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|_{\overline{\mathbb{K}}^{-1}} \|\nabla(\varphi-\overline{\varphi})\|_{\overline{\mathbb{K}}}\\ &\leqslant \frac{1}{2\alpha_2}\|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|^2_{\overline{\mathbb{K}}^{-1}} + \frac{\alpha_2}{2}\|\nabla(\varphi-\overline{\varphi})\|^2_{\overline{\mathbb{K}}},\end{aligned} \tag{4.20}$$

where $\alpha_1 \in (0, 2\beta_{1,L}]$ and $\alpha_2 \in\in (0, 2\beta_{1,K}]$. Therefore,

$$\begin{aligned}&(2\beta_{1,L}-\alpha_1)\|\boldsymbol{\varepsilon}(\mathbf{u}-\overline{\mathbf{u}})\|^2_{\overline{\mathbb{L}}} + (2\beta_{1,K}-\alpha_2)\|\nabla(\varphi-\overline{\varphi})\|^2_{\overline{\mathbb{K}}}\\ &\qquad\leqslant \frac{1}{\alpha_1}\|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|^2_{\overline{\mathbb{L}}^{-1}} + \frac{1}{\alpha_2}\|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|^2_{\overline{\mathbb{K}}^{-1}}\end{aligned} \tag{4.21}$$

We take supremum in (4.21) with respect to all possible data and find that

$$(2\beta_{1,L}-\alpha_1)R_{\Upsilon,L}+(2\beta_{1,K}-\alpha_2)R_{\Phi,K} \leqslant \sup_{\substack{\delta\mathbb{L}\in\delta\mathcal{L},\\ \delta\mathbb{K}\in\delta\mathcal{K},\\ \delta\mathbb{B}\in\delta\mathcal{B}}}\left\{\frac{1}{\alpha_1}\|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|^2_{\mathbb{L}^{-1}}+\frac{1}{\alpha_2}\|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|^2_{\mathbb{K}^{-1}}\right\} \tag{4.22}$$

Let $\alpha_2=2\beta_{1,K}$. Then (4.22) implies the estimate

$$R_{\Upsilon,L}\leqslant\inf_{\alpha_1<2\beta_{1,L}}\sup_{\substack{\delta\mathbb{L}\in\delta\mathcal{L},\\ \delta\mathbb{K}\in\delta\mathcal{K},\\ \delta\mathbb{B}\in\delta\mathcal{B}}}\frac{\|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|^2_{\mathbb{L}^{-1}}}{\alpha_1(2\beta_{1,L}-\alpha_1)}+\frac{\|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|^2_{\mathbb{K}^{-1}}}{2\beta_{1,K}(2\beta_{1,L}-\alpha_1)}. \tag{4.23}$$

Analogously, if $\alpha_1=2\beta_{1,L}$ then we obtain an upper bound for another quantity

$$R_{\Upsilon,K}\leqslant\inf_{\alpha_2<2\beta_{1,K}}\sup_{\substack{\delta\mathbb{L}\in\delta\mathcal{L},\\ \delta\mathbb{K}\in\delta\mathcal{K},\\ \delta\mathbb{B}\in\delta\mathcal{B}}}\frac{\|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|^2_{\mathbb{L}^{-1}}}{2\beta_{1,L}(2\beta_{1,K}-\alpha_2)}+\frac{\|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|^2_{\mathbb{K}^{-1}}}{\alpha_2(2\beta_{1,K}-\alpha_2)}. \tag{4.24}$$

We outline that finding supremum in (4.22), (4.23), and (4.24) is equivalent to solving a low dimensional algebraic problem.

A simpler upper bound can be deduced from (4.22) if we apply the estimates

$$\begin{aligned}\|\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{L},\delta\mathbb{B})\|^2_{\mathbb{L}^{-1}} &\leqslant \frac{\mu+1}{2}\|\delta\mathbb{L}\,\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\|^2_{\mathbb{L}^{-1}}+\frac{\mu+1}{2\mu}\|\nabla\overline{\varphi}\cdot\delta\mathbb{B}\|^2_{\mathbb{L}^{-1}}\\ &\leqslant \frac{(\mu+1)\epsilon_L^2}{2\gamma_1^2(\overline{L})}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\|^2+\frac{(\mu+1)\epsilon_B^2}{2\mu\gamma_1^2(\overline{L})}\|\nabla\overline{\varphi}\|^2,\quad \mu>0\end{aligned}$$

and

$$\|\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\delta\mathbb{K},\delta\mathbb{B})\|^2_{\mathbb{K}^{-1}}\leqslant\frac{(\nu+1)\epsilon_K^2}{2\gamma_1^2(\overline{K})}\|\nabla\overline{\varphi}\|^2+\frac{(\nu+1)\epsilon_B^2}{2\nu\gamma_1^2(\overline{K})}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\|^2,\quad \nu>0.$$

In this case, (4.22) implies the estimate

$$(2\beta_{1,L}-\alpha_1)R_{\Upsilon,L}+(2\beta_{1,K}-\alpha_2)R_{\Phi,K} \leqslant \inf_{\substack{\mu>0,\\ \nu>0}}\left\{\kappa_1(\mu,\nu)\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\|^2+\kappa_2(\mu,\nu)\|\nabla\overline{\varphi}\|^2\right\}, \tag{4.25}$$

where

$$\kappa_1(\mu,\nu) \;=\; \frac{(\mu+1)\epsilon_L^2}{2\alpha_1\gamma_1^2(\overline{L})} \;+\; \frac{(\nu+1)\epsilon_B^2}{2\nu\alpha_2\gamma_1^2(\overline{K})}$$

and

$$\kappa_2(\mu,\nu) \;=\; \frac{(\mu+1)\epsilon_B^2}{2\mu\alpha_1\gamma_1^2(\overline{L})} \;+\; \frac{(\nu+1)\epsilon_K^2}{2\alpha_2\gamma_1^2(\overline{K})}.$$

Estimate (4.25) suggests a practical way to estimate the size of the cloud containing all possible solutions avoiding large computations. Indeed, this estimate is coarser than (4.22). However, here the right hand side contains a very simple parametric optimisation problem for the parameters μ and ν, which can be easily solved provided that $\overline{\mathbf{u}}$ and $\overline{\varphi}$ are known. Even if these functions are known only approximately (e.g., as numerical approximations $\overline{\mathbf{u}}_h$ and $\overline{\varphi}_h$), there are still no fundamental difficulties arise. First of all we notice that if the approximations are sufficiently accurate, then the norms $\|\nabla\overline{\varphi}_h\|$ and $\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_h)\|$ would be almost the same as those in (4.25). Therefore, estimates of the quantities will be practically the same. If, nevertheless, we wish to get fully reliable estimates, then we use obvious estimates

$$\|\overline{\mathbf{u}}\| \leqslant \|\overline{\mathbf{u}}_h\| + \|\overline{\mathbf{u}} - \overline{\mathbf{u}}_h\| \quad \text{and} \quad \|\overline{\varphi}\| \leqslant \|\overline{\varphi}_h\| + \|\overline{\varphi} - \overline{\varphi}_h\|,$$

where $\|\overline{\mathbf{u}} - \overline{\mathbf{u}}_h\|$ and $\|\overline{\varphi} - \overline{\varphi}_h\|$ are estimated by the method discussed in Sec. 3.

4.3. Minorants of $R_{\Upsilon,L}$ and $R_{\Phi,K}$. Now turn to another quantitative characteristic of the solution cloud. We want to know how far various elements of this set can be located from each other. By this we get a presentation on the lower estimate of the diameter of the solutions cloud. This quantity has important meaning for computer simulation methods: it states the *accuracy limit* for all errors arising in computations. This limit always exist and must be taken into account when studying errors of approximations, which are typically considered as tending to zero if the dimensionality of the approximation space increases. In reality this purely theoretical scheme has no sense beyond the accuracy limit dictated by the accuracy of mathematical model used.

To deduce guaranteed and computable lower bounds for the accuracy limit, we take various matrices $\mathbb{L}_1$, $\mathbb{K}_1$, and $\mathbb{B}_1$ such that

$$|\mathbb{L}_1| = |\mathbb{K}_1| = |\mathbb{B}_1| = 1.$$

We consider matrices, which maximally differs from $\overline{\mathbb{L}}$, $\overline{\mathbb{K}}$, and $\overline{\mathbb{B}}$, respectively. More precisely, we define the parameters ξ_L, ξ_K, and ξ_B such that

$$\xi_L := \max_{\zeta\in\mathbb{R}}\big\{\overline{\mathbb{L}} + \zeta\mathbb{L}_1 \in \mathcal{L}\big\}, \tag{4.26}$$

$$\xi_K := \max_{\zeta\in\mathbb{R}}\big\{\overline{\mathbb{K}} + \zeta\mathbb{K}_1 \in \mathcal{K}\big\}, \tag{4.27}$$

$$\xi_B := \max_{\zeta\in\mathbb{R}}\big\{\overline{\mathbb{B}} + \zeta\mathbb{B}_1 \in \mathcal{B}\big\}. \tag{4.28}$$

Finding ξ_L, ξ_K, and ξ_B is a simple algebraic problem.

Let $(\widehat{\mathbf{u}}, \widehat{\varphi})$ solve the problem generated by $\overline{\mathbb{L}} + \xi_L\mathbb{L}_1$, $\mathbb{K} := \overline{\mathbb{K}} + \xi_K\mathbb{K}_1$, and $\mathbb{B} := \overline{\mathbb{B}} + \xi_B\mathbb{B}_1$, where the parameters ξ_L, ξ_K, and ξ_B are defined in accordance with (4.26)–(4.28). We set in (3.12) $\mathbf{v} = \overline{\mathbf{u}}$ and $\mathbf{u} = \widehat{\mathbf{u}}$ (see Fig. 1). Then the identity reads

$$\begin{aligned}
&\beta_{2,L}\frac{1}{2}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}} - \widehat{\mathbf{u}})\|^2_{\mathbb{L}} \geqslant \frac{1}{2}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}} - \mathbf{u})\|^2_{\mathbb{L}}\\
&\geqslant \sup_{\mathbf{w}\in V_0}\left\{F(\mathbf{w}) - \int\limits_\Omega \left((\mathbb{L}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) + \nabla\overline{\varphi}\cdot\mathbb{B}) : \boldsymbol{\varepsilon}(\mathbf{w}) + \frac{1}{2}\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{w})\right) dx\right\}\\
&\hfill \forall w \in V_0.
\end{aligned}$$

Since

$$F(\mathbf{w}) = -\int\limits_\Omega \mathbf{f}\cdot\overline{\mathbf{w}}dx = \int\limits_\Omega (\overline{\mathbb{L}}\varepsilon(\overline{\mathbf{u}}) + \overline{\mathbb{B}}\cdot\nabla\overline{\varphi}) : \varepsilon(\mathbf{w})dx,$$

we conclude that

$$\begin{aligned}
&\frac{\beta_{2,L}}{2}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}} - \widehat{\mathbf{u}})\|^2_{\mathbb{L}}\\
&\geqslant \sup_{\mathbf{w}\in V_0}\int\limits_\Omega \left(-\mathrm{T}(\overline{\mathbf{u}}, \overline{\varphi}; \zeta_L\mathbb{L}_1, \zeta_B\mathbb{B}_1) : \boldsymbol{\varepsilon}(\mathbf{w}) - \frac{1}{2}\mathbb{L}\boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{w})\right) dx.
\end{aligned} \tag{4.29}$$

Analogously, setting in (3.14) $\psi = \overline{\varphi}$ and $\varphi = \widehat{\varphi}$, we have

$$\begin{aligned}
&\frac{\beta_{2,K}}{2}\|\nabla(\overline{\varphi} - \widehat{\varphi})\|^2_{\mathbb{K}}\\
&\qquad\geqslant \sup_{\eta\in M_0}\int\limits_\Omega \left(-\mathrm{S}(\overline{\mathbf{u}}, \overline{\varphi}; \zeta_K\mathbb{K}_1, \zeta_B\mathbb{B}_1) : \nabla\eta - \frac{1}{2}\mathbb{K}\nabla\eta : \nabla\eta\right) dx.
\end{aligned} \tag{4.30}$$

Summing (4.29) and (4.30), we obtain

$$\begin{aligned}
&\beta_{2,L}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}}-\widehat{\mathbf{u}})\|^2_{\mathbb{L}}+\beta_{2,K}\|\nabla(\overline{\varphi}-\widehat{\varphi})\|^2_{\mathbb{K}}\\
&\quad\geqslant \sup_{\mathbf{w}\in V_0}\int\limits_\Omega \big(-2\mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\zeta_L\mathbb{L}_1,\zeta_B\mathbb{B}_1):\boldsymbol{\varepsilon}(\mathbf{w})-\beta_{2,L}\overline{\mathbb{L}}\boldsymbol{\varepsilon}(\mathbf{w}):\boldsymbol{\varepsilon}(\mathbf{w})\big)\,dx\\
&\quad+\sup_{\eta\in M_0}\int\limits_\Omega \big(-2\mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\zeta_K\mathbb{K}_1,\zeta_B\mathbb{B}_1):\nabla\eta-\beta_{2,K}\overline{\mathbb{K}}\nabla\eta:\nabla\eta\big)\,dx. \qquad (4.31)
\end{aligned}$$

Let $V_0^n\subset V_0$ be a finite dimensional space $\dim V_0^n=n$ and $M_0^m\subset M_0$ be another finite dimensional space $\dim M_0^m=m$. Define $\mathbf{w}^n\in V_0^n$ and $\varphi^m\in M_0$ such that

$$\int\limits_\Omega \overline{\mathbb{L}}\boldsymbol{\varepsilon}(\mathbf{w}^n):\boldsymbol{\varepsilon}(\mathbf{w})\,dx=\int\limits_\Omega \mathrm{T}(\overline{\mathbf{u}},\overline{\varphi};\zeta_L\mathbb{L}_1,\zeta_B\mathbb{B}_1):\boldsymbol{\varepsilon}(\mathbf{w})\,dx \quad \forall w\in V_0^n, \qquad (4.32)$$

$$\int\limits_\Omega \overline{\mathbb{K}}\nabla\varphi^m:\nabla\eta\,dx=\int\limits_\Omega \mathrm{S}(\overline{\mathbf{u}},\overline{\varphi};\zeta_K\mathbb{K}_1,\zeta_B\mathbb{B}_1):\nabla\eta\,dx \quad \forall\psi\in M_0^m. \qquad (4.33)$$

Problems (4.32) and (4.33) are linear finite dimensional problems. They can be easily solved by well known methods.

We set in (4.31) $\mathbf{w}=\mathbf{w}^n/\beta_{2,L}$ and $\eta=\varphi^m/\beta_{2,K}$. Then it yields the estimate

$$\beta_{2,L}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}}-\widehat{\mathbf{u}})\|^2_{\mathbb{L}}+\beta_{2,K}\|\nabla(\overline{\varphi}-\widehat{\varphi})\|^2_{\mathbb{K}}\geqslant\frac{1}{\beta_{2,L}}\|\boldsymbol{\varepsilon}(\mathbf{w}^n)\|^2_{\mathbb{L}}+\frac{1}{\beta_{2,K}}\|\nabla\varphi^m\|^2_{\mathbb{K}}, \qquad (4.34)$$

which gives a lower bound of the distance between the solution $\overline{\mathbf{u}}$ and $\overline{\varphi}$ generated by $\overline{\mathbb{L}}$, $\overline{\mathbb{K}}$, $\overline{\mathbb{B}}$ and $\mathbf{u}$ and φ generated by the tensors in (4.26), (4.27), and (4.28).

In conclusion, we draw attention to one important consequence of the above formulas. In computer modeling of scientific and technical problems it is usually assumed (explicitly or implicitly), that small inaccuracies in the data have little effect on the result and can be ignored. Using the estimates (4.31)–(4.34) we can show that such assumptions may be wrong. Consider a special but important case of isotropic elastic media, where

$$\mathbb{L}\varepsilon(\mathbf{w})=\lambda\,\mathrm{div}\mathbf{w}\,\mathbb{I}+2\mu\varepsilon(\mathbf{w})$$

and λ, μ are positive constants (Lamé moduli). They are often expressed throughout two the other material constants E and ν by the relations

$$\lambda = \frac{E\,\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{\mathrm{E}}{2(1+\nu)}. \tag{4.35}$$

Young's modulus E is a large positive constant and most materials have the Poisson's ratio ν in the interval $[0, 0.5]$. Materials are called *soft* if they have ν close to 0.5. It is easy to see that if ν tends to 0.5 then λ tends to $+\infty$. The case $\nu = 0.5$ corresponds to an incompressible media. Assume that the value of ν is not known exactly, i.e., the set of admissible data contains $\overline{\nu} < 0.5$ and $\widehat{\nu} < 0.5$ such that $|\overline{\nu} - \widehat{\nu}| \leqslant \delta$, where δ is a small positive number. The corresponding $\overline{\lambda}$ and $\widehat{\lambda}$ are large and tend to $+\infty$ if $\overline{\nu}$ and $\widehat{\nu}$ tend to 0.5. This situation may lead to catastrophic growth of errors generated by data uncertainty. In [10] this question was systematically analysed for linear elasticity problems with uncertainty in elasticity moduli. Similar phenomena arises in the coupled piesoelectric system (2.1)–(2.4).

Assume that all coefficients except the Lamé constant λ are defined exactly, so that $\widehat{K} = \overline{K}$, $\widehat{B} = \overline{B}$, and $\widehat{\mu} = \overline{\mu}$. Also, assume that $\overline{\lambda} > \widehat{\lambda}$ and these values grow as the corresponding $\widehat{\nu}$ and $\overline{\nu}$ tend to 0.5 (see (4.35). Then

$$\widehat{\mathbb{L}}\tau : \tau = \widehat{\lambda}\, I_1^2(\tau) + 2\mu|\tau|^2 \quad \text{and} \quad \overline{\mathbb{L}}\tau : \tau = \overline{\lambda} I_1^2(\tau) + 2\mu|\tau|^2.$$

In this case, $\beta_{2,L} = \widehat{\lambda}/\overline{\lambda} < 1$. Let $\overline{\lambda} = 2\widehat{\lambda}$. Then $\beta_{2,L} = 0.5$. Since $\delta\mathbb{B} = 0$, we consider the estimate (4.29) related to the elastic part of the solution only. The function $\widehat{\mathbf{u}}$ solves the problem with the Lamé modulus $\widehat{\lambda}$ and $\overline{\mathbf{u}}$ solves the problem with $\overline{\lambda}$. Notice that

$$\mathrm{T}(\overline{\mathbf{u}}, \overline{\varphi}; \delta\mathbb{L}, \delta\mathbb{B}) : \boldsymbol{\varepsilon}(\mathbf{w}) = (\mathbb{L} - \overline{\mathbb{L}})\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) : \boldsymbol{\varepsilon}(\mathbf{w}) = (\widehat{\lambda} - \overline{\lambda})\,(\mathrm{div}\overline{\mathbf{u}})(\mathrm{div}\mathbf{w}).$$

Hence we obtain the estimate

$$\beta_{2,L}\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}} - \widehat{\mathbf{u}})\|^2_{\mathbb{L}} \geqslant \sup_{\mathbf{w}\in V_0} \int\limits_\Omega \Big(2\widehat{\lambda}(\mathrm{div}\overline{\mathbf{u}})(\mathrm{div}\mathbf{w}) - \widehat{\lambda} d(\mathrm{div}\mathbf{w})^2 - 2\mu|\varepsilon(\mathbf{w})|^2\Big)\,\mathrm{d}x. \tag{4.36}$$

Assume that $\|\mathrm{div}\overline{\mathbf{u}}\| > 0$. It is not difficult to show that there exists a function $g_{\mathbf{u}}$ with zero mean such that

$$\int\limits_\Omega g_{\mathbf{u}}\, \mathrm{div}\overline{\mathbf{u}}\, dx = c_{\mathbf{u}} > 0. \tag{4.37}$$

Let $\mathbf{w}_0 \in V_0$ be such that $\mathrm{div}\mathbf{w}_0 = g_{\mathbf{u}}$ (existence of such a function follows from well known results in the theory of functions). In this case,

$$\int\limits_{\Omega} (\mathrm{div}\overline{\mathbf{u}})(\mathrm{div}\mathbf{w}_0)dx = c_{\mathbf{u}}.$$

We substitute $\mathbf{w} = \rho\mathbf{w}_0$ in (4.36) and obtain the estimate

$$\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}} - \widehat{\mathbf{u}})\|^2_{\mathbb{L}} \geqslant 2\widehat{\lambda}(2\rho c_{\mathbf{u}} - d\rho^2\|g_{\mathbf{u}}\|^2) - \int\limits_{\Omega} 2\mu\rho^2|\varepsilon(\mathbf{w}_0)|^2 dx.$$

In view of (4.37) $g_{\mathbf{u}} \not\equiv 0$ and, therefore, $\|g_{\mathbf{u}}\| > 0$. Hence we can set $\rho = \frac{1}{d\|g_{\mathbf{u}}\|^2} c_{\mathbf{u}}$. Then we arrive at the estimate

$$\|\boldsymbol{\varepsilon}(\overline{\mathbf{u}} - \widehat{\mathbf{u}})\|^2_{\mathbb{L}} \geqslant \frac{2\widehat{\lambda}c^2_{\mathbf{u}}}{d\|g_{\mathbf{u}}\|^2} - \frac{2\mu}{d^2\|g_{\mathbf{u}}\|^2}\|\varepsilon(\mathbf{w}_0)\|^2, \tag{4.38}$$

which shows that the set of possible solutions may contain very distant functions if $\widehat{\lambda}$ is large. Assume that $\Gamma^1_D = \Gamma$ and the boundary condition (2.9) for $\overline{\mathbf{u}}$ is defined by the function $\mathbf{u}_0$ such that

$$\int\limits_{\Gamma} \mathbf{u}_0 \cdot \mathbf{n} \mathrm{d}s \neq 0. \tag{4.39}$$

Then, the respective solution $\overline{\mathbf{u}}$ cannot be divergence free, i.e., $\|\mathrm{div}\overline{\mathbf{u}}\| > 0$. In this case, (4.38) shows that the distance between $\overline{\mathbf{u}}$ and $\widehat{\mathbf{u}}$ grows as $\widehat{\lambda} \to +\infty$. At the same time, the corresponding values of ν may be very close. This fact leads to the conclusion that for ν close to 0.5 numerical analysis of such type problems with Dirichlét boundary conditions (4.39) excluding divergence free fields is not entirely correct.

References

1. I. Babuška, J. Chleboun, I. Hlavacek, *Uncertain input data problems and the worst scenario method*, Elsevier, Amsterdam, 2004.
2. T. Buchukuri, O. Chkadua, D. Natroshvili, A.-M. Sändig, *Solvability and regularity results to boundary-transmission problems for metallic and piezoelectric elastic materials.* Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation, IANS, Preprint 2005/004 (2005).
3. C. Carstensen, J. Gedicke, *Guaranteed lower bounds for eigenvalues.* — Math. Comp. **83** (2014), 2605–2629.
4. I. Ekeland, d R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.

5. S. Kaliski, G. Patykiewicz, *Dynamical equation of motion coupled with the field of temperature and resolving function for elastic and inelastic anisotropic bodies in the magnetic fields.* — Proc. Vibr. Problem **1**, No. 4 (1960).
6. L. Knopoff, *The interaction between elastic wave motions and a magnetic field in electric conductors.* — J. Geophys. Res. **60**,No. 4 (1955).
7. Yu. Kuznetsov, S. Repin, *Guaranteed lower bounds of the smallest eigenvalues of elliptic differential operators.* — Russian J. Numer. Anal. Math. Mod. **21**, No. 2 (2003), 135–156.
8. U. Langer, S. Repin, T. Samrowski, *Reliable a posteriori estimates for coupled systems generated by piezoelectric models.* — Russian J. Numer. Anal. Math. Mod. **32**, No. 4 (2017), 259–267.
9. O. Mali, P. Nettaanmäki, S. Repin, *Accuracy verification methods,* Theory and algorithms. Springer, New York, 2014.
10. O. Mali, S. Repin, *Blowup of errors caused by inexact knowledge of the Poisson ration in some elasticity problems.* — Russian J. Numer. Anal. Math. Mod. **26**, No. 4 (2011), 413–425.
11. R. D. Mindlin, *On the equations of motion of piezoelectric crystals.* — In: J. R. M. Radok(ed): Problems of Continuum Mechanics. SIAM, 282–290 (1961).
12. R. D. Mindlin, *Polarization gradient in elastic dielectrics.* Int. J. Solids Struct. **6** (1968).
13. R. D. Mindlin, *Elasticity, piezoelasticity and crystal lattice dynamics.* — J. Elasticity, **2**, No. 4 (1972).
14. S. Repin, *A posteriori error estimation for nonlinear variational problems by duality theory.* — Zap. Nauch. Semin. POMI, **243** (1997), 201–214.
15. S. Repin, *A unified approach to a posteriori error estimation based on duality error majorants.* — Modelling '98 (Prague), Math. Comput. Simulation **50**, No. 1–4 (1999), 305–321.
16. S. Repin, *A posteriori estimates for partial differential equations.* — Radon Series on Computational and Applied Mathematics, **4** (2008), Walter de Gruyter GmbH & Co. KG, Berlin.
17. S. Repin, *Computable Majorants of constants in the Poincare and Friedrichs inequalities.* — J. Math. Sci. **186**, No. 2 (2012), 307–321.
18. S. Repin, T. Samrowski, S. Sauter, *Combined A Posteriori Modeling-Discretization Error Estimate for Elliptic Problems with Complicated Interfaces.* — ESAIM: Math. Model. Num. Anal. **46**, No. 6 (2012), 1389–1405.
19. S. Repin, S. Sauter, *Accuracy of Mathematical Models*, Tracts in Mathematics 33, European Mathematical Society, Berlin, 2020.
20. S. Repin, J. Valdman, *Functional a posteriori error estimates for problems with nonlinear boundary conditions.* —J. Numer. Math. **16**, No. 1 (2008), 51–81.
21. T. Samrowski, *Combined Error Estimates in the case of the Dimension Reduction.* — CMAM, **14** (2014), 113–134.
22. I. Šebetová, T. Vejchodský, *Two-sided bounds for eigenvalues of differential operators with applications to Friedrichs, Poincaré, trace, and similar constants.* — SIAM J. Numer. Anal. **52** (2014), 308–329.
23. R. C. Smith, *Uncertainty Quantification. Theory, Implementation, and Applications.* — SIAM, Philadelphia, 2014.

24. P. Steinhorst, *Anwendung adaptiver FEM für piezoelektrische und spezielle mechanische Probleme.* Dissertation, Fakultät für Mathematik, Technische Universit?at Chemnitz, 2009.
25. R. A. Toupin, *The elastic dielectrics.* — J. Rat. Mech. Analysis **5** (1956).
26. R. A. Toupin, *A dynamical theory of elastic dielectrics.* — Int. J. Engng. Sci. **1**, No. 1 (1963).
27. W. Voigt, *Lehrbuch der Kristall-Physik.* Leipzig: Teubner, 1910.

St.Petersburg Department
of the Steklov Institute of Mathematics,
Fontanka 27, St.Petersburg 191011, Russia
E-mail: repin@pdmi.ras.ru

Поступило 9 декабря 2024 г.

University of Zürich,
Institute of Mathematics,
8057, Zürich, Switzerland
E-mail: tatiana.samrowski@math.uzh.ch