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ON THE SUFFICIENT CONDITIONS FOR THE S-SHAPED BUCKLEY–LEVERETT FUNCTION

Abstract. The flux function in the Buckley–Leverett equation, that is, the function characterizing the ratio of the relative mobility functions of the two phases, is considered. The common conjecture stating that any convex mobilities result in an S-shaped Buckley–Leverett function is analyzed and disproved by a counterexample. Additionally, sufficient conditions for the S-shaped Buckley–Leverett function are given. The class of functions satisfying those conditions is proven to be closed under multiplication. Some functions from known relative mobility models are confirmed to be in that class.

With deep admiration and gratitude, this work is dedicated to Nina Nikolaevna Uraltseva on the occasion of her 90th birthday

1. Introduction

In fluid dynamics, the Buckley–Leverett equation is one of the simplest conservation laws used to model two-phase flow in porous media. The equation is given by:

$$s_t + f(s)_x = 0, \tag{1}$$

where $s = s(x,t)$ is the water saturation and f is the *fractional flow function*, also known as the Buckley–Leverett function. This function characterizes the ratio of relative mobilities of the two phases, which is expressed as:

$$f(s) = \frac{m_a(s)}{m_a(s) + m_b(1-s)},$$

where m_a and m_b represent the *relative phase mobilities*, typically water and oil mobilities in the context of petroleum sciences. These mobility functions are often assumed to be increasing and convex.

Key words and phrases: porous medium, relative phase mobility, Buckley–Leverett function, S-shaped functions, convex analysis.

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The Buckley–Leverett equation (1) could be paired with an arbitrary Cauchy problem or some other initial-boundary problem. There are ample results on existence, non-uniqueness (in general) and uniqueness (in certain classes of solutions) and even some explicit solution formulae for such problems (see e. g. [1, Chapter 2]). But the model problem for it is the Riemann problem

$$s(x,0) = \begin{cases} s^L, & x \leqslant 0, \\ s^R, & x > 0, \end{cases} \tag{2}$$

describing the evolution of a single discontinuity.

In the early works on the subject the assumption that the Buckley–Leverett function f is S-shaped is prevalent. This is likely due to the fact that an S-shaped function is used as the only example in the principal work by Buckley and Leverett [2], and this assumption is repeated in many papers thereafter for this and other more generalized problems. This assumption is no longer critical in the case of the Buckley–Leverett equation (1), since the Riemann problem (2) for it can be solved analytically for any sufficiently smooth function f by the convex hull construction given by Oleinik (see [3, 4] or [1, Sec. 2.4]). Nonetheless, it is still often important in more general conservation systems that include more phases or components, or have additional parameters such as temperature (see [5–10]). However, there is no comprehensive research on when f is actually S-shaped. The prevalent conjecture among engineers is that convex mobilities result in an S-shaped fractional flow function. Some mathematicians hold similar expectations. The only paper known to the author (and the one that inspired this work) investigating sufficient conditions for the S-shaped function is the paper by Castañeda [11]. That paper proves that when relative phase mobilities m_a and m_b are power functions with exponent greater than 1, the resultant Buckley–Leverett function f is S-shaped. It also states that the author could not find a counterexample to the convex conjecture.

Even in the context of the Riemann problem (2) for the equation (1) the S-shaped property still has significance. When we solve the problem by the convex hull construction, the straight segments on the convex hull of f correspond to shocks (travelling discontinuities) and strictly convex parts correspond to continuous rarefaction waves. Therefore, when f is S-shaped, the solution contains at most one shock, whereas for the function f with more than one inflection point, some Riemann problem solutions may contain two or more shocks. Noteworthy in this context is [7, Claim 17]. It does not give any conditions for when f is S-shaped, but instead assumes the mobilities are convex and proves the solution of the particular Riemann problem with $s^L = 1$, $s^R = 0$ always contains only one shock.

In this paper, we provide sufficient conditions for the Buckley–Leverett function to be S-shaped. The paper has the following structure: Sec. 2 defines a special subclass $\mathcal{M}$ of convex mobilities and proves the theorem asserting that mobilities m_a, m_b from that class always give an S-shaped fractional flow function f. Additionally, it proves the proposed class of convex functions is closed under multiplication. Sec. 3 presents several counterexamples of two convex functions outside $\mathcal{M}$ that produce a fractional flow function that is not S-shaped, and contains the graphs illustrating the provided counterexamples. Appendix A applies the proposed conditions to some known relative mobility models.

2. Sufficient conditions for the S-shaped function

Definition 1. Let $\mathcal{M}$ be a set of functions $m \in \mathcal{C}^2[0,1]$, such that:

(C1) Function m is fixed at zero and convex, i. e.

(C1.1) $m(0) = m'(0) = 0$;

(C1.2) $m''(s) > 0$ for $s > 0$.

(C2) Additionally, $\dfrac{m''}{m'}$ is a decreasing function on $(0,1)$.

Remark 1. Any power function $m(s) = As^a$ with $A > 0$ and power $a > 1$ is in $\mathcal{M}$, since $\dfrac{m''}{m'} = \dfrac{a-1}{s}$ is decreasing. Thus, Theorem 1 below covers the result of [11, Theorem 409].

Remark 2. It trivially follows from (C1) that every function $m \in \mathcal{M}$ is positive and increasing, i. e. $m(s), m'(s) > 0$ for $s > 0$. We use this property repeatedly without reference.

Lemma 1. *Let $m \in \mathcal{M}$. Then $\dfrac{m'}{m}$ is also a decreasing function. Therefore, for all $m \in \mathcal{M}$ the following variation of (C2) holds:*

(C2*) $\dfrac{m'}{m}$ and $\dfrac{m''}{m'}$ are decreasing functions on $(0,1)$.

Proof. To prove $\dfrac{m'}{m}$ is a decreasing function we need to demonstrate

$$\left(\frac{m'}{m}\right)' = \frac{m''m - m'^2}{m^2} < 0,$$

which is equivalent to

$$\frac{m'}{m} > \frac{m''}{m'},$$

which, considering (C1) and (C2), follows from Cauchy's Mean Value Theorem:

$$\frac{m'(x)}{m(x)} = \frac{m'(x) - m'(0)}{m(x) - m(0)} = \frac{m''(\widetilde{x})}{m'(\widetilde{x})} > \frac{m''(x)}{m'(x)}$$

holds for all $x \in (0, 1)$ and certain $\widetilde{x} \in (0, x)$. □

Remark 3. Let $m \in \mathcal{M}$. Then it is easy to see that $\lim\limits_{s\to 0} \frac{m''}{m} = +\infty$.

Remark 4. Though we require only $\mathcal{C}^2$ smoothness from our functions, (C2) guarantees that the derivative $\left(\frac{m''}{m'}\right)'$ in the sense of distributions exists and is negative, and thus exists the generalized derivative

$$m''' = m' \left(\frac{m''}{m'}\right)' + \frac{(m'')^2}{m'}.$$

Therefore, in the proofs below we operate the third derivative f''', examining its sign to discern the local monotonicity of the second derivative f''.

Theorem 1. *Let m_a and m_b be two mobility functions from the class $\mathcal{M}$. Then the fractional flow function $f(s) = \frac{m_a(s)}{m_a(s) + m_b(1-s)}$ is S-shaped, that is, there exists a unique inflection point $s^* \in (0, 1)$, such that $f''(s^*) = 0$.*

Proof. We note that $(m_b(1 - s))' = -m_b'(1 - s)$, thus m_b changes sign with every derivative. Keeping that in mind, we omit the variables s and $1 - s$ in the notation hereafter, implying that m_b and its derivatives are applied to the variable $1 - s$.

Denote $M = m_a + m_b$ and $h = m_a' m_b + m_a m_b'$. It is easy to calculate

$$f' = \frac{h}{M^2}, \qquad f'' = \frac{h'M - 2M'h}{M^3}, \tag{3}$$

where $h' = m_a'' m_b - m_a m_b''$. We solve the equation $f'' = 0$, or equivalently

$$g := h' - \frac{2M'}{M} h = 0, \tag{4}$$

and aim to demonstrate that it has a unique solution on $(0, 1)$.

Since $M, h > 0$ trivially, we note that if h' and $-M'$ have the same sign, then f'' and g have it too. Thus, any solution of (4) must satisfy

$$h'M' > 0 \quad \text{or} \quad h' = M' = 0. \tag{5}$$

Note also that M' is increasing due to (C1) (since $M'' = m_a'' + m_b'' > 0$) and it changes sign exactly one time from negative to positive (since $M'(0) = -m_b'(1) < 0$ and $M'(1) = m_a'(1) > 0$). Denote by s_M the sign change point:

$$M'(s) < 0, \quad 0 \leqslant s < s_M; \qquad M'(s_M) = 0; \qquad M'(s) > 0, \quad s_M < s \leqslant 1.$$

Similarly, due to (C2*) we deduce that $\dfrac{m_a''}{m_a} = \dfrac{m_a''}{m_a'}\dfrac{m_a'}{m_a}$ is decreasing and, keeping in mind the difference in the variable, $\dfrac{m_b''}{m_b} = \dfrac{m_b''}{m_b'}\dfrac{m_b'}{m_b}$ is increasing. Thus, $\dfrac{m_a''}{m_a} - \dfrac{m_b''}{m_b}$ is a decreasing function and has one sign change from positive to negative (due to Remark 3 the values near 0 are positive and the values near 1 are negative). We denote the sign change point s_h. Note that

$$\frac{m_a''}{m_a} - \frac{m_b''}{m_b} = \frac{m_a''m_b - m_a m_b''}{m_a m_b} = \frac{h'}{m_a m_b},$$

therefore

$$h'(s) > 0, \quad 0 < s < s_h; \qquad h'(s_h) = 0; \qquad h'(s) < 0, \quad s_h < s < 1.$$

From (5) we know that any solution of (4) must lie between s_M and s_h. If $s_M = s_h$, then $s^* = s_M = s_h$ is the unique solution of (4), and the theorem is proved. Otherwise, we will first consider the case $s_M < s_h$. The other case is very similar, but we list the differences at the end of this proof nonetheless.

Case $s_M < s_h$. Let us first prove the existence of s^*. It is easy to note that $M'(s_M) = h'(s_h) = 0$, therefore

$$f''(s_M) = \frac{h'(s_M)}{M^2(s_M)} > 0,$$
$$f''(s_h) = -\frac{2M'(s_h)h(s_h)}{M^3(s_h)} < 0,$$

thus a solution must exist.

In order to prove uniqueness, we intend to demonstrate that f'' is strictly decreasing at the point s^*, and thus changes sign from positive to negative:

$$f''(s^*) = 0 \quad \Longrightarrow \quad f'''(s^*) = g'(s^*)M^2(s^*) < 0. \tag{6}$$

It is clear, that there could only be one such change of sign, therefore the uniqueness proof would be complete. In order to prove (6), we note that from (4) we have

$$h(s^*) = h'(s^*)\frac{M(s^*)}{2M'(s^*)}.$$

Therefore, we calculate

$$\begin{aligned}
g'(s^*) = &\left(h' - \frac{2M'}{M}h\right)'\Bigg|_{s=s^*} \\
&= h''(s^*) - h'(s^*)\frac{M(s^*)}{2M'(s^*)}\left[\frac{2M''(s^*)}{M(s^*)} - \frac{2(M'(s^*))^2}{(M(s^*))^2}\right] - h'(s^*)\frac{2M'(s^*)}{M(s^*)} \\
&= h''(s^*) - h'(s^*)\left[\frac{M''(s^*)}{M'(s^*)} + \frac{M'(s^*)}{M(s^*)}\right].
\end{aligned}$$

All we need to do is demonstrate that

$$h'' - h'\left[\frac{M''}{M'} + \frac{M'}{M}\right] < 0 \tag{7}$$

holds for all s between s_M and s_h, and (6) will follow immediately.

By the definition of s_M and s_h, on (s_M, s_h) we have

$$M' > 0, \quad h' > 0. \tag{8}$$

Note that

$$\frac{h'}{m_a' m_b'} = \frac{m_a'' m_b - m_a m_b''}{m_a' m_b'} = \frac{m_a''}{m_a'}\frac{m_b}{m_b'} - \frac{m_b''}{m_b'}\frac{m_a}{m_a'}$$

is decreasing due to (C2*). Indeed, $\frac{m_a''}{m_a'}$ and $\frac{m_b}{m_b'}$ are both decreasing (recall that m_b has argument $1-s$), while $\frac{m_b''}{m_b'}$ and $\frac{m_a}{m_a'}$ are increasing. Thus

$$h'' - h'\left[\frac{m_a''}{m_a'} - \frac{m_b''}{m_b'}\right] = m_a' m_b' \left(\frac{h'}{m_a' m_b'}\right)' < 0. \tag{9}$$

Note also that

$$\frac{M''}{M'} + \frac{M'}{M} > \frac{M''}{M'} = \frac{m_a'' + m_b''}{m_a' - m_b'} > \frac{m_a''}{m_a'} > \frac{m_a''}{m_a'} - \frac{m_b''}{m_b'}. \tag{10}$$

Here, the first inequality holds because $M' > 0$ due to (8). The second inequality holds due to $m_b', m_b'' > 0$, keeping in mind that $m_a' > m_b'$ because $M' > 0$. And the third inequality holds similarly because $m_b', m_b'' > 0$, thus $\frac{m_b''}{m_b'} > 0$. Therefore, combining the relations (9) and (10) we obtain (7) and prove the uniqueness of s^*.

Case $s_M > s_h$. In this case on (s_h, s_M) we have

$$M' < 0, \quad h' < 0.$$

Now $f''(s_h) > 0$ and $f''(s_M) < 0$, thus a solution exists. The steps of the uniqueness proof are the same. We note that

$$\frac{M''}{M'} + \frac{M'}{M} < \frac{M''}{M'} = \frac{m_a'' + m_b''}{m_a' - m_b'} < -\frac{m_b''}{m_b'} < \frac{m_a''}{m_a'} - \frac{m_b''}{m_b'},$$

therefore (7) still follows from (9). Other than that, no modifications are required, thus the theorem is proved. □

In practice, the following theorem is very helpful in verifying condition (C2*) for some common functions.

Theorem 2. *Let m_1 and m_2 be positive and increasing functions on $(0,1)$ satisfying (C2*). Then their product m_1m_2 also satisfies (C2*).*

Remark 5. It is important for applications in Appendix A that this statement does not require (C1), therefore widening the set of possible multiplicators of $\mathcal{M}$.

Proof. In this proof, we will use the notation $i = 1, 2$. Note that since $\frac{m_i'}{m_i}$ and $\frac{m_i''}{m_i'}$ are decreasing, their product $\frac{m_i''}{m_i}$ is also decreasing. Consider the derivatives of these three fractions and we obtain

$$m_i''m_i - (m_i')^2 < 0, \quad m_i'''m_i' - (m_i'')^2 < 0, \quad m_i'''m_i - m_i''m_i' < 0.$$

It is easy to see that

$$\left(\frac{(m_1m_2)'}{m_1m_2}\right)' = (\ln(m_1m_2))'' = (\ln m_1)'' + (\ln m_2)'' = \left(\frac{m_1'}{m_1}\right)' + \left(\frac{m_2'}{m_2}\right)' < 0.$$

Sadly, the same simple trick does not work for the second fraction. Instead, we prove that $\frac{(m_1m_2)''}{(m_1m_2)'}$ is decreasing by expanding the derivatives and grouping some terms to achieve a similar estimate:

$$
\begin{aligned}
&(m_1m_2)'''(m_1m_2)' - ((m_1m_2)'')^2 \\
&\qquad = (m_1'''m_2 + 3m_1''m_2' + 3m_1'm_2'' + m_1m_2''')(m_1'm_2 + m_1m_2') \\
&\qquad - (m_1''m_2 + 2m_1'm_2' + m_1m_2'')^2 \\
&\qquad = m_2^2(m_1'''m_1' - (m_1'')^2) + m_1^2(m_2'''m_2' - (m_2'')^2) \\
&\qquad + m_2'm_2(m_1'''m_1 - m_1''m_1') + m_1'm_1(m_2'''m_2 - m_2''m_2') \\
&\qquad + (m_2')^2(m_1''m_1 - (m_1')^2) + (m_1')^2(m_2''m_2 - (m_2')^2) \\
&\qquad - 2(m_1''m_1 - (m_1')^2)(m_2''m_2 - (m_2')^2) < 0. \qquad \square
\end{aligned}
$$

Corollary 1. *The class $\mathcal{M}$ is closed under multiplication.*

3. Counterexample for the convex conjecture

Theorem 3. *Let $m_a = m_b$ be a mobility function satisfying (C1). Additionally, let*

$$
\left(\frac{m_a''}{m_a^3}\right)'\Bigg|_{s=0.5} > 0. \tag{11}
$$

Then the corresponding fractional flow function has more than one inflection point and therefore is not S-shaped.

Proof. In this proof we use the notations and conventions introduced in the proof of Theorem 1. Note that due to symmetry, we have

$$
M'(0.5) = h'(0.5) = 0.
$$

Therefore, using (3),

$$
f''(0.5) = \frac{h'M - 2M'h}{M^3}\Bigg|_{s=0.5} = 0,
$$

thus, $s = 0.5$ is an inflection point. However,

$$
\begin{aligned}
f'''(0.5) &= \left(\frac{h'M - 2M'h}{M^3}\right)'\Bigg|_{s=0.5} = \frac{h''M - 2M''h}{M^3}\Bigg|_{s=0.5} \\
&= \frac{m_a'''m_a^2 - 3m_a''m_a'm_a}{2m_a^3}\Bigg|_{s=0.5} = \frac{m_a^2}{2}\left(\frac{m_a''}{m_a^3}\right)'\Bigg|_{s=0.5} > 0,
\end{aligned}
$$

so f'' changes sign from negative to positive at $s = 0.5$. Therefore, there must exist at least two more inflection points, one before 0.5 and one after. $\square$

Here is an example of a mobility satisfying (C1) and (11):

$$m_a(s) = m_b(s) = s^{1.1}(1 + 15s^{10}).$$

See Fig. 1 and Fig. 2 for the corresponding plots. One can see that f'' has 3 zeroes and f is not S-shaped.

It is clear that (11) contradicts (C2). But the class of functions satisfying neither (C2), nor (11) is vast, and it leaves the following question: is it possible to construct a wider class $\mathcal{M}$, for which Theorem 1 holds, by weakening the (C2) restriction? We leave it an open problem for now. What is clear is that (11) is not the only way to construct a counterexample, just the most direct one. It is possible to construct mobilities that do not satisfy (11) but still lead to additional inflection points. To give an example, functions $m_a(s) = m_b(s) = s^{1.1}(1 + 15s^{30})$ result in $f'''(0.5) < 0$, but f still has 5 inflection points, as clearly shown on Fig. 3 and Fig. 4.

The graphs for the counterexample $m_a(s) = m_b(s) = s^{1.1}(1 + 15s^{10})$:

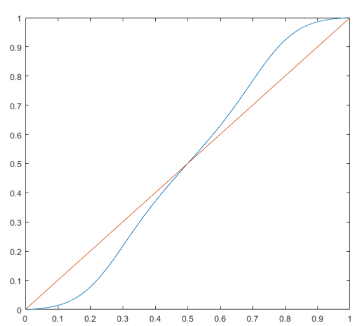

Figure 1. Function f.

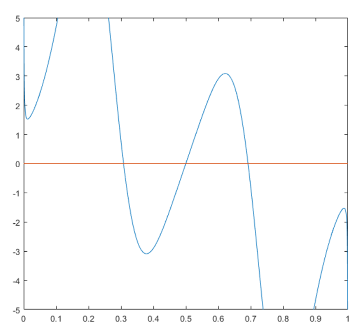

Figure 2. f'' with 3 zeroes.

The graphs for the counterexample $m_a(s) = m_b(s) = s^{1.1}(1 + 15s^{30})$:

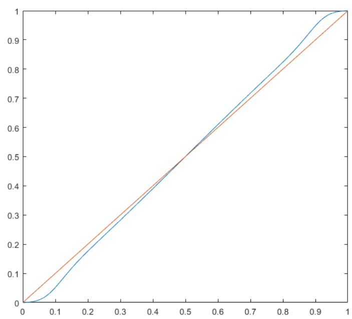

Figure 3. Function f.

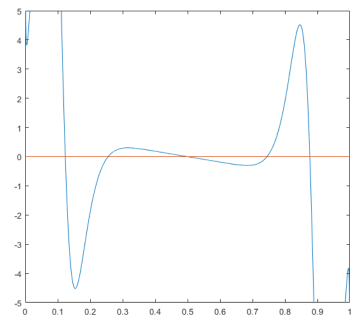

Figure 4. f'' with 5 zeroes.

Appendix A

In this appendix, we check various known relative mobility models against the conditions (C1), (C2).

A.1. The simplest Corey-type model. The simplest and the most commonly used relative mobility model (see [12]) is the power law model

$$m_a(s) = As^a, \quad m_b(s) = Bs^b,$$

where $A, B > 0$, $a, b > 1$. As we already noted in Remark 1, any power function with power greater than 1 satisfies (C1), (C2), therefore $m_a, m_b \in \mathcal{M}$.

A.2. Brooks and Corey model. The original model proposed by Corey (see [13]) is

$$m_a(s) = s^4, \quad m_b(s) = s^2(1 - (1 - s)^2).$$

Later, Brooks and Corey introduce a parameter to generalize that model (see [14]). That resulted in the following functions:

$$m_a(s) = s^{\frac{2+3\lambda}{\lambda}}, \quad m_b(s) = s^2 \left(1 - (1 - s)^{\frac{2+\lambda}{\lambda}}\right),$$

where λ is the pore size distribution index. This was further generalized in [15] to include an additional parameter:

$$m_a(s) = s^{\eta+\frac{2+\lambda}{\lambda}}, \quad m_b(s) = s^{\eta}\left(1-(1-s)^{\frac{2+\lambda}{\lambda}}\right).$$

In all variations, m_a is a power function, and thus was already considered previously. To study m_b we first analyze the expression in the brackets. Denote $\alpha = \frac{2+\lambda}{\lambda}$.

Lemma 2. *Function $n_\alpha(s) = 1-(1-s)^\alpha$ on $(0,1)$ is positive, increasing and satisfies (C2*) for all $\alpha > 1$.*

Proof.

$$n'_\alpha(s) = \alpha(1-s)^{\alpha-1} > 0, \quad n''_\alpha(s) = -\alpha(\alpha-1)(1-s)^{\alpha-2} < 0.$$

Therefore,

$$n''_\alpha n_\alpha - (n'_\alpha)^2 < 0,$$

so n'_α/n_α is decreasing and

$$\frac{n''_\alpha}{n'_\alpha} = \frac{1-\alpha}{1-s}$$

is also a decreasing function. □

Therefore, due to Theorem 2 we obtain (C1.1) and (C2*) for m_b for all $\eta > 1$ and $\lambda > 0$. The only condition left to check is (C1.2). We calculate the second derivative

$$m''_b(s) = s^{\eta-2}\Big[\eta(\eta-1) - P(s)(1-s)^{\alpha-2}\Big], \tag{12}$$

where

$$P(s) = \eta(\eta-1) - 2\eta\gamma s + \gamma(\gamma+1)s^2, \qquad \gamma = \eta+\alpha-1.$$

It is clear that for $1 < \alpha < 2$ (that is $\lambda > 2$) the last term blows up near $s = 1$, thus (C1.2) is broken. Otherwise, the following Lemma shows that m_b is convex for all $\eta \geqslant 2$.

Lemma 3. *Let $\alpha, \eta \geqslant 2$. Then $m_b(s) = s^\eta\left(1-(1-s)^\alpha\right)$ is convex on $(0,1)$.*

Proof. It is clear that m''_b is positive near 0 and 1. It is also easy to see that

$$(\eta(\eta-1) - P(s)(1-s)^{\alpha-2})' = (1-s)^{\alpha-3}\left[(\alpha-2)P(s) - (1-s)P'(s)\right],$$

and the function in square brackets

$$(\alpha-2)P(s)-(1-s)P'(s)=\alpha\left(\gamma(\gamma+1)s^2-2\gamma(\eta+1)s+\eta(\eta+1)\right) \quad (13)$$

has at most two zeroes on $(0,1)$. Therefore, the expression in the brackets in (12) has at most two extrema on $(0,1)$, and if we show them to be positive, the proof will be concluded. Let $z\in(0,1)$ be a root of (13). Note that (13) gives us

$$P(z)=2\gamma z-2\eta.$$

Using this and Bernoulli's inequality, we estimate

$$\begin{aligned}\eta(\eta-1)(1-z)^{2-\alpha}-P(z) &\geqslant \eta(\eta-1)+\eta(\eta-1)(\alpha-2)z-2\gamma z+2\eta \\ &= (\eta-2z)(\eta+1)+(\alpha-2)(\eta-2)(\eta+1)z>0.\end{aligned}$$

□

Therefore, $m_b\in\mathcal{M}$ for all $\alpha,\eta\geqslant 2$ (or equivalently $0<\lambda\leqslant 2, \eta\geqslant 2$).

Remark 6. For $1\leqslant\eta\leqslant 2$ there exists a separating value $\eta_0(\alpha)$, such that m_b is convex for $\eta>\eta_0(\alpha)$ and the convexity breaks for $\eta\leqslant\eta_0(\alpha)$. To obtain this separating value we need to calculate the largest root of (13):

$$z=\frac{\eta+1}{\gamma+1}+\sqrt{\frac{(\eta+1)^2}{(\gamma+1)^2}-\frac{\eta(\eta+1)}{\gamma(\gamma+1)}},$$

substitute this expression into the relation

$$(2\gamma z-2\eta)(1-z)^{\alpha-2}=\eta(\eta-1)$$

and solve the resulting equation for η as an implicit function of α.

Numerical experiment shows that this equation admits a unique solution for every α, but we omit any rigorous proof. The resulting curve is shown on Fig. 5. Notably, the curve itself has a limiting value as $\alpha\to+\infty$:

$$\lim_{\alpha\to+\infty}\eta_0(\alpha)=\eta_\infty,$$

where $\eta_\infty\approx 1.122183$ solves the implicit equation

$$(2+2\sqrt{\eta+1})e^{\eta+1+\sqrt{\eta+1}}=\eta(\eta-1).$$

Therefore, m_b is never convex for $1\leqslant\eta\leqslant\eta_\infty$.

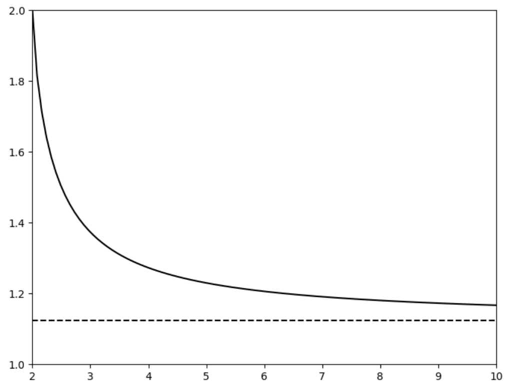

Figure 5. The graph of $\eta_0(\alpha)$ with η_∞ asymptotics.

A.3. Chierici model. Chierici (see [16]) proposed an exponential law for the relative mobility functions:

$$m_a(s) = A\exp\left[-B\left(\frac{s}{1-s}\right)^{-C}\right].$$

This expression is often not convex, so we are just going to provide an example of Chierici functions satisfying our conditions. Let $C = 1$, $B > 2$. Then

$$m_a(s) = A\exp\left[-B\,\frac{1-s}{s}\right], \quad m_a'(s) = \frac{AB}{s^2}\exp\left[-B\,\frac{1-s}{s}\right] > 0,$$

$$m_a''(s) = \frac{AB^2 - 2ABs}{s^4}\exp\left[-B\,\frac{1-s}{s}\right] > 0,$$

$$\left(\frac{m_a'}{m_a}\right)' = -\frac{2B}{s^3} < 0, \quad \left(\frac{m_a''}{m_a'}\right)' = \left(\frac{B-2s}{s^2}\right)' = \frac{2s-2B}{s^3} < 0.$$

Therefore $m_a \in \mathcal{M}$ for $C = 1$ and all $B > 2$.

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