Записки научных
семинаров ПОМИ
Том 538, 2024 г.

I. A. Panin

NICE TRIPLES IN THE DVR CONTEXT

Abstract. A theory of standard triples was invented by V. Voevodsky in [14] to construct the triangulated category of motives. Being inspired by that theory a theory of nice triples was invented in [7] in order to attack the Grothendieck–Serre conjecture and related problem. However both the mentioned theories were developed for smooth varieties over a field. In the present paper a theory of nice triples is developed for smooth schemes over a DVR. Theorem 1.4 is used in [12] as one of a major step in the proof of the Grothendieck–Serre conjecture in the constant mixed characteristic case.

§1. Introduction

In the present paper we work with schemes over a DVR ring D of mixed characteristic. The major our interest is the case of D having a *finite residue field*. The main aim of the present paper is to prove Theorem 1.4 (= Theorem 6.3). Our proof is based on theory of nice triples atapted to V-smooth schemes, where $V = \text{Spec } D$ is such that the residue field $k(v)$ at its closed point v is finite. If $X = \mathbf{P}^n_V$, then this result is proved in [3, Theorem 1.2] using a different approach.

If D has an *infinite residue field* it is not necessary to use Theorem 1.4 to approach the Grothendieck–Serre conjecture (see [12, Proof of Theorem 1.7]). This is *the basic reason* to focus on the case of DVR ring D with a finite residue field in the present paper.

We expect that Theorem 1.4 is true (and probably has a straight forward proof) in the infinite residue field case.

Notation 1.1. In this paper D is a DVR ring of mixed characteristic. We write V for Spec D, $v \in V$ for the closed point of V, $\eta \in V$ for the generic point of V. For each V-scheme S write S_v for the closed fibre of S and S_η for its generic fibre. It is supposed in this paper that the **residie field** $k(v)$ is **finite**.

Key words and phrases: nice triple, mixed characteristic, Grothendieck–Serre conjecture, smooth schemes.

Agreement 1.2. (Condition $(*)$). Let $M \subseteq \mathbf{A}^n_V$ be a closed subset. We will say that M satisfies the condition $(*)$ iff $\operatorname{codim}_{\mathbf{A}^n_v}(M_v) \geqslant 2$ and $\operatorname{codim}_{\mathbf{A}^n_\eta}(M_\eta) \geqslant 2$.

Notation 1.3. We will write $\mathbf{A}^{\circ;n}_V$ for any open subscheme of $\mathbf{A}^n_V$ of the form $\mathbf{A}^n_V - M$, where $M \subseteq \mathbf{A}^n_V$ is closed subjecting the condition $(*)$. Define open subschemes $\mathbf{P}^{\circ;n}_V$ of $\mathbf{P}^n_V$ similarly.

If $S \subseteq \mathbf{A}^n_V$ is a closed subset (say, a divisor), then for each open subset in $\mathbf{A}^n_V$ of the form $\mathbf{A}^{\circ;n}_V$ write S° for $S \cap \mathbf{A}^{\circ;n}_V$ and call S° the trace of S in $\mathbf{A}^{\circ;n}_V$.

Similarly, for each closed T in $\mathbf{P}^n_V$ and each open subset in $\mathbf{P}^n_V$ of the form $\mathbf{P}^{\circ;n}_V$ write T° for $T \cap \mathbf{P}^{\circ;n}_V$ and call T° the trace of T in $\mathbf{P}^{\circ;n}_V$.

Theorem 1.4 (Geometric)**.** *Let $X = \mathbf{P}^{\circ;n}_V$, that is the closed subset $M = \mathbf{P}^n_V - X$ enjoys the condition (*) as in Agreement* 1.2. *Let $x \in X_v$ be its closed point. Let $Z \subseteq \mathbf{P}^n_V$ be a divisor not containing $\mathbf{P}^n_v$ such that $x \in Z_v$. Write $\mathcal{O} = \mathcal{O}_{X,x}$ and $U = \operatorname{Spec}(\mathcal{O})$. Then there is a monic polynomial $h \in \mathcal{O}[t]$, a commutative diagram of V-schemes of the form*

$$\begin{array}{ccccc} (\mathbf{A}^1 \times U)_h & \xleftarrow{\ \tau_h\ } & Y_h := Y_{\tau^*(h)} & \xrightarrow{\ (p_X)|_{Y_h}\ } & X - Z \\ \downarrow{\scriptstyle \text{inc}} & & \downarrow{\scriptstyle \text{inc}} & & \downarrow{\scriptstyle \text{inc}} \\ (\mathbf{A}^1 \times U) & \xleftarrow{\ \tau\ } & Y & \xrightarrow{\ p_X\ } & X \end{array} \tag{1}$$

and a V-morphism $\delta : U \to Y$, which enjoy the following conditions

(i) *the left hand side square is an elementary* **distinguished** *square in the category of affine U-smooth schemes in the sense of* [5, Definition 3.1.3];

(ii) *the morphism δ is a section of the morphism $pr_U \circ \tau$ and $p_X \circ \delta = can : U \to X$, where can is the canonical morphism;*

(iii) *$\tau \circ \delta = i_0 : U \to \mathbf{A}^1 \times U$ is the zero section of the projection* $\operatorname{pr}_U : \mathbf{A}^1 \times U \to U$*;*

Our proof is based on a modification of the theory of nice triples invented in [7] and inspired by the Voevodsky theory of standard triples [14]. The proof is organized as follows.

(i) the Artin notion of elementary fibration is recalled and two results are stated and proved in Section 2 (point out that Theorem 2.4 is heavily based on [4, Theorem 1.4]);

(ii) the notion of a nice triple over a regular local base and certain related notions are recalled in Section 3;

(iii) Theorems 4.3 and 4.2 are stated and proved in Section 4, which allows to prove Theorem 6.1 and its Corollary 6.2 in Section 6;

(iv) a special basic nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over $U = \text{Spec } \mathcal{O}_{X,x}$ is constructed in Section 5 (see Proposition 5.1);

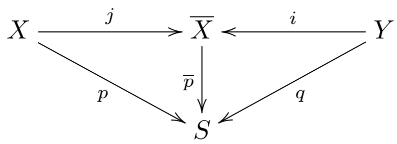
(v) Theorem 6.3 is derived from Corollary 6.2 in Section 6;

(vi) finally Theorem 1.4 is derived from Theorem 6.3 in Section 6.

§2. Some elementary fibration

Based on [4, Theorem 1.4] we extend in Theorem 2.4 a result of M. Artin from [1] concerning existence of nice neighborhoods. The following notion is introduced by Artin in [1, Exp. XI, Déf. 3.1].

Definition 2.1. *An elementary fibration is a morphism of schemes $p : X \to S$ which can be included in a commutative diagram*



(2)

of morphisms satisfying the following conditions:

(i) *j is an open immersion dense at each fibre of $\overline{p}$, and $X = \overline{X} - Y$*;

(ii) *$\overline{p}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one*;

(iii) *q is finite étale all of whose fibres are non-empty.*

Agreement 2.2. If $p : X \to S$ is an elementary fibration, and the diagram (2) enjoys the properties (i), (ii), (iii) in Definition 2.1, then the diagram (2) is called a *diagram of the elementary fibration $q\colon Y \to S$.*

Let $Z \subset X$ be a closed subscheme. A morphism $p\colon X \to S$ of schemes is called an *elementary Z-fibration* if it is an elementary fibration and the morphism $p|_Z\colon Z \to S$ is finite. In this case, we say that (2) is *a diagram of the elementary Z-fibration $p\colon X \to S$.*

Remark 2.3. Clearly, an elementary fibration is an almost elementary fibration in the sense of [7, Definition 2.1].

Now take again $X = \mathbf{P}_V^{\circ,n}$, Z, $x \in Z_v$ as in Theorem 1.4. A straightforward analysis of the proof of [4, Theorem 1.4] shows that the following result is true (it is a very partial extension of Artin's result [1, Example XI, Proposition 3.3] to schemes over a DVR.)

Theorem 2.4. *There is an open* $S \subset \mathbb{A}^{n-1}_V$*, an open neighborhood* $\dot{X}_S \subset \mathbf{P}^{\circ,n}_V = X$ *of the point* $x \in \mathbf{P}^{\circ,n}_v \subset \mathbf{P}^{\circ,n}_V = X$ *and an elementary fibration* $\dot{q}_S : \dot{X}_S \to S$ *and a commutative diagram of* S*-schemes*

$$\begin{array}{ccccc} \dot{X}_S & \xrightarrow{j_S} & \hat{X}_S & \xleftarrow{i} & \mathbb{W}_S \\ & {\scriptstyle \dot{q}_S} \searrow & \downarrow {\scriptstyle \hat{q}_S} & \swarrow {\scriptstyle pr_S} & \\ & & S & & \end{array} \tag{3}$$

which is a diagram of the elementary $\dot{Z}_S$*-fibration* ($\dot{Z}_S := Z \cap \dot{X}_S$) *and* $x \in (\dot{Z}_S)_v$.

Remark 2.5. Let $\dot{q}_S : \dot{X}_S \to S$, $\dot{Z}_S$, $x \in (\dot{Z}_S)_v$ as in Theorem 2.4, $s = \dot{q}_S(x) \in S \subset \mathbf{A}^{n-1}_V$ be the point in S. Put $\mathcal{S} = \operatorname{Spec} \mathcal{O}_{S,s} = \operatorname{Spec} \mathcal{O}_{\mathbf{A}^{n-1}_V, s}$. Taking the base change of the diagram (3) by means of the embedding $\mathcal{S} \hookrightarrow S$ we get an elementary $\dot{Z}_{\mathcal{S}}$-fibration $\dot{q}_{\mathcal{S}} : \dot{X}_{\mathcal{S}} \to \mathcal{S}$, where $\dot{Z}_{\mathcal{S}} := Z \cap \dot{X}_{\mathcal{S}}$ and a commutative diagram of $\mathcal{S}$-schemes

$$\begin{array}{ccccc} \dot{X}_{\mathcal{S}} & \xrightarrow{j_{\mathcal{S}}} & \hat{X}_{\mathcal{S}} & \xleftarrow{i_{\mathcal{S}}} & \mathbb{W}_{\mathcal{S}} \\ & {\scriptstyle \dot{q}_{\mathcal{S}}} \searrow & \downarrow {\scriptstyle \hat{q}_{\mathcal{S}}} & \swarrow {\scriptstyle \mathrm{pr}_{\mathcal{S}}} & \\ & & \mathcal{S} & & \end{array} \tag{4}$$

which is a diagram of the elementary $\dot{Z}_{\mathcal{S}}$-fibration $\dot{q}_{\mathcal{S}} : \dot{X}_{\mathcal{S}} \to \mathcal{S}$. Clearly, x is in $(\dot{Z}_{\mathcal{S}})_v$.

Similarly to [7, Proposition 2.4] we have the following result.

Proposition 2.6. *For the diagram* (4) *of the elementary* $\dot{Z}_{\mathcal{S}}$*-fibration one can find a commutative diagram of* $\mathcal{S}$*-schemes*

$$\begin{array}{ccccc} \dot{X}_{\mathcal{S}} & \xrightarrow{j_{\mathcal{S}}} & \hat{X}_{\mathcal{S}} & \xleftarrow{i_{\mathcal{S}}} & \mathbb{W}_{\mathcal{S}} \\ {\scriptstyle \pi} \downarrow & & \downarrow {\scriptstyle \bar{\pi}} & & \downarrow \\ \mathbf{A}^1 \times \mathcal{S} & \xrightarrow{\mathrm{in}} & \mathbf{P}^1 \times \mathcal{S} & \xleftarrow{i} & \{\infty\} \times \mathcal{S} \end{array} \tag{5}$$

such that the left-hand side square is Cartesian, the morphism $\bar{\pi}$ *is finite surjective and* π *is such that* $\mathrm{pr}_{\mathcal{S}} \circ \pi = \dot{q}_{\mathcal{S}}$*, where* $\mathrm{pr}_{\mathcal{S}}$ *is the projection* $\mathbf{A}^1 \times \mathcal{S} \to \mathcal{S}$.

In particular, $\pi : \dot{X}_{\mathcal{S}} \to \mathbf{A}^1 \times \mathcal{S}$ is a finite surjective morphism of $\mathcal{S}$-schemes, where $\dot{X}_{\mathcal{S}}$ and $\mathbf{A}^1 \times \mathcal{S}$ are regarded as $\mathcal{S}$-schemes via the morphism $\dot{q}_{\mathcal{S}}$ and the projection $\mathrm{pr}_{\mathcal{S}}$, respectively. Note that the scheme $\dot{X}_{\mathcal{S}}$ is affine, since the morphism π is finite.

Remark 2.7. Let $\dot{Z}_{\mathcal{S}} \subset \dot{X}_{\mathcal{S}}$ be the divisor and $x \in \dot{Z}_{\mathcal{S}}$ be the point as in Remark 2.5. Then the morphism $\pi|_{\dot{Z}_{\mathcal{S}}} : \dot{Z}_{\mathcal{S}} \to \mathbf{A}^1 \times \mathcal{S}$ is finite since the morphism π is finite. Regarding the closed subset $\pi(\dot{Z}_{\mathcal{S}})$ as a reduced closed subscheme in $\mathbf{A}^1_{\mathcal{S}}$ one can find a monic polynomial $g \in \Gamma(\mathcal{S}, \mathcal{O}_{\mathcal{S}})[t]$ such that the closed subscheme $\pi(\dot{Z}_{\mathcal{S}})$ of $\mathbf{A}^1_{\mathcal{S}}$ is defined by the principle ideal $(g) \subset \Gamma(\mathcal{S}, \mathcal{O}_{\mathcal{S}})[t]$. Put $\mathrm{f} = \pi^*(g) \in \Gamma(\dot{X}_{\mathcal{S}}, \mathcal{O}_{\dot{X}_{\mathcal{S}}})$. Clearly, the f vanishes at the point x and the $\mathcal{S}$-morphism $\dot{q}_{\mathcal{S}}|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to \mathcal{S}$ is finite. Also $\dot{Z}_{\mathcal{S}} \subset \{\mathrm{f} = 0\}$.

The following result is pretty closed to Theorem 1.4. It will be proved below (see Theorem 6.3) and it will be used to prove Theorem 1.4.

Theorem 2.8 (Intermediate)**.** *Let $\dot{q}_{\mathcal{S}} : \dot{X}_{\mathcal{S}} \to \mathcal{S}$ be the elementary $\dot{Z}_{\mathcal{S}}$-fibration and $x \in \dot{Z}_{\mathcal{S}}$ be the point as in Remark* 2.5. *Let $U = \mathrm{Spec}(\mathcal{O}_{\dot{X}_{\mathcal{S}},x})$. Then there is a monic polynomial $h \in \mathcal{O}_{\dot{X}_{\mathcal{S}},x}[t]$, a commutative diagram of schemes with an irreducible affine U-smooth Y*

$$\begin{array}{ccccc}
(\mathbf{A}^1 \times U)_h & \xleftarrow{\tau_h} & Y_h := Y_{\tau^*(h)} & \xrightarrow{(\dot{p}_X)|_{Y_h}} & (\dot{X}_{\mathcal{S}})_{\mathrm{f}} \\
\downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} \\
(\mathbf{A}^1 \times U) & \xleftarrow{\tau} & Y & \xrightarrow{\dot{p}_X} & \dot{X}_{\mathcal{S}}
\end{array} \qquad (6)$$

and a morphism $\delta : U \to Y$ subject to the following conditions:

(i) *the left hand side square is an elementary* **distinguished** *square in the category of affine U-smooth schemes in the sense of* [5, Definition 3.1.3];

(ii) *the morphism δ is a section of the morphism $pr_U \circ \tau$ and $\dot{p}_X \circ \delta = \dot{can} : U \to \dot{X}_{\mathcal{S}}$, where $\dot{can}$ is the canonical embedding;*

(iii) *$\tau \circ \delta = i_0 : U \to \mathbf{A}^1 \times U$ is the zero section of the projection $\mathrm{pr}_U : \mathbf{A}^1 \times U \to U$;*

§3. Nice triples over a local regular base

Let W be a local regular scheme with the closed point w, $\mathcal{O} = \Gamma(W, \mathcal{O}_W)$.

Definition 3.1. *A* nice triple *over W consists of the data*

(i) *a smooth morphism $q_W : \mathcal{X} \to W$, where $\mathcal{X}$ is irreducible,*
(ii) *an element $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,*
(iii) *a section Δ of the morphism q_W,*

subject to the following conditions:

(a) *each irreducible component of each fibre of the morphism q_W has dimension one,*
(b) *the module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite as a $\mathcal{O}$-module,*
(c) *there exists a finite surjective W-morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times W$,*
(d) $\Delta^*(f) \neq 0 \in \Gamma(W, \mathcal{O}_W) = \mathcal{O}$.

There are many choices of the morphism Π. Any of them is regarded as assigned to the nice triple.

Definition 3.2. *A* morphism *between two nice triples over W*

$$(q' : \mathcal{X}' \to W, f', \Delta') \to (q : \mathcal{X} \to W, f, \Delta)$$

is an étale morphism of W-schemes $\theta : \mathcal{X}' \to \mathcal{X}$ such that

(1) $q'_W = q_W \circ \theta$,
(2) $f' = \theta^*(f) \cdot h'$ *for an element* $h' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,
(3) $\Delta = \theta \circ \Delta'$.

The item (2) implies that $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a finite $\mathcal{O}$-module.

Note also that no conditions are imposed on the interrelation of Π' and Π.

Definition 3.3. *A nice triple $(q_W : \mathcal{X} \to W, \Delta, f)$ over W is called* special nice triple *if the closed point of $\Delta(W)$ is contained in the set of closed points of $\{f = 0\}$.*

Remark 3.4. Let $(\mathcal{X}, f, \Delta)$ be a special nice triple over W and let $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ be a morphism between nice triples over W. Then the triple $(\mathcal{X}', f', \Delta')$ is a special nice triple over W.

§4. Two crucial results

Over the rest of the paper we suppose that the residue field $k(w)$ is **finite**. Let us state two crucial results which are used in Section 6 to prove Theorem 6.1. Their proofs are given in the present section. If W as in Definition 3.1 then *for any W-scheme Y and the closed point $w \in W$* set

$$Y_w = Y \times_W w.$$

For a finite set A denote by $\sharp A$ the cardinality of A.

Definition 4.1. *Let $(\mathcal{X}, f, \Delta)$ be a special nice triple over the W. Recall that the residue field $k(w)$ at the closed point $w \in W$ is supposed to be finite. We say that the triple $(\mathcal{X}, f, \Delta)$ is $(*)$-special nice triple (or shortly $(*)$-special), if the following conditions hold:*

(1*) *for $\mathcal{Z} = \{f = 0\} \subset \mathcal{X}$ and for any closed point $w \in W$, any integer $d \geqslant 1$ one has*

$$\sharp\{z \in \mathcal{Z}_w | \deg[k(z) : k(w)] = d\} \leqslant \sharp\{x \in \mathbf{A}^1_w | \deg[k(x) : k(w)] = d\};$$

(2*) *for the vanishing locus $\mathcal{Z}$ of f and for the closed point $w \in W$ the point $\Delta(w) \in \mathcal{Z}_w$ is the only $k(w)$-rational point of $\mathcal{Z}_w = \mathcal{Z} \times_W w$.*

The following theorem is very much similar to [8, Theorem 3.8]. In more details. In [8, Theorem 3.8] the base U is an essentially smooth semi-local scheme over a field. In Theorem 4.2 below the base W is an essentially smooth local scheme over the DVR scheme V. Clearly, as U, so W are regular semi-local schemes. This is the reason why as the statement, so the proof of Theorem 4.2 repeats literally as the statement, so the proof of Theorem [8, Theorem 3.8].

Theorem 4.2. *Let W and $\mathcal{O}$ be as in Section 3. Let $(q'_W : \mathcal{X}' \to W, f', \Delta')$ be a $(*)$-special nice triple over W. Let $\mathcal{Z}'$ be the closed subscheme of $\mathcal{X}$ defined by* **the principal ideal** (f'). *Then there exists a finite surjective morphism*

$$\mathbf{A}^1 \times W \xleftarrow{\sigma} \mathcal{X}'$$

of W-schemes which enjoys the following properties:

(a) *the morphism $\mathbf{A}^1 \times W \xleftarrow{\sigma|_{\mathcal{Z}'}} \mathcal{Z}'$ is a closed embedding;*
(b) *σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta'(W)$;*
(c) *$\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$ scheme theoretically and $\mathcal{Z}'' \cap \Delta'(W) = \varnothing$;*
(d) *$\sigma^{-1}(\{0\} \times W) = \Delta'(W) \coprod \mathcal{D}$ scheme theoretically and $\mathcal{D} \cap \mathcal{Z}' = \varnothing$;*
(e) *for $\mathcal{D}_1 := \sigma^{-1}(\{1\} \times W)$ one has $\mathcal{D}_1 \cap \mathcal{Z}' = \varnothing$.*
(f) *there is a monic polynomial $h \in \mathcal{O}[t]$ such that*

$$(h) = \mathrm{Ker}[\mathcal{O}[t] \xrightarrow{-\circ\sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')]$$

The following theorem is a slight extension of [8, Theorem 3.9]

Theorem 4.3. *Let W be as in Section 3. Let $(\mathcal{X}, f, \Delta)$ be a special nice triple over W. Then there exists a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples* **over** *W such that $(\mathcal{X}', f', \Delta')$ is a $(*)$-special nice triple over W.*

To prepare the proof of Theorem 4.3 we use Notation as in [8, Section 4] and the notation of the present section. Particularly, the scheme S below in this prove has nothing common with the one from Section 1. We also use Lemma 4.4 stated right below.

Let W be as in Definition 3.1. Let S be an **irreducible** regular semi-local V-scheme strictly flat over V and $p : S \to W$ be a **strictly flat** V-morphism. Let $T \hookrightarrow S$ be a closed sub-scheme of S such that the restriction $p|_T : T \to W$ is an isomorphism. Let $\delta : W \to T$ be the inverse to $p|_T$. We will assume below that $\dim(T) < \dim(S)$, where dim is the Krull dimension.

Lemma 4.4 ([8, Appendix A])**.** *Suppose that all the closed points of S have* **finite** *residue fields. Suppose that for the closed point $w \in W$ the scheme S_w is a* **semi-local Dedekind scheme.** *Then there exists a finite étale morphism $\rho : S' \to S$ (with an irreducible scheme S') and a section $\delta : T \to S'$ of ρ over T such that the following holds*

(1) *for the closed point $w \in W$ let $w' \in T$ be a unique point such that $p(w') = w$, then the point $\delta(w') \in S'_w$ is the only $k(w)$-rational point of S'_w,*
(2) *for the closed point $w \in W$ and any integer $d \geqslant 1$ one has*

$$\sharp\{z \in S'_w | [k(z) : k(w)] = d\} \leqslant \ \sharp\{x \in \mathbf{A}^1_w | [k(x) : k(w)] = d\}.$$

Proof of Theorem 4.3. In this proof given our special nice triple $(\mathcal{X}, f, \Delta)$ over W we construct a new one $(\mathcal{X}', f', \Delta')$ over W and a morphism

$$\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$$

of nice triples over W, which has the following property:
if $\Pi : \mathcal{X} \to \mathbf{A}^1 \times W$ is a finite surjective morphism assinged to the nice triple, and $\mathcal{Y} = \Pi^{-1}(\Pi(\mathcal{Z} \cup \Delta(W)))$ is the closed subset in $\mathcal{X}$, and $y_1, \dots, y_m$ are all its closed points, and $S = \mathrm{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \dots, y_m})$, and $S' = \theta^{-1}(S)$, *then S' is étale and finite over S, irreducible and the set of closed points of the closed subset* $\{f' = 0\}$ in $\mathcal{X}'$ *is contained in the set of closed points of* S'.

Let $\Pi : \mathcal{X} \to \mathbf{A}^1 \times W$ be the finite surjective W-morphism consider the following diagram

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & & \downarrow & & \\ \mathcal{X}-\mathcal{Z} & \hookrightarrow & \mathcal{X} & \xrightarrow{\Pi} & \mathbf{A}^1 \times W \\ & & {\scriptstyle\Delta}\uparrow\downarrow{\scriptstyle q_W} & & \\ & & W & & \end{array}$$

Here and in the Construction 4.5 below $\mathcal{Z}$ is the closed **subset** defined by the equation $f = 0$. By the assumption, $\mathcal{Z}$ is finite over W.

Construction 4.5. (compare with [6, the proof of Lemma 8.1]) Let W be as in Definition 3.1 and let $(\mathcal{X}, f, \Delta)$ be a nice triple over W. Since Δ is a section of q_W, hence $\Delta(W)$ is a closed subset in $\mathcal{X}$. Let $\Pi : \mathcal{X} \to \mathbf{A}^1 \times W$ a finite surjective morphism of W-schemes, which exists, since $(\mathcal{X}, f, \Delta)$ is a nice triple over W. Let $\mathcal{Y} = \Pi^{-1}(\Pi(\mathcal{Z} \cup \Delta(W)))$ be the closed subset in $\mathcal{X}$. Since $\mathcal{Z}$ and $\Delta(W)$ are both finite over W and since Π is a finite morphism of W-schemes, $\mathcal{Y}$ is also finite over W. Denote by $y_1, \dots, y_m$ its closed points and let $S = \mathrm{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \dots, y_m})$. Since $\mathcal{Y}$ is also finite over W it is closed in $\mathcal{X}$. Since $\mathcal{Y}$ is contained in S it is also closed in S. Set $T = \Delta(W) \subseteq S$.

So, **this construction** converts **our** nice triple $(\mathcal{X}, f, \Delta)$ and **our** morphism Π to certain data $(\mathcal{Z}, \mathcal{Y}, S, T)$, where $\mathcal{Z}, \mathcal{Y}, T$ are closed subsets as in $\mathcal{X}$, so in S. Point out that S is a semi-local subscheme in $\mathcal{X}$. Particularly, the set of all closed points of $\mathcal{Z}$ *is contained* in the set of all closed points of S.

The closed points of S are the points $y_1, ..., y_m$. For each index i the residue field extension $k(y_i)/k(w)$ is finite. The residue field $k(w)$ is **finite**. Thus, for each index i the residue field $k(y_i)$ is **finite**. So, we are in a position to apply Lemma 4.4.

Let $p = q_W|_S : S \to W$ and $\delta = \Delta : W \to S$. Applying Lemma 4.4 to S, T and δ we get S', $\rho : S' \to S$, and $\delta : T \to S'$ subjecting to the conditions (1) and (2) from the lemma 4.4. Recall that $\rho : S' \to S$ is a finite étale morphism (with an irreducible scheme S') and δ is a section of ρ over $T \subseteq S$.

Applying the second part of the construction [8, Construction 4.2] and also [8, Proposition 4.3] to the special nice triple $(\mathcal{X}, f, \Delta)$, the finite surjective morphism Π and to the finite étale morphism ρ and to its section δ over T we get

(i′) the nice triple $(\mathcal{X}', f', \Delta')$ over W;

(ii′) the morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ between the special nice triples;

(iii′) the equality $f' = (\theta)^*(f)$;

(iv′) the vanishing locus $\mathcal{Z}'$ of f' on $\mathcal{X}'$ such that its set of closed points is contained in the set of closed points of the subscheme S'.

The property (ii′) shows particularly that the triple $(\mathcal{X}', f', \Delta')$ is a special nice triple.

The properties (1) and (2) of the W-scheme S' (see Lemma 4.4) show that the special nice triple $(\mathcal{X}', f', \Delta')$ over W is $(*)$-special. That follows from the property (iv′) mentioned just above.

This completes the proof of the theorem. □

§5. Basic nice triple

Let $X = \mathbf{P}^{\circ,n}_V$. That is the closed subset $M = \mathbf{P}^n_V - X$ enjoys the condition (*) as in Agreement 1.2. Let $x \in X_v$ be its closed point. Let $Z \subseteq \mathbf{P}^n_V$ be a divisor not containing $\mathbf{P}^n_v$ such that $x \in Z_v$. By Theorem 2.4, Remark 2.5, Proposition 2.6 and Remark 2.7 there are the following preliminary data

(a) the affine open S in $\mathbf{A}^{n-1}_V$;

(b) the affine open $\dot{X}_S$ in $X = \mathbf{P}^{\circ,n}_V$, its closed subset $\dot{Z}_S = Z \cap \dot{X}_S$, the point $x \in (\dot{Z}_S)_v$;

(c) the elementary $\dot{Z}_S$-fibration $\dot{q}_S : \dot{X}_S \to S$ and the point $s = \dot{q}_S(x) \in S \subset \mathbf{A}^{n-1}_V$;

(d) the subscheme $\mathcal{S} = \operatorname{Spec} \mathcal{O}_{S,s} = \operatorname{Spec} \mathcal{O}_{\mathbf{A}^{n-1}_V, s}$ of the scheme S;

and

(i) the elementary $\dot{Z}_{\mathcal{S}}$-fibration $\dot{q}_{\mathcal{S}} : \dot{X}_{\mathcal{S}} \to \mathcal{S}$, which is the base change of the $\dot{q}_S$ by means of the inclusion $\mathcal{S} \hookrightarrow S$ and the point x is in $(\dot{Z}_{\mathcal{S}})_v$;

(ii) the finite surjective $\mathcal{S}$-morphism $\pi : \dot{X}_{\mathcal{S}} \to \mathbf{A}^1 \times \mathcal{S}$;

(iii) the function $\mathrm{f} \in \Gamma(\dot{X}_{\mathcal{S}}, O_{\dot{X}_{\mathcal{S}}})$ with $\dot{Z}_{\mathcal{S}} \subset \{\mathrm{f} = 0\}$ (so, the f vanishes at the point x);

(iv) the locus $\{\mathrm{f} = 0\} \cap \dot{X}_{\mathcal{S}}$ is $\mathcal{S}$-finite.

Put $U := \operatorname{Spec} \mathcal{O}_{X,x} = \operatorname{Spec} \mathcal{O}_{\dot{X}_{\mathcal{S}},x} \subset \dot{X}_{\mathcal{S}}$. We will regard U as an $\mathcal{S}$-scheme via the morphism $U \hookrightarrow X_{\mathcal{S}} \xrightarrow{\dot{q}_{\mathcal{S}}} \mathcal{S}$. The following proposition provides us with a basic nice triple over U.

Proposition 5.1. *Let $\dot{X}_{\mathcal{S}}$, $x \in \dot{X}_{\mathcal{S}}$, $\mathrm{f} \in \Gamma(\dot{X}_{\mathcal{S}}, O_{\dot{X}_{\mathcal{S}}})$ be as just above in this Section. Then one can construct a special nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U and an essentially smooth morphism $q_{\dot{X}_{\mathcal{S}}} : \mathcal{X} \to \dot{X}_{\mathcal{S}}$ such that $q_{\dot{X}_{\mathcal{S}}} \circ \Delta = \dot{c}an$, $f = q^*_{\dot{X}_{\mathcal{S}}}(\mathrm{f})$ (here $\dot{c}an : U \hookrightarrow \dot{X}_{\mathcal{S}}$ is the inclusion).*

Proof. Consider the V-schemes U and $\mathcal{X} = U \times_{\mathcal{S}} \dot{X}_{\mathcal{S}}$, the projections $q_U : \mathcal{X} \to U$ and $q_{\dot{X}_{\mathcal{S}}} : \mathcal{X} \to \dot{X}_{\mathcal{S}}$. Put $f = q^*_{\dot{X}_{\mathcal{S}}}(\mathrm{f}) \in \Gamma(\mathcal{X}, O_{\mathcal{X}})$. Consider also the diagonal section $\Delta : U \to \mathcal{X}$ of the projection q_U. Check that

$$(q_U : \mathcal{X} \to U, \Delta, f)$$

is a special nice triple over U. First note that the scheme $\mathcal{X}$ is regular. Thus, it is a disjoint union of its irreducible components. If $\mathcal{X}$ is not irreducible, then replace it with its irreducible component containing $\Delta(U)$. Nevertheless write $\mathcal{X}$ for that specific component and write f for $f|_{\text{that component}}$. Now we are ready indeed to check that the $(q_U : \mathcal{X} \to U, \Delta, f)$ is a special nice triple over U.

First check requirements on the data: indeed, the scheme $\mathcal{X}$ is irreducible, the morphism q_U is smooth, Δ is its section, $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Secondly check that these data enjoys conditions (a)–(d). The morphism q_U is the base change of the morphism $\dot{q}_{\mathcal{S}} : \dot{X}_{\mathcal{S}} \to \mathcal{S}$ and the morphism $\dot{q}_{\mathcal{S}}$ is our elementary fibration. Thus the q_U enjoys the condition (a). The $\mathcal{S}$-morphism $\dot{q}_{\mathcal{S}}|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to \mathcal{S}$ is finite. Thus the function f enjoys the condition (b). Now check the condition (d). Since $q_{\dot{X}_{\mathcal{S}}} \circ \Delta = \dot{c}an : U \hookrightarrow \dot{X}_{\mathcal{S}}$ one has $\Delta^*(f) = \mathrm{f}|_U$. It is sufficient to show that $\mathrm{f} \neq 0$ in $\Gamma(\dot{X}_{\mathcal{S}}, O_{\dot{X}_{\mathcal{S}}})$. This is the case since the closed subset $\{\mathrm{f} = 0\}$ of $\dot{X}_{\mathcal{S}}$ is finite over $\mathcal{S}$ (and hence this set does not coincides with $\dot{X}_{\mathcal{S}}$). The condition (d) is checked. Check the condition (c). Consider the finite surjective morphism $\pi : \dot{X}_{\mathcal{S}} \to \mathbf{A}^1 \times \mathcal{S}$ of $\mathcal{S}$-schemes as in Proposition 2.6. Then the U-morphism $\Pi = id_U \times_{\mathcal{S}} \pi : \mathcal{X} \to U \times_{\mathcal{S}} \mathbf{A}^1_{\mathcal{S}} = \mathbf{A}^1_U$ is finite surjective as the base change of the morphism π. So, the triple $(q_U : \mathcal{X} \to U, \Delta, f)$ is a nice triple over U. It is *a special* nice triple, since the function f vanishes at the point x.

The morphism $q_{\dot{X}_{\mathcal{S}}} : \mathcal{X} \to \dot{X}_{\mathcal{S}}$ is the base change of the composite morphism $U \hookrightarrow \dot{X}_{\mathcal{S}} \xrightarrow{\dot{q}_S} S$. Thus, the $q_{\dot{X}_{\mathcal{S}}}$ is essentially smooth. The equality $q_{\dot{X}_{\mathcal{S}}} \circ \Delta = \dot{can}$ is obvious. Finally, $f = q_X^*(\mathrm{f})$ by the very definition of f. $\square$

§6. Theorem 6.1 and Theorems 2.8 and 1.4

The main aim of this section is *to prove* Theorem 1.4. Theorem 6.1 below is a purely geometric one. If the residue field $k(v)$ consists of two elements and if the closed point of $u \in U$ is $k(v)$-rational and the scheme $\mathcal{Z}'$ below is such that its closed fibre $\mathcal{Z}'_u$ contains three $k(v)$-rational points, then *there are no* closed embedding $\mathcal{Z}'$ into $\mathbf{A}^1 \times U$. So, one of the main problem in the proof of the next theorem is to find an $\mathcal{X}'$, a morphism $q'_{X_{\mathcal{S}}}$ and a function f' to overcome the mentioned difficulties.

Theorem 6.1. *Let* $\dot{X}_{\mathcal{S}}$, $x \in \dot{X}_{\mathcal{S}}$, $\mathrm{f} \in \Gamma(\dot{X}_{\mathcal{S}}, \mathcal{O}_{\dot{X}_{\mathcal{S}}})$ *be as in Proposition* 5.1. *Let* $U = \mathrm{Spec}(\mathcal{O}_{\dot{X}_{\mathcal{S}},x})$. *Then there is a diagram of the form*

$$\begin{array}{ccccc} \mathbf{A}^1 \times U & \xleftarrow{\ \sigma\ } & \mathcal{X}' & \xrightarrow{\ q'_{X_{\mathcal{S}}}\ } & \dot{X}_{\mathcal{S}} \\ & \searrow^{\mathrm{pr}_U} & q'_U \downarrow \uparrow \Delta' & \nearrow^{\mathrm{can}} & \\ & & U & & \end{array} \qquad (7)$$

with an irreducible affine scheme $\mathcal{X}'$, *a smooth morphism* q'_U, *a finite surjective* U*-morphism* σ, *an essentially smooth morphism* $q'_{X_{\mathcal{S}}}$ *and* $f' = (q'_{X_{\mathcal{S}}})^*(\mathrm{f}) \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$, *which enjoys the following properties:*

(a) *if* $\mathcal{Z}'$ *is the closed subscheme of* $\mathcal{X}'$ *defined by the principal ideal* (f'), *then the morphism* $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathbf{A}^1 \times U$ *is a closed embedding and the morphism* $q'_U|_{\mathcal{Z}'} : \mathcal{Z}' \to U$ *is finite;*

(a′) $q'_U \circ \Delta' = id_U$ *and* $q'_{X_{\mathcal{S}}} \circ \Delta' = \dot{can} : U \hookrightarrow \dot{X}_{\mathcal{S}}$ *and* $\sigma \circ \Delta' = i_0$ *(the first equality shows that* $\Delta'(U)$ *is a closed subscheme in* $\mathcal{X}'$*);*

(b) σ *is étale in a neighborhood of* $\mathcal{Z}' \cup \Delta'(U)$*;*

(c) $\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$ *scheme theoretically for some closed subscheme* $\mathcal{Z}''$ *and* $\mathcal{Z}'' \cap \Delta'(U) = \varnothing$*;*

(d) $\mathcal{D}_0 := \sigma^{-1}(\{0\} \times U) = \Delta'(U) \coprod \mathcal{D}'_0$ *scheme theoretically for some closed subscheme* $\mathcal{D}'_0$ *and* $\mathcal{D}'_0 \cap \mathcal{Z}' = \varnothing$*;*

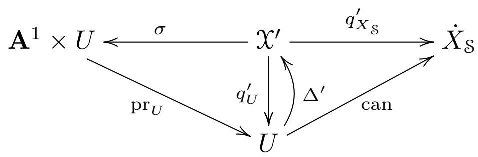
(e) *for* $\mathcal{D}_1 := \sigma^{-1}(\{1\} \times U)$ *one has* $\mathcal{D}_1 \cap \mathcal{Z}' = \varnothing$;

(f) *there is a monic* $h \in \mathcal{O}[t]$ *such that* $(h) = \mathrm{Ker}[\mathcal{O}[t] \xrightarrow{\sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \xrightarrow{\bar{}} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')]$, *where* $\mathcal{O} := \mathcal{O}_{\dot{X}_{\mathcal{S}},x}$ *and the map bar takes any* $g \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ *to* $\bar{g} \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$.

Proof of Theorem 6.1. Let $\dot{X}_{\mathcal{S}}$, $x \in \dot{X}_{\mathcal{S}}$, $\mathrm{f} \in \Gamma(\dot{X}_{\mathcal{S}}, O_{\dot{X}_{\mathcal{S}}})$, the special nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U, the essentially smooth morphism $q_{\dot{X}_{\mathcal{S}}} : \mathcal{X} \to \dot{X}_{\mathcal{S}}$ be as in Proposition 5.1. Then $q_{\dot{X}_{\mathcal{S}}} \circ \Delta = \dot{can}$, where $\dot{can} : U \hookrightarrow \dot{X}_{\mathcal{S}}$ is the embedding, $f = q^*_{\dot{X}_{\mathcal{S}}}(\mathrm{f})$ by the same Proposition.

By Theorem 4.3 there exists a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ such that the triple $(\mathcal{X}', f', \Delta')$ is a $(*)$-special nice triple over U.

The triple $(\mathcal{X}', f', \Delta')$ is a $(*)$-special nice triple **over** U. Thus by Theorem 4.2 there is a finite surjective morphism $\mathbf{A}^1 \times U \xleftarrow{\sigma} \mathcal{X}'$ of the U-schemes satisfying the conditions (a) to (f) from that theorem. Hence one has a diagram



(8)

with the irreducible scheme $\mathcal{X}'$, the smooth morphism $q'_U := q_U \circ \theta$, the finite surjective morphism σ and the essentially smooth morphism $q'_{X_{\mathcal{S}}} := q_{\dot{X}_{\mathcal{S}}} \circ \theta$ and with the function $f' \in (q'_{X_{\mathcal{S}}})^*(\mathrm{f}) \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$, which enjoy the properties (a) to (f) from Theorem 6.1. Whence the Theorem 6.1. □

To formulate a first consequence of the Theorem 6.1 (see Corollary 6.2) note that using the items (b) and (c) of Theorem 6.1 one can find an element $g \in I(\mathcal{Z}'')$ such that
(1) $(f') + (g) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,
(2) $\mathrm{Ker}((\Delta')^*) + (g) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,
(3) $\sigma_g = \sigma|_{\mathcal{X}'_g} : \mathcal{X}'_g \to \mathbf{A}^1_U$ is étale.

Corollary 6.2 (Corollary of Theorem 6.1)**.** *The function f' from Theorem* 6.1, *the polynomial h from the item* (*f*) *of that theorem, the morphism* $\sigma : \mathcal{X}' \to \mathbf{A}^1_U$ *and the function* $g \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ *defined just above enjoy the following properties:*

(i) *the morphism* $\sigma_g = \sigma|_{\mathcal{X}'_g} : \mathcal{X}'_g \to \mathbf{A}^1 \times U$ *is étale*;

(ii) *data* $(\mathcal{O}[t], \sigma_g^* : \mathcal{O}[t] \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g, h)$ *satisfies the hypotheses of* [2, Proposition 2.6], *i.e.,* $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ *is a finitely generated* $\mathcal{O}[t]$-*algebra, the element* $(\sigma_g)^*(h)$ *is not a zero-divisor in* $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ *and* $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$;

(iii) $(\Delta'(U) \cup \mathcal{Z}') \subset \mathcal{X}'_g$ *and* $\sigma_g \circ \Delta' = i_0 : U \to \mathbf{A}^1 \times U$,

(iv) $\mathcal{X}'_{gh} \subseteq \mathcal{X}'_{gf'} \subseteq \mathcal{X}'_{f'} \subseteq \mathcal{X}'_{(q'_{X_S})^*(\mathrm{f})}$;

(v) $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f')$, $h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) = (f') \cap I(\mathcal{Z}'')$, *and* $(f') + I(\mathcal{Z}'') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$.

Proof of Corollary 6.2. We use notation from Theorem 6.1. Since $\mathcal{X}$ is a regular affine irreducible scheme and $\sigma : \mathcal{X}' \to \mathbf{A}^1_U$ is a finite surjective morphism, the induced $\mathcal{O}$-algebra homomorphism $\sigma^* : \mathcal{O}[t] \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a monomorphism. We will regard the $\mathcal{O}$-algebra $\mathcal{O}[t]$ as a subalgebra via σ^*.

The assertions (i) and (iii) of the Corollary hold by our choice of g. The assertion (iv) holds, since $\sigma^*(h)$ is in the principal ideal (f') (use the properties (a) and (f) from Theorem 6.1). Prove now the assertion (ii). The morphism σ is finite. Hence the $\mathcal{O}[t]$-algebra $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ is finitely generated. The scheme $\mathcal{X}'$ is regular and irreducible. Thus, the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a domain. The homomorphism σ^* is injective. Hence, the element h is not zero and is not a zero divisor in $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$.

It remains to check that $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$. Firstly, by the choice of h and by the item (a) of Theorem 6.1 one has $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$. Secondly, by the property (1) of the element g one has $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / f'\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$. Finally, by the items (c) and (a) of Theorem 6.1 one has

$$\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') \times \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/I(\mathcal{Z}'') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(h). \tag{9}$$

Localizing both sides of (9) in g one gets an equality

$$\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / f'\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g.$$

Finally, we have a chain of equalities

$$\begin{aligned}\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') &= \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / f'\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g \\ &= \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g.\end{aligned}$$

Whence the assertion (ii).

Prove the assertion (v). The equality $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$ is checked in the proof of the assertion (ii). The item (c) of Theorem 6.1

yields the equality $(f') + I(\mathcal{Z}'') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Finally, the equality (9) yields the one $h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) = (f') \cap I(\mathcal{Z}'')$. □

Set $Y := \mathcal{X}'_g$, $\dot{p}_X = q'_{\dot{X}_{\mathcal{S}}}|_Y : Y \to \dot{X}_{\mathcal{S}}$, $p_U = q'_U|_Y : Y \to U$, $\tau = \sigma_g$, $\tau_h = \sigma_{gh}$, $\delta = \Delta'$ and note that $pr_U \circ \tau = p_U$. Take the monic polinomial $h \in \mathcal{O}[t]$ from the item (f) of Theorem 6.1. With this replacement of notation and with the element h we arrive to the following

Theorem 6.3 (=Theorem 2.8). *Let $\dot{X}_{\mathcal{S}}$, $x \in \dot{X}_{\mathcal{S}}$, $\mathrm{f} \in \Gamma(\dot{X}_{\mathcal{S}}, \mathcal{O}_{\dot{X}_{\mathcal{S}}})$ be as in Proposition* 5.1. *Put $U = \mathrm{Spec}(\mathcal{O}_{\dot{X}_{\mathcal{S}},x})$ and $\mathcal{O} = \mathcal{O}_{\dot{X}_{\mathcal{S}},x}$. Then the monic $h \in \mathcal{O}[t]$, the commutative diagram of V-schemes together with the irreducible affine U-smooth scheme Y*

$$\begin{array}{ccccc} (\mathbf{A}^1 \times U)_h & \xleftarrow{\tau_h} & Y_h := Y_{\tau^*(h)} & \xrightarrow{(\dot{p}_X)|_{Y_h}} & (\dot{X}_{\mathcal{S}})_{\mathrm{f}} \\ \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} \\ (\mathbf{A}^1 \times U) & \xleftarrow{\tau} & Y & \xrightarrow{\dot{p}_X} & \dot{X}_{\mathcal{S}} \end{array} \qquad (10)$$

and the morphism $\delta : U \to Y$ enjoy the conditions stated in Theorem 2.8.

Proof. The items (iii) and (iv) of the Corollary 6.2 show that the morphisms $\delta : U \to Y$ and $(\dot{p}_X)|_{Y_h} : Y_h \to (\dot{X}_{\mathcal{S}})_{\mathrm{f}}$ are well defined. The items (i), (ii) of that Corollary show that the left hand side square in the diagram (10) is an elementary **distinguished** square in the category of smooth U-schemes in the sense of [5, Definition 3.1.3]. The equalities $\dot{p}_X \circ \delta = \dot{can}$ and $\tau \circ \delta = i_0$ are clear. □

Proof of Theorem 1.4. Consider the following commutative diagram

$$\begin{array}{ccccccc} (\mathbf{A}^1 \times U)_h & \xleftarrow{\tau_h} & Y_h := Y_{\tau^*(h)} & \xrightarrow{(\dot{p}_X)|_{Y_h}} & (\dot{X}_{\mathcal{S}})_{\mathrm{f}} & \xrightarrow{in} & X - Z \\ \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} \\ (\mathbf{A}^1 \times U) & \xleftarrow{\tau} & Y & \xrightarrow{\dot{p}_X} & \dot{X}_{\mathcal{S}} & \xrightarrow{in} & X \end{array} \qquad (11)$$

where the left square and the middle one equals to the left and the right square respectively of the diagram (10). Clearly, Theorem 6.3 yields Theorem 1.4. Theorem 1.4 is proved. □

References

1. M. Artin, *Comparaison avec la cohomologie classique: cas d'un préschéma lisse*, in: *Théorie des topos et cohomologie étale des schémas* (SGA 4). Tome 3. Lect. Notes Math., vol. 305, Exp. XI, Springer-Verlag, Berlin-New York, 1973.
2. J.-L. Colliot-Thélène, M. Ojanguren, *Espaces Principaux Homogènes Localement Triviaux.* — Publ. Math. IHÉS **75** No. 2 (1992), 97–122.
3. N. Guo, I. Panin, *A Presentation theorem for smooth projective schemes over discrete valuation rings*, `arxiv:2302.02818v1` (2023).
4. N. Guo, I. Panin, *Weak elementary fibrations*, `arxiv:2302.02837v1` (2023) (to appear in St.Petersburg Math. Journal).
5. F. Morel, V. Voevodsky, *A^1-homotopy theory of schemes.* — Publ. Math. IHÉS **90** (1999), 45–143.
6. M. Ojanguren, I. Panin, *Rationally trivial hermitian spaces are locally trivial.* — Math. Z. **237** (2001), 181–198.
7. I. Panin, A. Stavrova, N. Vavilov, *On Grothendieck—Serre's conjecture concerning principal G-bundles over reductive group schemes*: I. — Compositio Math. **151** (2015), 535–567.
8. I. Panin, *Nice triples and moving lemmas for motivic spaces.* — Izv. Math. **83**, No. 4 (2019), 796–829.
9. I. Panin, *Proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing a field.* — Izvestiya Mathematics **84**, No. 4 (2020), 780–795.
10. I. Panin, *On Grothendieck–Serre conjecture concerning principal bundles.* — Proceedings of the International Congress of Mathematicians, vol. 2 (Rio de Janeiro 2018), World Sci. Publ., Hackensack, NJ 2018, pp. 201–221.
11. I. Panin, *Moving lemmas in mixed characteristic and applications*, `arXiv: 2202.00896v1`.
12. I. A. Panin, A. K. Stavrova, *Constant case of the Grothendieck–Serre conjecture in mixed characteristic*, Preprint, November (2024).
13. I. A. Panin, A. K. Stavrova, *On the Gille theorem for the relative projective line*, `arxiv: 2305.16627v2`.
14. V. Voevodsky, *Cohomological theory of presheaves with transfers.* — In: *Cycles, Transfers, and Motivic Homology Theories*, Ann. Math. Studies, 2000, Princeton University Press.

Steklov Mathematical Institute at St. Petersburg
E-mail: `paniniv@gmail.com`

Поступило 19 ноября 2024 г.