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CHEVALLEY GROUPS OVER LAURENT POLYNOMIAL RINGS

Abstract. Let G be a simply connected Chevalley–Demazure group scheme without SL_2-factors. For any unital commutative ring R, we denote by $E(R)$ the standard elementary subgroup of $G(R)$, that is, the subgroup generated by the elementary root unipotent elements. Set $K_1^G(R) = G(R)/E(R)$. We prove that the natural map

$$K_1^G(R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) \to K_1^G\big(R((x_1))\ldots((x_n))\big)$$

is injective for any $n \geqslant 1$, if R is either a Dedekind domain or a Noetherian ring that is geometrically regular over a Dedekind domain with perfect residue fields. For $n = 1$ this map is also an isomorphism. As a consequence, we show that if D is a PID such that $SL_2(D) = E_2(D)$ (e. g. $D = \mathbb{Z}$), then

$$G(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) = E(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]).$$

This extends earlier results for special linear and symplectic groups due to A. A. Suslin and V. I. Kopeiko.

§1. Introduction

For any commutative (unital) ring R, let $E_N(R)$ denote the elementary subgroup of $\mathrm{SL}_N(R)$, i.e., the subgroup generated by elementary matrices $I + te_{ij}$, $1 \leqslant i, j \leqslant N$, $i \neq j$, $t \in R$. A. Suslin [20, Corollary 7.10] established that for any regular ring R such that $SK_1(R) = 1$, one has

$$\mathrm{SL}_N(R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m]) = E_N(R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m])$$

for any $N \geqslant \max(3, \dim(R) + 2)$ and any $n, m \geqslant 0$. The corresponding statement for $N = 2$ is well known to be wrong: for any field k one has

$$\mathrm{SL}_2(k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m]) \neq E_2(R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m]),$$

as soon as $m \geqslant 2$ or $m = 1$ and $n \geqslant 1$, see e. g. [3]; to the best of our knowledge, it is not known whether $\mathrm{SL}_2(k[x_1^{\pm 1}, x_2^{\pm 1}]) = E_2(k[x_1^{\pm 1}, x_2^{\pm 1}])$.

Let D be a principal ideal domain (PID for short). Following [9], we say that D is a *special PID*, if $\mathrm{SL}_n(D) = E_n(D)$ for all $n \geqslant 2$. (In fact, it is

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enough to require that $\mathrm{SL}_2(D) = E_2(D)$, see Lemma 2.8 below.) Clearly, a special PID D satisfies $SK_1(D) = 1$. Examples of special PIDs are Euclidean domains and localizations of 2-dimensional regular local rings at a regular parameter, including the rings $A(x)$ for a discrete valuation ring A [9, Corollaries 6.2 and 6.3]. Any localization of a special PID is also a special PID [9, Corollary 6.4]. The following lemma of V. I. Kopeiko provides one more class of examples.

Lemma 1.1 ([8, Lemma 4])**.** *If D is a special PID, then $D((x))$ is a special PID.*

The above-mentioned theorem of A. Suslin implies that for any special PID D one has

$$\mathrm{SL}_N(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m]) = E_N(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m])$$

for any $N \geqslant 3$ and any $n, m \geqslant 0$. The main result of [8] is that for any special PID D one has

$$\mathrm{Sp}_{2N}(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m]) = Ep_{2N}(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m])$$

for any $N \geqslant 2$ and any $n, m \geqslant 0$. Our aim is to extend these two results to all simply connected semisimple Chevalley–Demazure group schemes of isotropic rank $\geqslant 2$.

By a Chevalley–Demazure group scheme we mean a split reductive group scheme in the sense of [4]. These group schemes are defined over $\mathbb{Z}$. Their groups of points are usually called just Chevalley groups. We say that a Chevalley–Demazure group scheme G has *isotropic rank* $\geqslant n$ if and only if every irreducible component of its root system has rank $\geqslant n$. For any commutative ring R with 1 and any fixed choice of a pinning, or épinglage of G in the sense of [4], we denote by E the elementary subgroup functor of G. That is, $E(R)$ is the subgroup of $G(R)$ generated by elementary root unipotent elements $x_\alpha(r)$, $\alpha \in \Phi$, $r \in R$, in the notation of [2, 10], where Φ is the root system of G. If G has isotropic rank $\geqslant 2$, then $E(R)$ is independent of the choice of the pinning and normal in $G(R)$ [23, 11]. If a simply connected Chevalley–Demazure group scheme G has isotropic rank 1, it means that G is a direct product of several simply connected Chevalley–Demazure group schemes including at least one factor isomorphic to SL_2. If this is the case, then we always assume that such an isomorphism is fixed throughout every specific argument, and the corresponding direct factor of $E(R)$ is a fixed standard elementary subgroup $E_2(R)$ of $\mathrm{SL}_2(R)$; this is to take care of the fact that $E_2(R)$ is not, in general, a normal subgroup of $\mathrm{SL}_2(R)$. We also denote

$$K_1^G(R) = G(R)/E(R);$$

this is a group if the isotropic rank of G is $\geqslant 2$, and a pointed set otherwise.

Our main result is the following theorem.

Theorem 1.2. *Let D be a special PID. Let G be a simply connected Chevalley–Demazure group scheme of isotropic rank $\geqslant 2$. Then*

$$K_1^G(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_m]) = K_1^G(D) = 1$$

for any $m, n \geqslant 0$.

If $n = 0$, this result is a special case of our earlier results [16, Theorems 1.1 and 1.5] (see also Theorem 2.6 below). If $n \geqslant 1$, and if D is semilocal and contains a field, or if D is itself a field, the claim of Theorem 1.2 is already known by [17, Corollary 3.3]. In fact, it was even proved there for arbitrary simply connected semisimple reductive group schemes G of isotropic rank $\geqslant 2$ and arbitrary equicharacteristic semilocal regular rings D. However, if $n \geqslant 1$, there is no suitable local-global principle that would allow to deduce any result for non-semilocal rings from the semilocal case. Instead, in order to prove Theorem 1.2 we prove the following result.

Theorem 1.3. *Let A be a Dedekind ring, or a Noetherian ring which is geometrically regular over a Dedekind ring D with perfect residue fields. Let G be a simply connected Chevalley–Demazure group scheme of isotropic rank $\geqslant 2$. Then the natural map*

$$K_1^G(A[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) \to K_1^G\big(A((x_1))\ldots((x_n))\big)$$

is injective for any $n \geqslant 1$. If $n = 1$, this map is an isomorphism.

§2. Proof of Theorems 1.2 and 1.3

Recall that a pair (A, I), where A is a commutative ring and I is an ideal of A, is called a *Henselian pair* if I is contained in the Jacobson radical of A and for any monic polynomial $f \in A[x]$ and any factorization $\bar{f} = g_0 h_0$, where $\bar{f}$ is the image of f in $A/I[x]$ and g_0, h_0 are two monic polynomials in $A/I[x]$ generating the unit ideal, there exists a factorization $f = gh$ in $A[x]$ with g, h monic and $\bar{g} = g_0$, $\bar{h} = h_0$.

Proposition 2.1. *Let G be a simply connected Chevalley–Demazure group scheme. Let A be a commutative ring and let I be an ideal of A.*

(1) *If I is contained in the Jacobson radical of A, then the natural map $K_1^G(A) \to K_1^G(A/I)$ is injective.*
(2) *If (A, I) is a Henselian pair, then $K_1^G(A) \cong K_1^G(A/I)$.*

Proof. (1) Since $G(A) \to G(A/I)$ is a group homomorphism and $E(A) \to E(A/I)$ is surjective, it is enough to show that any element $g \in G(A)$ belongs to $E(A)$, once it is mapped to 1 under $G(A) \to G(A/I)$. Let B, B^- be a pair of standard opposite Borel subgroups of G, let U_B, U_{B^-} be their unipotent radicals, and let $T = B \cap B^-$ be their common maximal torus. The group scheme G contains an open $\mathbb{Z}$-subscheme $\Omega_B = U_B \cdot T \cdot U_{B^-}$, isomorphic to the direct product of schemes $U_B \times_{\mathbb{Z}} T \times_{\mathbb{Z}} U_{B^-}$, and this subscheme Ω_B is a principal open subscheme [2]. That is, there an element $d \in \mathbb{Z}[G]$ such that $g \in G(A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ belongs to $U_B(A)T(A)U_{B^-}(A)$ if and only if $g(d) \in A^\times$. Since G is simply connected, we have $T(A) \leqslant E(A)$. Then, if $g \in G(A)$ is mapped to $1 \in G(A/I)$, it follows that $g(d) \in A$ is mapped to $(A/I)^\times$. Since I is contained in the Jacobson radical of I, it follows that $g(d) \in A^\times$. Then $g \in \Omega_B(A)$. Since G is simply connected, we have $T(A) \leqslant E(A)$. Hence $g \in E(A)$.

(2) By (1) the map $K_1^G(A) \to K_1^G(A/I)$ is injective. Since G is affine and smooth, the map $G(A) \to G(A/I)$ is surjective [6, Th. I.8]. It follows that $K_1^G(A) \to K_1^G(A/I)$ is surjective. □

Remark 2.2. An analog of Proposition 2.1 for isotropic reductive groups G was established in [5, §7] under the additional assumption that G is defined over a semilocal ring C such that A is a C-algebra. The proof for non-split groups is much more complicated.

The following corollary generalizes [7, Lemma 1 and Remark on p. 1112] for SL_n, $n \geqslant 2$.

Corollary 2.3. *Let G be a simply connected Chevalley–Demazure group scheme. Then $K_1^G(A) = K_1^G(A[[x]])$ for any commutative ring A.*

Proof. The claim follows from Proposition 2.1 since $(A[[x]], xA[[x]])$ is a Henselian pair. □

Theorem 2.4. *Let G be a simply connected Chevalley–Demazure group scheme. Let A be a commutative ring. Then*

$$G(A((x))) = G(A[x^{\pm 1}])E(A[[x]]).$$

In particular, $K_1^G(A[x^{\pm 1}]) \to K_1^G(A((x)))$ is surjective.

Proof. By [17, Corollary 4.4] we have $G(A((x))) = G(A[x^{\pm1}])G(A[[x]])$. By Proposition 2.1 we have $G(A[[x]]) = G(A)E(A[[x]])$. The claim follows. $\square$

Lemma 2.5. *Let G be a simply connected Chevalley–Demazure group scheme of isotropic rank $\geqslant 2$. Let A be a commutative ring such that $K_1^G(A) = K_1^G(A[x])$. Then $K_1^G(A[x^{\pm1}]) = K_1^G\big(A((x))\big)$.*

Proof. By Theorem 2.4 the map $K_1^G(A[x^{\pm1}]) \to K_1^G(A((x)))$ is surjective. By [15, Corollary 3.4] this map is injective. $\square$

Let $\phi : R \to A$ be a homomorphism of commutative rings. Following [22] we will say that ϕ is *geometrically regular*, if ϕ is flat and for every prime ideal p of R, and every prime ideal q of A lying over p, $A_q/pA_q = k(p)\otimes_A A_q$ is a geometrically regular $k(p)$-algebra, i. e. if for any purely inseparable finite field extension $k'/k(p)$ the ring $k' \otimes_{k(p)} A_q/pA_q = k' \otimes_A A_q$ is regular in the usual sense. We will just say that A is a *geometrically regular R-algebra*, if the structure homomorphism $\phi : R \to A$ is clear from context.

The following theorem is a slight extension of the main result of [16].

Theorem 2.6 ([16, Theorems 1.1, 1.5])**.** *Assume that either A is a Dedekind ring, or A is a Noetherian ring geometrically regular over a Dedekind ring with perfect residue fields. Let G be a simply connected Chevalley–Demazure group scheme of isotropic rank $\geqslant 2$. Then*

$$K_1^G(A) = K_1^G(A[x_1, \dots, x_n]) \quad \textit{for any} \quad n \geqslant 1.$$

Proof. If A is a Dedekind ring, the claim is contained in [16, Theorem 1.1]. Assume that A is geometrically regular over a Dedekind ring D with perfect residue fields. Since $A[x_1, \dots, x_n]$ is also geometrically regular over D for any $n \geqslant 1$, it is enough to show that $K_1^G(A) = K_1^G(A[x])$. By the generalized Quillen-Suslin local-global principle (see [20, Theorem 3.1], [21, Corollary 4.4], [11, Lemma 17], [19, Theorem 5.4]) in order to show that $K_1^G(A) = K_1^G(A[x])$, it is enough to show that $K_1^G(A_m) = K_1^G(A_m[x])$ for every maximal ideal m of A. By the very definition of a geometrically regular ring homomorphism given above, every maximal localization A_m is geometrically regular over the corresponding prime localization D_p of D. By Popescu's theorem [13] (see [22, Theorem 1.1]), it follows that A_m is a filtered direct limit of smooth D_p-algebras. Since K_1^G commutes with filtered direct limits, we can assume that A_m is actually a localization of a smooth D_p-algebra, and thus essentially smooth over D_p. Then $K_1^G(A_m[x]) = K_1^G(A_m)$ by [16, Theorem 1.5]. $\square$

Lemma 2.7. *Let D be a Dedekind ring with perfect residue fields, and let A be a Noetherian ring which is geometrically regular over D. Then $A[[x]]$ and $A((x))$ are Noetherian rings geometrically regular over D.*

Proof. Since A is Noetherian, the ring $A[[x]]$ is flat over A, and hence its localization $A((x))$ is also flat over A. Then both these rings are flat over D. Since A is Noetherian and regular, both these rings are also Noetherian and regular. Set $B = A[[x]]$ for brevity, and denote by $\phi : D \to A$ the structure morphism of A over D. It remains to check that for every prime ideal p of D, and every prime ideal q of B lying over p, $B_q/\phi(p)B_q = k(p) \otimes_B B_q$ is a geometrically regular $k(p)$-algebra. Since $k(p)$ is perfect by assumption, it is enough to know that $B_q/\phi(p)B_q$ is a regular local ring. Now if $\phi^{-1}(q \cap A) = p = 0$, then B_q contains the field $D_p = k(p)$, and hence $k(p) \otimes_B B_q = B_q$, which is obviously regular. If $\phi^{-1}(q \cap A) = p = (\pi)$, where π is a prime element of D, then let n be a maximal ideal of A containing $\phi(\pi)$, and let $m = n + xB$ be the corresponding maximal ideal of B. Since $k(p) \otimes_A A_n = A_n/\phi(p)A_n$ is regular, $\phi(\pi)$ is a regular element of A_n. Hence $\phi(\pi)$ is a regular element of $B_m = A_n[[x]]$, and hence $B_m/\phi(p)B_m$ is also regular. Then $B_q/\phi(p)B_q = (B_m/\phi(p)B_m)_{q/\phi(p)B_m}$ is also regular. This shows that $A[[x]]$ is geometrically regular over D. Since $A((x))$ is a localization of $A[[x]]$, it follows that $A((x))$ is also geometrically regular over D. □

Proof of Theorem 1.3. By Theorem 2.6 we have

$$K_1^G(A[x_2, \dots, x_n][x_1]) = K_1^G(A[x_2, \dots, x_n]). \qquad (1)$$

Now if $n = 1$, the claim of the theorem follows from Lemma 2.5. We prove the rest of the claim by induction on n.

Since $A[x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ is a localization of $A[x_2, \dots, x_n]$, by [14, Lemma 4.6] (or [1, Lemma 4.2]) the equality (1) implies that

$$K_1^G(A[x_2^{\pm 1}, \dots, x_n^{\pm 1}][x_1]) = K_1^G(A[x_2^{\pm 1}, \dots, x_n^{\pm 1}]).$$

Hence by Lemma 2.5 the map

$$K_1^G(A[x_1^{\pm 1}, \dots, x_n^{\pm 1}]) \to K_1^G\big(A[x_2^{\pm 1}, \dots, x_n^{\pm 1}]((x_1))\big) \qquad (2)$$

is an isomorphism. Since the map (2) factors through the map

$$K_1^G(A[x_1^{\pm 1}, \dots, x_n^{\pm 1}]) \to K_1^G\big(A((x_1))[x_2^{\pm 1}, \dots, x_n^{\pm 1}]\big),$$

the latter map is also injective.

Now if A is a Dedekind ring, then $A[[x_1]]$ is a regular ring of dimension 2, and, since x_1 belongs to every maximal ideal of $A[[x_1]]$, we conclude that $A((x_1))$ is a regular ring of dimension 1, hence also Dedekind. If A is Noetherian and geometrically regular over D, by Lemma 2.7 the ring $A((x_1))$ is also Noetherian and geometrically regular over D. Summing up, the induction assumption applies to $A((x_1))$, and we are done. □

For the proof of Theorem 1.2 we need the following lemma which follows from the stability theorems of M. R. Stein and E. B. Plotkin [18, 12].

Lemma 2.8. [16, Lemma 3.1] *Let R be a Noetherian ring of Krull dimension $\leqslant 1$. If $\mathrm{SL}_2(R) = E_2(R)$, then $G(R) = E(R)$ for any simply connected Chevalley–Demazure group scheme G.*

Proof of Theorem 1.2. By Theorem 2.6 we have

$$K_1^G(D[x_1, \ldots, x_n, y_1, \ldots, y_m]) = K_1^G(D[x_1, \ldots, x_n]).$$

Since $D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = D[x_1, \ldots, x_n]_{x_1 \ldots x_n}$ is a localization of $D[x_1, \ldots, x_n]$, by [14, Lemma 4.6] (or [1, Lemma 4.2]) this implies that

$$K_1^G(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}][y_1, \ldots, y_m]) = K_1^G(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]).$$

By Theorem 1.3 the map

$$K_1^G(D[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) \to K_1^G\big(D((x_1)) \ldots ((x_n))\big)$$

is injective. By Lemma 1.1 $A = D((x_1)) \ldots ((x_n))$ is a special PID. By definition, it means that $\mathrm{SL}_2(A) = E_2(A)$. Then by Lemma 2.8 we have $K_1^G(A) = 1$ for any simply connected Chevalley–Demazure group scheme of isotropic rank $\geqslant 2$. This finishes the proof. □

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