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A POSTERIORI ERROR IDENTITIES FOR THE EVOLUTIONARY STOKES PROBLEM

Abstract. This paper is concerned with functional identities that control distances between the exact solution of the evolutionary Stokes problem and a function from the corresponding energy space. Left hand sides of the identities contain norms of errors associated with velocity and stress fields error and the right hand ones contain known data and integrals that can be either directly computed or estimated via known quantities. It is shown that identities yield guaranteed and fully computable bounds of errors. A posteriori error identities and estimates are derived in the most general form. They do not use Galerkin orthogonality, divergence free property, or other special features of functions compared with the exact solution. Therefore, they are applicable for a wide variety of approximations, regardless of the method by which they were obtained.

Dedicated to the jubilee of Nina Nikolaevna Uraltseva

§1. Introduction

1.1. A posteriori error identities. The goal of this paper is to deduce functional relations that hold for solutions of the evolutionary Stokes problem and any function from the corresponding energy space viewed as an approximation of this solution. They can take the form of equalities (identities) or inequalities and represent the most general relationships that connect measures of errors and values (quantities) that can be really observed in a numerical experiment.

In general terms, the basic problem is as follows. Let $\mathbf{u} \in \mathbf{V}$ be a (generalised) solution of a certain problem and $\mathbf{v}$ be an approximation of $\mathbf{u}$. The only condition imposed on $\mathbf{v}$ is that it must belong to the same energy space $\mathbf{V}$ that contains the unique solution $\mathbf{u}$. Notice that $\mathbf{u}$ may denote not a unique function, but a set of functions characterising the exact solution. Accordingly, $\mathbf{v}$ denotes approximations of these functions. Reliable quantitative analysis of a mathematical model must use *fully controllable*

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computations that provide explicit and realistic estimate of the distance between $\mathbf{v}$ and $\mathbf{u}$. The best solution to this problem would be establishing the identity

$$\boldsymbol{\mu}(\mathbf{e}) = \mathrm{F}(\mathbf{v}, \mathcal{D}) \quad \forall v \in V, \tag{1.1}$$

where $\mathbf{e} := \mathbf{v} - \mathbf{u}$ is the deviation (error), $\boldsymbol{\mu} : \mathbf{V} \to \mathbb{R}_{\geqslant 0}$ is an error measure, and $\mathcal{D}$ denotes the set of problem data (coefficients, domain, boundary and initial conditions, etc.) and F is a nonnegative functional, which is directly computable if the arguments are known.

If $\mathbf{v}$ comes from a numerical experiment, then (1.1) is a tool of a posteriori error control (it can be called *an a posteriori error identity*). If $\mathbf{v}$ is a solution of a different (e.g., simplified) mathematical model, then (1.1) suggests a way to evaluate the corresponding *modelling error*. For some classes of boundary value problems, identities of the type (1.1) have been derived (e.g., for some elliptic equations, see [25, 31]). However, in the majority of cases a posteriori error identities have a more complicated form:

$$\boldsymbol{\mu}(\mathbf{e}) = \mathrm{F}(\mathbf{v}, \mathcal{D}) + \mathcal{L}(\mathbf{v}, \mathbf{e}), \tag{1.2}$$

where $\mathcal{L}$ is a continuous functional that includes known functions together with the unknown function $\mathbf{e}$, while F is a computable integral type functional. If the term $\mathcal{L}(\mathbf{v}, \mathbf{e})$ is estimated such that the unknown quantities are controlled by the measure $\boldsymbol{\mu}(\mathbf{e})$, then (1.2) yields fully computable two–sided bounds of the error. For elliptic boundary value problems estimates obtained by this method are well known (e.g., see [25, 27] and publications cited therein). For the parabolic heat equation, error identities and related estimates were derived in [24, 26]. These results were recently generalised to a class of nonlinear evolutionary problems with monotone spatial operators [28] and to parabolic – hyperbolic problems [30]. In all the above cases, (1.2) holds with $\mathcal{L}(\mathbf{v}, \mathbf{e})$, which is linear with respect to $\mathbf{e}$.

This article should be considered as a continuation of [26], where identities of the type (1.2) were studied for the Cauchy problem associated with the parabolic heat equation. Now, the method is extended to a class of initial-boundary value problems that has one essential feature: solutions lie in a subspace formed by *divergence free* (solenoidal) functions. The need to take into account the condition $\mathrm{div} u = 0$ creates significant difficulties in all parts of quantitative analysis. In particular, generation of successful numerical approximations usually requires methods adapted to this specific feature of the solution (e.g., see [4–6, 9, 20, 34] and references cited therein).

The difficulties also arise in both main components of error analysis, i.e., in a priori and a posteriori estimates. In the context of a posteriori estimates, close (but different) results for incompressible viscous flow problems were earlier obtained in [7,21,22,29]. The estimates obtained in this article differ from them since they are derived for a combined deviation measure that includes errors in terms of the velocity and stress.

Main results are as follows. Theorems 1 and 2 establish error identities of the type (1.2) for two different error measures. The identity (3.2) holds with minimal restrictions on the solution and its approximations. A narrower class is considered in Theorem 2. Here we assume additional differentiability in time for u, v, σ, and the identity operates with a norm stronger than that in (3.2).

Right hand sides of the error identities (3.2) and (4.3) contain unknown function e. Therefore, they cannot be directly used for quantitative analysis of errors. However, the identities create a basis for getting fully computable error estimates. These estimates are derived in the next section. They are valid for any admissible velocity field including the case of non–solenoidal fields. Estimates account possible violations of the divergence free condition in an integral sense by the terms that depend on the distance to a solenoidal field closest to v. Essentially, the latter question is no longer connected with solving an initial boundary value problem and represents a specific problem of the approximation theory. This problem was studied in [23,29]. Combining these results with the estimates derived in the paper suggests a method of guaranteed and fully computable error control for a wide set of approximations independently on the way by which they were obtained.

§2. Notation and problem statement

Let Ω be an open bounded domain in $\mathbb{R}^d$, $d \geqslant 2$ with Lipschitz continuous boundary Γ and $T > 0$. Then $Q_T := \Omega \times (0,T)$ is the space–time cylinder and $S_T := \Gamma \times (0,T)$ is its lateral surface.

To make formulas compact, we use special notation for the terms generated by the difference of quantities at $t = T$ and $t = 0$, i.e.,

$$\Big[g(t)\Big]_0^T := g(T) - g(0).$$

For example, we write $\Big[\|v\|_\Omega\Big]_0^T$ instead of $\|v(x,T)\|_\Omega - \|v(x,0)\|_\Omega$.

In the relations below, $v_{,i}$ means the spatial derivativ of v with respect to x_i and time derivative is denoted by v_t or $\partial_t v$. Spatial gradient and divergence are denoted by ∇ and div, respectively, i.e., $\nabla v = \{v_{i,j}\}$, $\mathrm{div} v \ := \ v_{i,i}$, $i = 1, ..., n$, and $\mathrm{Div}\sigma := \{\sigma_{ij,j}\}$.

Since no confusion may arise, for the L_2 norms of scalar, vector, and tensor valued functions defined in Ω we use the same notation $\|\cdot\|_\Omega$ and define the subspace $\widetilde{L}(\Omega) := \{g \in L_2(\Omega) \ | \ \int\limits_\Omega g dx = 0\}$ containing functions with zero mean values.

For the vector valued functions defined in Ω, we use the spaces

$$Y(\Omega) := L_2(\Omega, \mathbb{R}^d) \quad \text{and} \quad V(\Omega) \ := \ H^1(\Omega, \mathbb{R}^d)$$

and the following subspaces:

$$\begin{aligned} Y_{\mathrm{div}}(\Omega) &:= \{y \in Y(\Omega) \ | \ \mathrm{div} y \in L_2(\Omega)\}, \\ Y_{\nabla}(\Omega) &:= \{y \in Y(\Omega) \ | \ \exists \phi \in H^1(\Omega) : \ y = \nabla \phi\}, \\ V_0(\Omega) &:= \{v \in V(\Omega) \ | \ v = 0 \text{ on } \Gamma\}. \end{aligned}$$

The closure of smooth solenoidal functions compactly supported in Ω with respect to the norm of $V(\Omega)$ forms the space $\overset{\circ}{S}{}^1(\Omega)$ and

$$S_{0,\Gamma}(\Omega) := \{v \in Y_{\mathrm{div}}(\Omega) \ | \ \mathrm{div} v = 0, \ v \cdot n = 0 \text{ on } \Gamma\}$$

denotes the set of solenoidal functions with zero normal traces on Γ. In view of the Helmgholtz theorem, we have the orthogonal decomposition

$$Y(\Omega) = S_{0,\Gamma}(\Omega) \oplus Y_{\nabla}(\Omega).$$

For the symmetric tensor–valued functions we define the space

$$\Sigma(\Omega) := L_2(\Omega, \mathbb{M}_s^{d\times d})$$

and its subspace

$$\Sigma_{\mathrm{Div}}(\Omega) := \left\{\tau \in L_2(\Omega, \mathbb{M}_s^{d\times d}) \ \middle| \ \mathrm{Div}\tau \in L_2(\Omega, \mathbb{R}^d)\right\},$$

which is a Hilbert space supplied with the norm

$$\|\tau\|_{\mathrm{Div},\Omega} := \left(\|\tau\|^2_\Omega + \|\mathrm{Div}\tau\|^2_\Omega\right)^{1/2}.$$

We use standard notation for Bochner spaces containing time dependent functions. For any $t < +\infty$ and any separable Banach space X supplied with the norm $\|\cdot\|_X$, $L_2(0, t; X)$ denotes the space of measurable functions v such that $\|v\|^2_{L_2(0,t;X)} := \int_0^t \|v\|^2_X \, dt < \infty$. By $L_{2,1}(Q_T)$, we denote the space of functions v such that $\int_0^T \|v\|^2_\Omega dt < +\infty$.

Let

$$\begin{aligned}
&W_2^{1,0}(Q_T,\mathbb{R}^d) := L_2(0,T,V(\Omega)), \quad \overset{\circ}{W}{}_2^{1,0}(Q_T,\mathbb{R}^d) := L_2(0,T,V_0(\Omega)),\\
&W_2^{1,1}(Q_T,\mathbb{R}^d) := H^1(Q_T,\mathbb{R}^d),\\
&\overset{\circ}{W}{}_2^{1,1}(Q_T,\mathbb{R}^d) := \{v \in H^1(Q_T,\mathbb{R}^d) \mid v = 0 \text{ on } S_T\},
\end{aligned}$$

and $\overset{\circ}{S}{}_2^{1,1}(Q_T,\mathbb{R}^d)$ denote a subspace of $\overset{\circ}{W}{}_2^{1,1}(Q_T,\mathbb{R}^d)$ containing solenoidal vector functions. In addition to the standard norm, we also use weighted norms

$$\|w\|_{\nu,\Omega} := \left(\int\limits_\Omega \nu|w|^2 dx\right)^{1/2} \quad \text{and} \quad \|w\|_{\nu,Q_T} := \left(\int\limits_0^T \|\varepsilon(w)\|^2_{\nu,\Omega} dt\right)^{1/2},$$

where $\varepsilon(v) := \frac{1}{2}\Big(\nabla(v) + (\nabla(v))^T\Big)$ is the small strains tensor.

For any $v \in V_0(\Omega)$, it holds the inequality

$$\|w\|_\Omega \leqslant C\|\varepsilon(w)\|_\Omega, \tag{2.1}$$

which follows from the Friedrihs and Korn's inequalities. The respective constant C depends on Ω and can be estimated from above by known methods (e.g., see Chapter 2 of [31]).

We consider the classical Stokes problem: find a vector valued function $u(x,t)$ (velocity), a tensor valued function $\sigma(x,t)$ (stress), and a scalar valued function $p(x,t)$ (pressure) such that

$$u_t - \mathrm{Div}\sigma = f \qquad \text{in } Q_T, \tag{2.2}$$

$$\sigma = \nu\varepsilon(u) - p\mathbb{I} \qquad \text{in } Q_T, \tag{2.3}$$

$$\mathrm{div} u = 0 \qquad \text{in } Q_T, \tag{2.4}$$

$$u = 0 \qquad \text{on } S_T, \tag{2.5}$$

$$u(x,0) = u_0(x) \qquad x \in \Omega. \tag{2.6}$$

Here ν is a positive constant (viscosity), $f(x,t) \in L_{2,1}(\Omega,\mathbb{R}^d)$ for $t \in [0,T]$ is a given function, $u_0 \in V_0(\Omega)$ is a given divergence free function, and $\mathbb{I}$ denotes the unit tensor in $\mathbb{M}^{d\times d}$.

The identity

$$\int\limits_{Q_T} (u_t \cdot w + \nu\varepsilon(u) : \varepsilon(w) - p\mathrm{div} w)\, dxdt = \int\limits_{Q_T} f \cdot w dxdt \tag{2.7}$$

follows from (2.2)–(2.3) and holds for any test function $w \in \overset{\circ}{W}{}_2^{1,0}(Q_T, \mathbb{R}^d)$. It defines the generalised solution u (e.g., see [13]) as a function in $\overset{\circ}{S}{}_2^{1,1}(\Omega, \mathbb{R}^d)$ that satisfies (2.6) and (2.7). If we use a narrower set of test functions and $w \in \overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d)$, then the identity can be represented in a somewhat different form

$$\int\limits_{Q_T} (\nu\varepsilon(u) : \varepsilon(w) - u \cdot w_t)\, dxdt + \Big[\int\limits_{\Omega} u \cdot w dx\Big]_0^T = \int\limits_{Q_T} (f \cdot w + p\mathrm{div} w) dxdt. \tag{2.8}$$

If the set of test functions is reduced to divergence free fields, then (2.8) reads as follows:

$$\int\limits_{Q_T} (\nu\varepsilon(u) : \varepsilon(w) - u \cdot w_t) dxdt + \Big[\int\limits_{\Omega} u \cdot w dx\Big]_0^T = \int\limits_{Q_T} f \cdot w dxdt \quad \forall w \in \overset{\circ}{S}{}_2^{1,1}(Q_T, \mathbb{R}^d). \tag{2.9}$$

§3. The main error identity

Henceforth the functions $v(x,t)$, $\tau(x,t)$, and $q(x,t)$ are considered as certain approximations of $u(x,t)$, $\sigma(x,t)$, and $p(x,t)$, respectively. It does not matter to us how they were obtained. The only condition is

$$v \in \overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d), \quad \tau \in \Sigma(\mathrm{Div}, Q_T), \quad q \in L_2(0,T; \widetilde{L}_2(\Omega)). \tag{3.1}$$

We assume that $v(x,t)$, $\tau(x,t)$, and $q(x,t)$ are known and, therefore, the functions (residuals of the equations (2.2) and (2.3))

$$\mathbb{R}(v,\tau) := \mathrm{Div}\tau - v_t + f \quad \text{and} \quad \mathbb{T}(v,\tau,q) := \tau - \nu\varepsilon(v) + q\mathbb{I}$$

are computable also known.

3.1. Error measure and residuals. Together with σ and τ, we define $\widehat{\sigma} := \nu\varepsilon(u)$ (hence $\sigma = \widehat{\sigma} - p\mathbb{I}$) and $\widehat{\tau} := \tau + q\mathbb{I}$. Then

$$e := v - u, \quad \eta := \tau - \sigma, \quad \text{and} \quad \widehat{\eta} := \widehat{\tau} - \widehat{\sigma}$$

are the corresponding error functions. Define the quantity

$$\boldsymbol{\mu}(e, \widehat{\eta}) := \|\varepsilon(e)\|^2_{\nu, Q_T} + \|\widehat{\eta}\|^2_{\nu^{-1}, Q_T} + \Big[\|e\|^2_{\Omega}\Big]_0^T.$$

If $v(x,0) = u_0(x,0)$, then $\boldsymbol{\mu}(e,\widehat{\eta})$ can be considered as a natural measure of the distance to the exact solution. It contains three parts: velocity error (measured by the norm equivalent to the norm of $\overset{\circ}{W}{}_2^{1,1}(Q_T,\mathbb{R}^d)$), error associated with stresses (measured by the norm equivalent to the norm of $\Sigma(\Omega)$), and the term $\|v(x,T)-u(x,T)\|^2_\Omega$ that shows the velocity error at $t=T$.

Our goal is to establish functional relations that connect $\boldsymbol{\mu}(e,\widehat{\eta})$ and computable quantities $\mathbb{T}(v,\tau,q)$ and $\mathbb{R}(v,\tau)$. Theorem below gives the key relation.

Theorem 3.1. *For any v, τ, and q satisfying* (3.1), *it holds*

$$\boldsymbol{\mu}(e,\widehat{\eta}) = \|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T} - 2\int\limits_{Q_T}\mathbb{H}(v,\tau,\psi)\cdot e\,dxdt + 2\mathbb{L}(\mathring{v},\,e,\widehat{\eta}), \quad (3.2)$$

where $\mathring{v} := v - v_0$, v_0 is any function in $\overset{\circ}{S}{}_2^{1,1}(Q_T,\mathbb{R}^d)$, ψ is any function in $L_2(0,T;H^1(\Omega))$,

$$\mathbb{H}(v,\tau,\psi) := \mathbb{R}(v,\tau) - \nabla\psi,$$

and

$$\mathbb{L}(\mathring{v},\,e,\widehat{\eta}) := \Big[\int\limits_\Omega \mathring{v}\cdot e\,dx\Big]_0^T + \int\limits_{Q_T}\Big(\varepsilon(\mathring{v}):\widehat{\eta} - \mathring{v}_t\cdot e\Big)dxdt + \int\limits_{Q_T}\mathbb{H}(v,\tau,\psi)\cdot\mathring{v}dxdt.$$

Proof. From (2.7), it follows that the error function $e(x,t)$ satisfies the relation

$$\int\limits_{Q_T}\left((u_t-v_t)\cdot w + \nu\varepsilon(u-v):\varepsilon(w) - p\mathrm{div}w\right)dxdt$$
$$= \int\limits_{Q_T}(fw - v_t w - \nu\varepsilon(v):\varepsilon(w))dxdt. \quad (3.3)$$

Set in (3.3) $w = u - v$. We obtain

$$\int\limits_{Q_T}\left(e_t\cdot e + \nu|\varepsilon(e)|^2\right)dxdt$$
$$= \int\limits_{Q_T}(v_t\cdot e + \nu\varepsilon(v):\varepsilon(e) - f\cdot e - p\mathrm{div}e)dxdt. \quad (3.4)$$

Since

$$\int_{Q_T} e \cdot e_t \, dxdt = \frac{1}{2} \int_{\Omega} \big(e(x,T)^2 - e(x,0)^2\big) dx = \frac{1}{2} \Big[\|e\|_{\Omega}^2 \Big]_0^T,$$

from (3.4) it follows that

$$\int_{Q_T} \nu |\varepsilon(e)|^2 dxdt + \frac{1}{2} \Big[\|e\|_{\Omega}^2 \Big]_0^T$$
$$= \int_{Q_T} (\nu\varepsilon(v) : \varepsilon(e) + v_t \cdot e - q \mathrm{div} e - f \cdot e) \; dx \, dt + \int_{Q_T} (q-p) \mathrm{div} e dxdt. \quad (3.5)$$

For any $\tau \in \Sigma_{\mathrm{Div}}(\Omega)$, it holds

$$\int_{\Omega} (\mathrm{Div}\tau \cdot e + \tau : \varepsilon(e)) \, dx = \int_{\Gamma} e \cdot (\tau \mathbf{n}) ds = 0,$$

where $\mathbf{n}$ is the unit outward normal to Γ. Hence, (3.5) can be rewritten in the form

$$\|\varepsilon(e)\|_{\nu,Q_T}^2 + \frac{1}{2} \Big[\|e\|_{\Omega}^2 \Big]_0^T$$
$$= \int_{Q_T} (q-p) \mathrm{div} e dxdt - \int_{Q_T} \Big(\mathbb{T}(v,\tau,q) : \varepsilon(e) + \mathbb{R}(v,\tau) \cdot e \Big) \, dxdt. \quad (3.6)$$

Notice that

$$\widehat{\eta} = \widehat{\tau} - \widehat{\sigma} = \tau + q\mathbb{I} - \nu\varepsilon(u) = \tau + q\mathbb{I} - \nu\varepsilon(v) + \nu\varepsilon(v-u) = \mathbb{T}(v,\tau,q) + \nu\varepsilon(e)$$

and, therefore,

$$\frac{1}{2} \|\widehat{\eta}\|_{\nu^{-1},Q_T}^2 .$$
$$= \frac{1}{2} \|\mathbb{T}(v,\tau,q)\|_{\nu^{-1},Q_T}^2 + \int_{Q_T} \mathbb{T}(v,\tau,q) : \varepsilon(e) dxdt + \frac{1}{2} \|\varepsilon(e)\|_{\nu,Q_T}^2 \quad (3.7)$$

Summation of (3.6) and (3.7) yields the identity

$$\frac{1}{2}\|\varepsilon(e)\|^2_{\nu,Q_T} + \frac{1}{2}\|\widehat{\eta}\|^2_{\nu^{-1},Q_T} + \frac{1}{2}\Big[\|e\|^2_\Omega\Big]_0^T$$
$$= \frac{1}{2}\|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T} - \int\limits_{Q_T} \mathbb{R}(v,\tau)\cdot e\,dxdt + \int\limits_{Q_T}(q-p)\mathrm{div}e dxdt. \quad (3.8)$$

It contains unknown pressure p, which is to be excluded. Let v_0 be a function in $\overset{\circ}{S}{}^{1,1}_2(Q_T,\mathbb{R}^d)$ and $\mathring{v} = v - v_0$. It is easy to see that

$$\int\limits_\Omega (q-p)\mathrm{div}(v-u)dx = \int\limits_\Omega (q-p)\mathrm{div}\mathring{v}dx.$$

By (2.8) we reform the last integral as follows:

$$\int\limits_{Q_T}(q-p)\mathrm{div}\mathring{v}dxdt$$
$$= \int\limits_{Q_T}\left(-\nu\varepsilon(u):\varepsilon(\mathring{v}) + f\cdot\mathring{v} + u\cdot\mathring{v}_t + q\mathrm{div}\mathring{v}\right)dxdt - \Big[\int\limits_\Omega u\cdot\mathring{v}dx\Big]_0^T$$
$$= \int\limits_{Q_T}\left(f\cdot\mathring{v} + v\cdot\mathring{v}_t + q\mathrm{div}\mathring{v} - \widehat{\tau}:\varepsilon(\mathring{v})\right)dxdt - \Big[\int\limits_\Omega u\cdot\mathring{v}dx\Big]_0^T$$
$$+ \int\limits_{Q_T}\left((\widehat{\tau}-\widehat{\sigma}):\varepsilon(\mathring{v}) + (u-v)\cdot\mathring{v}_t\right)dxdt.$$

Since

$$\int\limits_{Q_T} v\cdot\mathring{v}_t dxdt - \Big[\int\limits_\Omega u\cdot\mathring{v}dx\Big]_0^T = \int\limits_\Omega e\cdot\mathring{v}_t dxdt + \int\limits_{Q_T} u\cdot\mathring{v}_t dxdt - \Big[\int\limits_\Omega u\cdot\mathring{v}dx\Big]_0^T$$
$$= \Big[\int\limits_\Omega e\cdot\mathring{v}dx\Big]_0^T - \int\limits_{Q_T}(v_t-u_t)\cdot\mathring{v}dxdt + \int\limits_{Q_T} u\cdot\mathring{v}_t dxdt - \Big[\int\limits_\Omega u\cdot\mathring{v}dx\Big]_0^T$$
$$= \Big[\int\limits_\Omega e\cdot\mathring{v}dx\Big]_0^T - \int\limits_\Omega v_t\cdot\mathring{v}dx,$$

we have

$$\int\limits_{Q_T} (q-p)\mathrm{div}\mathring{v}dxdt = \int\limits_{Q_T} \left(f\cdot\mathring{v} + q\mathrm{div}\mathring{v} - v_t\cdot\mathring{v} - \widehat{\tau}:\varepsilon(\mathring{v})\right)dxdt$$
$$+ \Big[\int\limits_{\Omega} e\cdot\mathring{v}dx\Big]_0^T + \int\limits_{Q_T} \left(\widehat{\eta}:\varepsilon(\mathring{v}) - e\cdot\mathring{v}_t\right)dxdt. \quad (3.9)$$

Notice that the first integral in the right hand side of (3.9) has a simpler form

$$\int\limits_{Q_T} \left(f\cdot\mathring{v} + q\mathbb{I}:\varepsilon(\mathring{v}) - (\tau+q\mathbb{I}):\varepsilon(\mathring{v}) - v_t\cdot\mathring{v}\right)dxdt$$
$$= \int\limits_{Q_T} \left(f\cdot\mathring{v} - \tau:\varepsilon(\mathring{v}) - v_t\cdot\mathring{v}\right)dxdt$$
$$= \int\limits_{Q_T} (\mathrm{Div}\tau + f - v_t)\cdot\mathring{v}dxdt = \int\limits_{Q_T} \mathbb{R}(v,\tau)\cdot\mathring{v}dxdt.$$

Hence

$$\int\limits_{Q_T} (q-p)\mathrm{div}\mathring{v}dxdt = \int\limits_{Q_T} \mathbb{R}(v,\tau)\cdot\mathring{v}dxdt$$
$$+ \Big[\int\limits_{\Omega} e\cdot\mathring{v}dx\Big]_0^T + \int\limits_{Q_T} \left(\widehat{\eta}:\varepsilon(\mathring{v}) - e\cdot\mathring{v}_t\right)dxdt. \quad (3.10)$$

By (3.8) and (3.10), we obtain

$$\frac{1}{2}\|\varepsilon(e)\|^2_{\nu,Q_T} + \frac{1}{2}\|\widehat{\eta}\|^2_{\nu^{-1},Q_T} + \frac{1}{2}\Big[\|e\|^2_{\Omega}\Big]_0^T$$
$$= \frac{1}{2}\|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T} - \int\limits_{Q_T} \mathbb{R}(v,\tau)\cdot(e-\mathring{v})\,dxdt$$
$$\Big[\int\limits_{\Omega} e\cdot\mathring{v}dx\Big]_0^T + \int\limits_{Q_T} \left(\widehat{\eta}:\varepsilon(\mathring{v}) - e\cdot\mathring{v}_t\right)dxdt. \quad (3.11)$$

To complete the proof we use specifics of the term containing $\mathbb{R}(v,\tau)$. The function $e-\mathring{v}$ is equal to $v_0 - u$. Hence for any $t\in[0,T]$ it belongs to

$\overset{\circ}{S}{}^1(\Omega)$, so that for any $\psi(x,t) \in H^1(Q_T)$, it holds

$$\int\limits_{\Omega} \nabla\psi \cdot (e - \mathring{v})dx = 0. \tag{3.12}$$

Now (3.2) follows from (3.11) and (3.12). □

3.2. Comments. It is worth adding several comments regarding the structure and meaning of the identity (3.2).

1. Let $v(x,0) = u_0(x)$. In view of (3.7), the left hand side of (3.2) has the form

$$\|\nu\varepsilon(u) - \tau - q\mathbb{I}\|^2_{\nu^{-1},Q_T} + \|\varepsilon(v-u)\|^2_{\nu,Q_T} + \|v(x,T) - u(x,T)\|^2_{\Omega}.$$

It is nonnegative and vanishes if and only if $v = u$ and $\tau{+}q\mathbb{I}{=}\nu\varepsilon(u){=}\sigma{+}p\mathbb{I}$. Hence the left hand side of (3.2) is a *consistent measure* of the distance to the exact solution. Moreover, it is continuous with respect to convergence of v in $W^{1,0}_2(Q_T,\mathbb{R}^d)$, τ in $L_2(0,T,\Sigma(\Omega))$, and $q \in L_2(0,T,\widetilde{L}_2(\Omega))$.

It is not difficult to see that (3.2) has the form of (1.2). Here $\mathbf{v}$ stands for the triple of functions (v,τ,q),

$$\mathrm{F}(\mathbf{v},\mathcal{D}) = \|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T},$$

the error function $\mathbf{e}$ has two components e and $\widehat{\eta}$, and the functional $\mathcal{L}$ is defined by the relation

$$\mathcal{L}(\mathbf{v},\mathbf{e}) = 2\Big(\mathbb{L}(v - v_0,\, e, \widehat{\eta}) - \int\limits_{Q_T} \mathbb{H}(v,\tau,\psi)\cdot e\, dxdt\Big),$$

where the functions ψ and v_0 are at our disposal and can be selected to make the corresponding estimates sharper (see Section 5.2).

2. Let $q = 0$, τ be zero matrix and v be zero vector. Then, the left hand side of (3.2) is reduced to

$$2\|\varepsilon(u)\|^2_{\nu} + \Big[\|u\|^2_{\Omega}\Big]_0^T.$$

Since $\mathbb{T}(v,\tau,q) = 0$, $\mathring{v} = 0$, and $\mathbb{L}(\mathring{v},\, e, \widehat{\eta}) = 0$, the right hand side contains only one term:

$$2\int\limits_{Q_T} (\mathbb{R}(v,\tau) - \nabla\psi)\cdot u dxdt = 2\int\limits_{Q_T} f u dxdt.$$

Hence for zero functions v, τ and q, the identity (3.2) is reduced to the *energy balance relation*

$$\int\limits_{Q_T} (\nu|\varepsilon(u)|^2 + u_t \cdot u)dxdt = \int\limits_{Q_T} fudxdt$$

that follows from (2.7) if we set $w = u$.

3. Let $\widetilde{u}(x,t)$ solves (2.2)–(2.5) for the same f and satisfies the initial condition $\widetilde{u}(x,t) = \widetilde{u}_0(x) \neq u_0(x)$. The respective stress and pressure fields are denoted by $\widetilde{\sigma}$ and $\widetilde{p}$, respectively. We set $\widehat{\tau} = \nu\varepsilon(\widetilde{u})$ and $q = \widetilde{p}$. Then $\widehat{\eta} = \nu\varepsilon(\widetilde{u} - u)$,

$$\mathbb{R}(v,\tau) = \mathrm{Div}\widetilde{\sigma} - \widetilde{u}_t + f = 0 \quad \text{and} \quad \mathbb{T}(\widetilde{v},\widetilde{\tau},\widetilde{q}) = \widetilde{\sigma} - \nu\varepsilon(\widetilde{u}) + \widetilde{p}\mathbb{I} = 0.$$

Since $\mathring{v} = 0$, the identity (3.2) is reduced to

$$2\|\varepsilon(e)\|^2_{\nu,Q_T} + \|e(x,T)\|^2_\Omega = \|u(x,0) - \widetilde{u}(x,0)\|^2_\Omega,$$

where $e(x,t) = \widetilde{u}(x,t) - u(x,t)$. This relation shows that $\|e(x,T)\|_\Omega \to 0$ as $T \to +\infty$ (stabilisation phenomenon).

3.3. Divergence free approximations. Assume that

$$v \in \overset{\circ}{S}{}^{1,1}_2(Q_T, \mathbb{R}^d). \tag{3.13}$$

We set $v_0 = v$ and arrive at a simplified version of the identity

$$\|\varepsilon(e)\|^2_{\nu,Q_T} + \|\widehat{\eta}\|^2_{\nu^{-1},Q_T} + \Big[\|e\|^2_\Omega\Big]_0^T = \\ = \|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T} - 2\int\limits_{Q_T} \mathbb{R}(v,\tau)\cdot edxdt, \tag{3.14}$$

which looks similar to the identity (3.3) derived in [26] for the parabolic heat equation.

Let (in addition to (3.13)) v and τ be coordinated in such a way that

$$\mathbb{R}(v,\tau) \in Y_\nabla(\Omega) \quad \text{for a.e. } (x,t) \in Q_T. \tag{3.15}$$

Then, after selecting a suitable ψ, the integral in the right hand side of (3.14) vanishes and we arrive at the following corollary of Theorem 3.1.

Corollary 3.1. *For any v and τ satisfying the conditions (3.13) and (3.15), it holds the identity*

$$\int\limits_{Q_T}\left(\nu\|\varepsilon(v-u)\|^2+\frac{1}{\nu}\|\tau+q\mathbb{I}-\nu\varepsilon(u)\|^2\right)dxdt+\Big[\|v-u\|^2_\Omega\Big]_0^T$$
$$=\int\limits_{Q_T}\frac{1}{\nu}|\tau+q\mathbb{I}-\nu\varepsilon(v)|^2dxdt. \qquad (3.16)$$

Identity (3.16) is a very special case of the identity (3.2). It can be viewed as a form of the so–called *hypercircle theorem*, which was derived for linear elasticity problems in [19] by means of Helmgholtz type decomposition and by variational methods for problems with quadratic energy functionals in [12]. In [26], such an identity was derived for the parabolic equation $u_t - \Delta u = f$. It reads

$$\|\nabla(v-u)\|^2_{Q_T}+\|y-p\|^2_{Q_T}+\Big[\|v-u\|^2_\Omega\Big]_0^T=\|\nabla v-y\|^2_{Q_T},$$

where $v(x,t)$ is an approximation of $u(x,t)$ and $y(x,t)$ is an approximation of $p(x,t)=\nabla u(x,t)$ such that

$$\int\limits_{Q_T}(y\cdot\nabla w-fw+v_tw)dxdt=0 \qquad \forall w\in \overset{\circ}{W}{}^{1,1}_2(Q_T). \qquad (3.17)$$

In fact, (3.17) means that y and v must satisfy the pointwise condition

$$\mathbb{R}(v,y):=\mathrm{div}y-v_t+f=0 \quad \text{a.e. in } Q_T.$$

Identity (3.16) holds under a much weaker condition (3.15), which claims that the residual $\mathrm{Div}\tau-v_t+f$ must belong to $Y_\nabla(\Omega)$ (orthogonal complement to the subspace of solenoidal fields). This difference is caused by specific features of the Stokes problem whose solution belongs to the subspace of divergence free functions.

The right hand side of (3.16) contains only known functions and it is fully computable what, from the first glance, makes this identity attractive for practical purposes. However, on this way we are faced with two essential technical problems stipulated by the conditions (2.4) and (3.15). There exist various ways to generate conforming (internal) approximations of the set $\overset{\circ}{S}{}^{1,1}_2(Q_T,\mathbb{R}^d)$ (e.g., see [10,32]), but, regrettably, the corresponding constructions may be rather cumbersom (especially if $d=3$). At the same time, approximate solutions obtained by simple approximations in combination with commonly used numerical procedures satisfy (2.4) only approximately. For them, (3.16) is not applicable. Besides, satisfying (3.15) may also generate difficulties. For these reasons, we suggest another method, which is free from the above restrictions.

§4. Identity using additional regularity

Now we consider the case, where the solution and approximations possess additional differentiability in time, i.e.,

$$u, v \in \overset{\circ}{W}{}_2^{1,1,+}(Q_T) := \Big\{ w \in \overset{\circ}{W}{}_2^{1,1}(Q_T, \mathbb{R}^d) \ \mid \ w_t \in \overset{\circ}{W}{}_2^{1,0}(Q_T, \mathbb{R}^d) \Big\} \quad (4.1)$$

and

$$\sigma, \tau \ \in \ \Sigma^+_{\rm Div}(Q_T) := \Big\{ \tau \in \Sigma_{\rm Div}(Q_T) \ \mid \ \tau_t \in \Sigma(Q_T) \Big\}. \qquad (4.2)$$

For these functions, we define

$$\boldsymbol{\mu}^+(e, e_t, \widehat{\eta}) := \|\varepsilon(e)\|^2_{\nu,Q_T} + \Big[\|e\|^2_\Omega \Big]_0^T + 2\|e_t\|^2_{Q_T} + \Big[\|\widehat{\eta}\|_{\nu^{-1},\Omega} \Big]_0^T + \|\widehat{\eta}\|^2_{\nu^{-1},Q_T}.$$

Theorem 4.1. *In addition to the conditions of Theorem* 3.1 *let the functions satisfy* (4.1) *and* (4.2). *Then*

$$\boldsymbol{\mu}^+(e, e_t, \widehat{\eta}) = \|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T} + \Big[\|\mathbb{T}(v,\tau,q)\|^2_{\nu,\Omega} \Big]_0^T$$
$$+ 2 \int\limits_{Q_T} \Big(\mathbb{T}_t(v,\tau,q) : \varepsilon(e) - \mathbb{H}(v,\psi)\cdot(e+e_t) \Big) dxdt + 2\mathcal{L}^+(\mathring{v},\, e, \widehat{\eta}), \quad (4.3)$$

where $\mathbb{H}$ *and* ψ *are the same as in Theorem* 1 *and*

$$\mathcal{L}^+(\mathring{v},\, e, \widehat{\eta}) := \int\limits_{Q_T} \Big(\widehat{\eta} : \varepsilon(\mathring{v} + \mathring{v}_t) + e_t\cdot(\mathring{v}+\mathring{v}_t) \Big) dxdt + \int\limits_{Q_T} (\mathbb{H}(v,\tau,\psi))\cdot(\mathring{v}+\mathring{v}_t) dxdt.$$

Proof. We use (3.3) and set $w = u_t - v_t$. Then, (3.3) reads

$$\int\limits_{Q_T} \Big(|e_t^2| + \nu\varepsilon(e) : \varepsilon(e_t) \Big) dxdt$$
$$= \int\limits_{Q_T} (v_t\cdot e_t + \nu\varepsilon(v) : \varepsilon(e_t) - f\cdot e_t - p{\rm div} e_t)\, dxdt. \qquad (4.4)$$

Since $e_t = 0$ on S_T, it holds

$$\int\limits_{Q_T} (e_t \cdot \mathrm{Div}\tau + \tau : \varepsilon(e_t))dxdt = \int\limits_0^T \int\limits_\Gamma (\tau\, n) \cdot e_t \, dxdt = 0. \tag{4.5}$$

From (4.4) and (4.5), we obtain

$$\int\limits_{Q_T} \Big(|e_t^2| + \nu\varepsilon(e) : \varepsilon(e_t) \Big) dxdt$$
$$= \int\limits_{Q_T} (q - p)\mathrm{div} e_t dxdt - \int\limits_{Q_T} (\mathbb{T}(v,\tau,q) : \varepsilon(e_t) + \mathbb{R}(v,\tau) \cdot e_t) \, dxdt. \tag{4.6}$$

It is easy to see that

$$\int\limits_{Q_T} (\mathbb{T}(v,\tau,q) : \varepsilon(e_t) \, dxdt = \int\limits_{Q_T} \frac{1}{\nu} \mathbb{T}(v,\tau,q) : (\nu\varepsilon(v_t) - \sigma_t - p_t\mathbb{I}) \, dxdt$$
$$= \int\limits_{Q_T} \frac{1}{\nu} \mathbb{T}(v,\tau,q) : (\tau_t + q_t\mathbb{I} - \sigma_t - p_t\mathbb{I}) dxdt - \int\limits_{Q_T} \frac{1}{\nu} \mathbb{T}(v,\tau,q) : \mathbb{T}_t(v,\tau,q) dxdt$$
$$= \int\limits_{Q_T} \frac{1}{\nu} \mathbb{T}(v,\tau,q) : (\widehat{\tau}_t - \widehat{\sigma}_t) dxdt - \frac{1}{2} \Big[\|\mathbb{T}(v,\tau,q)\|^2_{\nu,\Omega} \Big]_0^T. \tag{4.7}$$

Now, we rearrange the integral term

$$\int\limits_{Q_T} \frac{1}{\nu} \mathbb{T}(v,\tau,q) : (\widehat{\tau}_t - \widehat{\sigma}_t) dxdt = \int\limits_{Q_T} \frac{1}{\nu} (\widehat{\tau} - \nu\varepsilon(v)) : (\widehat{\tau}_t - \widehat{\sigma}_t) dxdt$$
$$= \int\limits_{Q_T} \frac{1}{\nu} (\widehat{\sigma} - \nu\varepsilon(v)) : (\widehat{\tau}_t - \widehat{\sigma}_t) dxdt + \int\limits_{Q_T} \frac{1}{\nu} (\widehat{\tau} - \widehat{\sigma}) : (\widehat{\tau}_t - \widehat{\sigma}_t) dxdt.$$

Here

$$\int\limits_{Q_T} \frac{1}{\nu} (\widehat{\sigma} - \nu\varepsilon(v)) : (\widehat{\tau}_t - \widehat{\sigma}_t) dxdt = \int\limits_{Q_T} \varepsilon(u - v) : (\widehat{\tau}_t - \widehat{\sigma}_t) dxdt$$
$$= \int\limits_{Q_T} \varepsilon(e) : (\nu\varepsilon(v_t) - \widehat{\tau}_t) dxdt + \int\limits_{Q_T} \nu\varepsilon(e) : \varepsilon(u_t - v_t) dxdt$$

$$= \int\limits_{Q_T} \varepsilon(e) : (\nu\varepsilon(v_t) - \widehat{\tau}_t)dxdt - \int\limits_{Q_T} \nu\varepsilon(e) : \varepsilon(e_t)dxdt.$$

Hence

$$\int\limits_{Q_T} \frac{1}{\nu}\mathbb{T}(v,\tau,q) : (\widehat{\tau}_t - \widehat{\sigma}_t)dxdt = \frac{1}{2}\Big[\|\widehat{\eta}\|_{\nu^{-1},\Omega}\Big]_0^T - \int\limits_{Q_T} \varepsilon(e) : (\tau + q\mathbb{I} - \nu\varepsilon(v))_t \, dxdt - \int\limits_{Q_T} \nu\varepsilon(e) : \varepsilon(e_t)dxdt. \tag{4.8}$$

Using (4.7) and (4.8), we rewrite (4.6) as follows:

$$\int\limits_{Q_T} \Big(|e_t^2| + \nu\varepsilon(e) : \varepsilon(e_t)\Big)dxdt = \frac{1}{2}\Big[\|\mathbb{T}(v,\tau,q)\|^2_{\nu,\Omega}\Big]_0^T - \int\limits_{Q_T} \mathbb{R}(v,\tau)\cdot e_t \, dxdt + \int\limits_{Q_T} \nu\varepsilon(e) : \varepsilon(e_t)dxdt - \frac{1}{2}\Big[\|\widehat{\eta}\|_{\nu^{-1},\Omega}\Big]_0^T + \int\limits_{Q_T} \varepsilon(e) : \mathbb{T}_t(v,\tau,q)\, dxdt + \int\limits_{\Omega} (q-p)\mathrm{div} e_t dxdt.$$

We arrive at the identity

$$\|e_t\|^2_{Q_T} + \frac{1}{2}\Big[\|\widehat{\eta}\|_{\nu^{-1},\Omega}\Big]_0^T = \frac{1}{2}\Big[\|\mathbb{T}(v,\tau,q)\|^2_{\nu,\Omega}\Big]_0^T - \int\limits_{Q_T} \mathbb{R}(v,\tau) e_t \, dxdt + \int\limits_{Q_T} \varepsilon(e) : \mathbb{T}_t(v,\tau,q)\, dxdt + \int\limits_{\Omega} (q-p)\mathrm{div} e_t dxdt. \tag{4.9}$$

Notice that for any $v_0 \in \overset{\circ}{S}{}^{1,1}(Q_T, \mathbb{R}^d)$

$$\int\limits_{\Omega} (p-q)\mathrm{div}(v_t - u_t)dx \ = \ \int\limits_{\Omega} (p-q)\mathrm{div}\mathring{v}_t dx.$$

In view of (2.7), with $w = (v - v_0)_t$

$$\int\limits_{Q_T} (p - q)\mathrm{div}\mathring{v}_t dx = \int\limits_{Q_T} (\nu\varepsilon(u) : \varepsilon(\mathring{v}_t) - f \cdot \mathring{v}_t + u_t \cdot \mathring{v}_t - q\mathrm{div}\mathring{v}_t)\, dxdt$$
$$= -\int\limits_{Q_T} \mathbb{R}(v, \tau) \cdot \mathring{v}_t dxdt - \int\limits_{Q_T} (\widehat{\eta} : \varepsilon(\mathring{v}_t) + e_t \cdot \mathring{v}_t)dxdt. \tag{4.10}$$

From (4.9) and (4.10) it follows that

$$2\|e_t\|^2_{Q_T} + \Big[\|\widehat{\eta}\|_{\nu^{-1},\Omega}\Big]_0^T$$
$$= \Big[\|\mathbb{T}(v, \tau, q)\|^2_{\nu,\Omega}\Big]_0^T - 2\int\limits_{Q_T} (\mathbb{R}(v, \tau) - \nabla\psi) \cdot (e_t - \mathring{v}_t)\, dxdt$$
$$+ 2\int\limits_{Q_T} \varepsilon(e) : \mathbb{T}_t(v, \tau, q)\, dxdt + 2\int\limits_{Q_T} (\widehat{\eta} : \varepsilon(\mathring{v}_t) + e_t \cdot \mathring{v}_t)dxdt, \tag{4.11}$$

where $\psi \in L_2(0, T; H^1(\Omega))$. Summation of (3.2) and (4.11) yields the identity

$$\boldsymbol{\mu}_1^+(e, e_t, \widehat{\eta}) = \|\mathbb{T}(v, \tau, q)\|^2_{\nu^{-1},Q_T} + \Big[\|\mathbb{T}(v, \tau, q)\|^2_{\nu,\Omega}\Big]_0^T$$
$$+2\int\limits_{Q_T}\Big(\varepsilon(e) : \mathbb{T}_t(v, \tau, q) - \mathbb{H}(v, \tau, \psi) \cdot (e - \mathring{v}) - \mathbb{H}(v, \tau, \psi) \cdot (e_t - \mathring{v}_t)\Big)dxdt$$
$$+2\int\limits_{Q_T} (\widehat{\eta} : \varepsilon(\mathring{v}_t) + e_t \cdot \mathring{v}_t)dxdt + 2\int\limits_{Q_T}\Big(\widehat{\eta} : \varepsilon(\mathring{v}) - e \cdot \mathring{v}_t\Big)dxdt + 2\Big[\int\limits_{\Omega} \mathring{v} \cdot e\, dx\Big]_0^T.$$

The last three integral terms are converted to

$$2\int\limits_{Q_T}\Big(\widehat{\eta} : \varepsilon(\mathring{v} + \mathring{v}_t) + e_t \cdot (\mathring{v} + \mathring{v}_t)\Big)dxdt$$

and we arrive at the identity (4.3). □

§5. Estimates of the distance to the solution

First term in the right hand side of (3.2) contains only known functions and can be directly computed. Other terms include unknown error functions e and $\widehat{\eta}$. These terms are linear functionals (with respect to e and $\widehat{\eta}$) and can be estimated from above by the norms that contained in $\boldsymbol{\mu}$.

The advanced identity (4.3) has a similar structure. Its right hand side has two computable terms and a linear functional depending on e, e_t, and $\widehat{\eta}$. These functions also appear at the left side in the form of squared norms. Due to this structure, the identities (3.2) and (4.3) imply fully computable bounds of deviations from the exact solutions.

Below we use a simple way to estimate the terms containing unknown functions. The results are formulated as Corollaries 5.1 and 5.2 to Theorems 3.1 and 4.1, respectively. Of course, this rather straightforward estimation method is not the only possible one. Using more sophisticated methods and additional information on properties of approximations (or a certain post–processing of them) one may try to get much sharper bounds. Here we do not discuss these rather special questions (the reader will find several variants of advanced estimates in [25,27,31]). The estimates derived in this section are intended to show that error identities (3.2) and (4.3) provide a basis for *a posteriori error control* of numerical approximations. At this point, it should be said that there exist alternative approaches to a posteriori error estimation used by numerical analysts solving viscous flow problems. The corresponding literature is very large, we cite just a few publications [1, 8, 14, 15, 17]. Most of them use special properties of a particular approximation method combined with certain post–processing procedures (e.g., regularisation or averaging) in order to obtain error indicators for mesh adaptive algorithms. These estimates are mainly focused not at guaranteed bounds of the global error, but at identifying those subregions that make the greatest contribution to the total error. Estimates derived within the framework of the functional approach (such as those presented below) are essentially different. They do not use special properties of approximations and exact solutions (such as Galerkin orthogonality and extra regularity of solutions or meshes) and are valid for any approximation from the basic energy space.

5.1. Estimates generated by the identity (3.2). By (2.1) and Young's inequality, we have

$$2\left|\int_\Omega \mathbb{H}(v,\tau,\psi)\cdot e dx\right| \leqslant \beta_1(t)\|\varepsilon(e)\|^2_{\nu,\Omega} + \frac{C^2}{\nu\beta_1(t)}\|\mathbb{H}(v,\tau,\psi)\|^2_\Omega, \quad (5.1)$$

where $\beta_1(t)$ is a positive bounded function. Consider the term $\mathbb{L}(\mathring{v}, e, \widehat{\eta})$. Without a loss of generality we assume that $v(x,0)$ is a divergence free function and set $v_0(x,0) = v(x,0)$. Then, the first term of $\mathbb{L}(\mathring{v}, e, \widehat{\eta})$ has a simpler form:

$$\Big[\int_\Omega \mathring{v}\cdot e\, dx\Big]_0^T = \int_\Omega \mathring{v}(x,T)\cdot e(x,T)\, dx. \quad (5.2)$$

Next, we have

$$2\left|\int_\Omega \mathring{v}(x,T)\cdot e(x,T)\, dx\right| \leqslant \alpha(t)\|e(\cdot,T)\|^2_\Omega + \frac{1}{\alpha(t)}\|\mathring{v}(\cdot,T)\|^2_\Omega, \ \ \alpha > 0, \quad (5.3)$$

and

$$2\left|\int_\Omega \Big(\widehat{\eta} : \varepsilon(\mathring{v}) - e\cdot \mathring{v}_t\Big)dx\right|$$
$$\leqslant \beta_2(t)\|\widehat{\eta}\|^2_\Omega + \beta_3(t)\|\varepsilon(e)\|^2_\Omega + \frac{1}{\beta_2(t)}\|\varepsilon(\mathring{v})\|^2_\Omega + \frac{C^2}{\beta_3(t)}\|\mathring{v}_t\|^2_\Omega, \quad (5.4)$$

where C is a constant in (2.1) and $\alpha(t)$, $\beta_2(t)$, and $\beta_3(t)$ are positive functions bounded for any $t \in [0,T]$. The last term of $\mathbb{L}(\mathring{v}, e, \widehat{\eta})$ is bounded from above by the estimate analogous to (5.1). Thus, (5.1)–(5.4), lead us to the following corollary of the main identity.

Corollary 5.1. *For any v, τ, and q satisfying* (3.1)*, it holds*

$$\mathbf{m}(e,\widehat{\eta};\alpha,\beta_1,\beta_2,\beta_3) \leqslant \|e(\cdot,0)\|^2_\Omega + \|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T}$$
$$+ \int_0^T \frac{C^2+\nu}{\nu\beta_1}\|\mathbb{H}(v,\tau,\psi)\|^2_\Omega dt \, + \mathrm{I}(\mathring{v}), \quad (5.5)$$

where

$$\mathbf{m}(e,\widehat{\eta};\alpha,\beta_1,\beta_2,\beta_3):=\int\limits_0^T\Big((1-\beta_1-\beta_3)\|\varepsilon(e)\|^2_{\nu,\Omega}+(1-\beta_2)\|\widehat{\eta}\|^2_{\nu^{-1},\Omega}\Big)dt$$
$$+(1-\alpha)\|e(\cdot,T)\|^2_\Omega,$$

$\alpha\in(0,1]$, $\beta_2(t)\leqslant 1$, $\beta_1(t)+\beta_3(t)\leqslant 1$, *and*

$$\mathrm{I}(\mathring{v}):=\frac{1}{\alpha}\|\mathring{v}(\cdot,T)\|^2_\Omega+\int\limits_0^T\left(\frac{1}{\beta_1(t)}\|\mathring{v}\|^2_\Omega+\frac{1}{\beta_2(t)}\|\varepsilon(\mathring{v})\|^2_\Omega+\frac{C^2}{\beta_3(t)}\|\mathring{v}_t\|^2_\Omega\right)dt.$$

The right hand side of (5.5) is nonnegative. Assume that it is equal to zero. Then $\mathrm{I}(\mathring{v})=0$ and $v(x,0)=u_0(x)$ so that v satisfies the initial condition. Since $\mathrm{I}(\mathring{v})=0$, there exists a divergence free field v_0 such that $v-v_0=0$ so that v is also divergence free. Also,

$$\mathbb{T}(v,\tau,q)=\mathbb{H}(v,\tau,\psi)=0, \tag{5.6}$$

what means that

$$\mathrm{Div}\tau+f-v_t=\nabla\psi\quad\text{and}\quad\tau=\nu\varepsilon(v)-q\mathbb{I}.$$

Hence for any $w\in\mathring{W}^{1,0}_2(Q_T,\mathbb{R}^d)$ it holds

$$\int\limits_{Q_T}(v_t\cdot w+\nu\varepsilon(v):\varepsilon(w)-(q+\psi)\mathrm{div}w)\,dxdt=\int\limits_{Q_T}f\cdot w dxdt.$$

Comparing this relation with (2.7) we conclude that the majorant attains zero if an only if $v=u$, $\widehat{\tau}=\widehat{\sigma}$, $q=p-\psi$, and $\tau=\sigma+\psi\mathbb{I}$.

By (5.2)–(5.4), we also deduce a computable lower bound of the combined error norm:

$$\int\limits_0^T\Big((1+\beta_1+\beta_3)\|\varepsilon(e)\|^2_{\nu,\Omega}+(1+\beta_2)\|\widehat{\eta}\|^2_{\nu^{-1},\Omega}\Big)dt+(1+\alpha)\|e(\cdot,T)\|^2_\Omega$$
$$\geqslant\|e(\cdot,0)\|^2_\Omega+\|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T}-\int\limits_0^T\frac{C^2+\nu}{\nu\beta_1}\|\mathbb{H}(v,\tau,\psi)\|^2_\Omega dt-\mathrm{I}(\mathring{v}).$$

This bound is sensible provided that the last two terms in the right hand side are essentially smaller then the first two ones.

5.2. Finding ψ and v_0. Nonnegative functional in the right hand sids of (5.5) depends on v, τ, q, ψ, and v_0. The first three functions are known (they are certain approximations) while ψ and v_0 are at our disposal. The choice of these functions must be subject to a natural requirement: reduce the value of the right side of (5.5).

The best function ψ must minimise the norm $\|\mathbb{H}(v,\tau,\psi)\|_{Q_T}$. For any $t \in (0,T)$ such a function is defined by the relation

$$\int\limits_{\Omega} \nabla\psi_* \cdot \nabla\psi dx = \int\limits_{\Omega} \mathbb{R}(v,\tau) \cdot \nabla\psi dx \qquad \forall \psi \in \overset{\circ}{H}{}^1(\Omega)$$

and

$$\|\mathbb{H}(v,\tau,\psi_*)\|^2_{\Omega} = \|\mathbb{R}(v,\tau)\|^2_{\Omega} - \|\nabla\psi_*\|^2_{\Omega}.$$

In practice ψ_* is replaced by a certain approximation $\psi_{*,h}$ found in a finite dimensional subspace $V_h \subset \overset{\circ}{H}{}^1(\Omega)$. It generates a computable upper bound

$$\|\mathbb{H}(v,\tau,\psi_*)\|^2_{Q_T} \leqslant \|\mathbb{H}(v,\tau,\psi_{*,h})\|^2_{Q_T} = \|\mathbb{R}(v,\tau)\|^2_{\Omega} - \|\nabla\psi_{*,h}\|^2_{\Omega}.$$

Theorem 3.1 is valid for any function $\mathring{v} = v - v_0$, where $v_0 \in \overset{\circ}{S}{}^{1,1}_2(Q_T, \mathbb{R}^d)$. In other words, the identity (3.2) is indifferent to the choice of the divergence free counterpart v_0 for the function v. This is not the case for the majorant (5.5), which contains the term $\mathrm{I}(\mathring{v})$. There are two ways to evaluate this term. The first (and the simplest) way is to project v to a certain finite dimensional subspace of $\overset{\circ}{S}{}^{1,1}_2(Q_T, \mathbb{R}^d)$ and compute the norms of $\mathring{v}$. Then, the majorant suggests a tool for fully reliable error control avoiding knowledge of the constant $\mathbb{C}(\Omega)$ in the inf-sup (LBB) condition associated with the domain Ω.

Another way is to deduce an upper bound of $\mathrm{I}(\mathring{v})$ using the orthogonal projector to $\overset{\circ}{S}{}^1(\Omega)$, i.e., we set $v_0 = \pi_S(v)$, where

$$\|\varepsilon(v - \pi_S v)\|_{\Omega} \leqslant \|\varepsilon(v - w_0)\|_{\Omega} \qquad \forall\, w_0 \in \overset{\circ}{S}{}^1(\Omega).$$

In this case, the constant $\mathbb{C}(\Omega)$ arises in the estimate

$$\|\varepsilon(v - \pi_S v)\|_{\Omega} \leqslant \mathbb{C}(\Omega)\|\mathrm{div} v\|_{\Omega}. \tag{5.7}$$

Getting sharp estimates of $\mathbb{C}(\Omega)$ is an important and complex problem that has been studied by a number of authors (e.g., see [2,3,11,16,18,33]). However, such estimates are known only for several simple domains (mainly in $\mathbb{R}^2$). Advanced forms of the estimate (5.7) are studied in [23,29]. They are based on decomposition of Ω into a collection of simple subdomains, for which the constants $\mathbb{C}$. Using them we can estimate the norms in I and find a computable upper bound for this term.

5.3. Particular cases. Choosing $\alpha(t)$ and $\beta_i(t)$, $i = 1, 2, 3$, we obtain particular forms of (5.5). Consider three examples where the estimates are related to each of the three norms that jointly form the combined error norm in (5.5). For simplicity, we assume that $v(x, 0) = u_0(x)$. Then $\|e(\cdot, 0)\|_\Omega = 0$.

1. If $\alpha = \frac{1}{2}$, $\beta_1 = \beta_3 = \frac{1}{2}$ and $\beta_2 = 1$, then (5.5) implies a bound for the maximum norm

$$\frac{1}{2} \max_{t\in(0,T)} \|e(\cdot, t)\|^2_\Omega \leqslant \|\mathbb{T}(v, \tau, q)\|^2_{\nu^{-1},Q_T} + 2\int\limits_0^T \left(1 + \frac{C^2}{\nu}\right) \|\mathbb{H}(v, \tau, \psi)\|^2_\Omega dt$$
$$+ 2\|\mathring{v}(\cdot, T)\|^2_\Omega + \int\limits_0^T \left(2\|\mathring{v}\|^2_\Omega + \|\varepsilon(\mathring{v})\|^2_\Omega + 2C^2\|\mathring{v}_t\|^2_\Omega\right) dt. \quad (5.8)$$

2. If $\alpha = 1$, $\beta_1 = \beta_2 = \beta_3 = \frac{1}{2}$, then we deduce an upper bound for the error in terms of stresses

$$\frac{1}{2}\|\widehat{\eta}\|^2_{Q_T} \leqslant \|\mathbb{T}(v, \tau, q)\|^2_{\nu^{-1},Q_T} + 2\int\limits_0^T \left(1 + \frac{C^2}{\nu}\right) \|\mathbb{H}(v, \tau, \psi)\|^2_\Omega dt$$
$$+ \|\mathring{v}(\cdot, T)\|^2_\Omega + 2\int\limits_0^T \left(\|\mathring{v}\|^2_\Omega + \|\varepsilon(\mathring{v})\|^2_\Omega + C^2\|\mathring{v}_t\|^2_\Omega\right) dt. \quad (5.9)$$

3. Finally, if $\alpha = 1$, $\beta_1 = \beta_3 = \frac{1}{4}$, $\beta_2 = 1$, then we have another error majorant

$$\frac{1}{2}\|\varepsilon(e)\|^2_{Q_T} \leqslant \|\mathbb{T}(v, \tau, q)\|^2_{\nu^{-1},Q_T} + 4\int\limits_0^T \left(1 + \frac{C^2}{\nu}\right) \|\mathbb{H}(v, \tau, \psi)\|^2_\Omega dt$$
$$+ \|\mathring{v}(\cdot, T)\|^2_\Omega + \int\limits_0^T \left(4\|\mathring{v}\|^2_\Omega + \|\varepsilon(\mathring{v})\|^2_\Omega + 4C^2\|\mathring{v}_t\|^2_\Omega\right) dt. \quad (5.10)$$

5.4. Estimates generated by the identity (4.3). Theorem 4.1 also implies computable error bounds provided that the solution and approximations satisfy necessary regularity conditions. For positive bounded functions $\alpha_i(t)$, $i = 1, 2$ and $\beta_j(t)$, $j = 1, 2, 3$, we define

$$\mathbf{m}^+(e, \widehat{\eta};\, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) := \Big[\|e\|^2_\Omega\Big]_0^T + \Big[\|\widehat{\eta}\|_{\nu^{-1},\Omega}\Big]_0^T$$

$$+ \int\limits_0^T \Big((1-\beta_1-\beta_3)\|\varepsilon(e)\|^2_{\nu,\Omega} + (2-\alpha_1-\alpha_2)\|e_t\|^2_\Omega + (1-\beta_2)\|\widehat{\eta}\|^2_{\nu^{-1},\Omega}\Big)dt,$$

which is a nonnegative measure of the distance to the functions u and $\widehat{\sigma}$. The functional

$$\mathrm{I}^+(\mathring{v}) := 2\int\limits_\Omega (\mathbb{H}(v,\tau,\psi))\cdot(\mathring{v}+\mathring{v}_t)dxdt + \left(\frac{1}{\alpha_2}+\frac{1}{\beta_2}\right)\|\mathring{v}+\mathring{v}_t\|^2_\Omega$$

controls the distance between v and the set of divergence free fields. By the same arguments as before, we obtain the following result.

Corollary 5.2. *If the assumptions (4.1) and (4.2) hold, then*

$$\mathbf{m}^+(e, \widehat{\eta};\, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \leqslant \|\mathbb{T}(v,\tau,q)\|^2_{\nu^{-1},Q_T} + \Big[\|\mathbb{T}(v,\tau,q)\|^2_{\nu,\Omega}\Big]_0^T$$

$$+ \int\limits_0^T \left(\Big(\frac{C^2}{\nu\beta_1}+\frac{1}{\alpha_1}\Big)\|\mathbb{H}(v,\tau,\psi)\|^2_\Omega + \frac{1}{\nu\beta_3}\|T_t^2(v,\tau,q)\|_\Omega\right)dt + \mathrm{I}^+(\mathring{v}),$$

where $\alpha_1+\alpha_2 \leqslant 2$, $\beta_1+\beta_3\leqslant 1$, $\beta_2 \leqslant 1$.

As in the previous case, by varying the parameters we can obtain various particular forms of this estimate, which highlight one or another part of the measure $\mathbf{m}^+$.

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