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A. Mikhaylov, V. Mikhaylov

INVERSE PROBLEM FOR SEMI-INFINITE JACOBI MATRICES AND ASSOCIATED HILBERT SPACES OF ANALYTIC FUNCTIONS

Abstract. We consider the dynamic problems for the discrete systems with discrete time associated with finite and semi-infinite Jacobi matrices. The result of the paper is a procedure of association of special Hilbert spaces of functions, namely de Branges space, playing an important role in the inverse spectral theory, with these systems. We point out the relationships with the classical moment problems theory and compare properties of classical Hankel matrices associated with moment problems with properties of matrices of connecting operators associated with dynamical systems.

Dedicated to the anniversary of Nina Nikolaevna Uraltseva

§1. Introduction

For a given sequence of positive numbers $\{a_0, a_1, \dots\}$ (in what follows we assume $a_0 = 1$) and real numbers $\{b_1, b_2, \dots\}$, we denote by A the semi-infinite Jacobi matrix

$$A = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1}$$

For $N \in \mathbb{N}$, by A^N we denote the $N \times N$ Jacobi matrix which is a block of (1) consisting of the intersection of first N columns with first N rows of A. We consider the dynamical system corresponding to a semi-infinite Jacobi matrix:

$$\begin{cases} u_{n,t+1} + u_{n,t-1} - a_n u_{n+1,t} - a_{n-1} u_{n-1,t} - b_n u_{n,t} = 0, \ t \geqslant 0, \ n \geqslant 1, \\ u_{n,-1} = u_{n,0} = 0, \quad n \geqslant 1 \\ u_{0,t} = g_t, \quad t \geqslant 0, \end{cases} \tag{2}$$

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where $g = (g_0, g_1, \ldots)$ is a *boundary control*, $g_i \in \mathbb{C}$, $i = 0, 1, 2, \ldots$, the solution to (2) is denoted by u^g.

De Branges spaces [8, 9, 17] play an important role in the inverse spectral theory. In [10, 11] the authors show how to associate finite-dimensional de Branges spaces with the dynamical systems of the form (2). Note that our approach differs from the classical one and potentially admits the generalization to the multidimensional systems [15]. The algorithm proposed in [10, 11] is as follows: fixing some finite time $t = T$ one introduces the *reachable set* of the dynamical system at this time:

$$U^T := \left\{ u^g_{\cdot,T} \,|\, g \in \mathcal{F}^T \right\}.$$

Then one needs to apply the Fourier transform associated with the operator corresponding to the matrix A to elements from U^T and get a linear manifold $\mathcal{F}U^T$. Then this linear manifold is equipped with the norm defined by the *connecting operator* C^T associated with the system (2), which resulted in the finite-dimensional de Branges space associated with A^T.

Thus for the system (2), due to the finiteness of the speed of wave propagation, the described procedure leads to the finite dimensional Hilbert space of analytic functions associated with A^T (not with the whole matrix A!). The natural question then is to try to associate some infinite-dimensional functional spaces with the dynamical system (2) with semi-infinite matrix, taking $T \to \infty$.

For a given sequence of *moments* $s_0, s_1, s_2, \ldots$ a solution of a *Hamburger moment problem* [1, 18] is a Borel measure $d\rho(\lambda)$ on $\mathbb{R}$ such that

$$s_k = \int_{-\infty}^{\infty} \lambda^k \, d\rho(\lambda), \quad k = 0, 1, 2, \ldots. \tag{3}$$

If in a Hamburger moment problem an additional constraint $\operatorname{supp} d\rho \subset [0, +\infty)$ or $\operatorname{supp} d\rho \subset [0, 1]$ is imposed on the measure, then such a problem is called *Stieltjes moment problem* or *Hausdorff moment problem*. The set of solutions of Hamburger moment problem is denoted by $\mathcal{M}_H$. The moment problem (moment sequence $s_0, s_1, \ldots$) is called *determinate* if a solution exists and is unique, if a solution is not unique, it is called *indeterminate*, the same notations are used in respect to corresponding measures. The relationships of classical moments problems and dynamic inverse problem for the system (2) are described in [13, 14].

The following semi-infinite Hankel matrix is associated with the moment problem (see [1])

$$S=\begin{pmatrix} s_0 & s_1 & s_2 & s_3 & \dots \\ s_1 & s_2 & s_3 & \dots & \dots \\ s_2 & s_3 & \dots & \dots & \dots \\ s_3 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{4}$$

The $N\times N$ block of this matrix is denoted by S_N.

The *connecting operator* C^T associated with the system (2) (for fixed time $t=T$) plays a central role in the Boundary Control method [3,4], an approach to inverse dynamic problems. In [13,14] the authors shown the simple relationship between C^T and S_T. As C^T was used in the procedure of the construction of finite-dimensional spaces, the natural question then is to study properties of "semi-infinite" connecting operator C in a way the semi-infinite Hankel matrix S was studied in [5,7,20,21].

In the second section we provide all the necessary information on de Branges spaces. In the third section we list the results for the finite and semi-infinite operators S_N, S according to [5,7,20,21]. In the forth section we briefly outline the results for the dynamical system (2) and for dynamical system associated with A^N according to [12]. In the fifth section we describe the procedure of the de Branges spaces construction in the finite dimensional case according to [10, 11]. After that in the last section we introduce the semi-infinite matrix C and compare its properties with ones of the "moment problem" counterpart Hankel matrix S. All these give us the possibility to introduce the infinite-dimensional de Branges spaces in the limit circle (indeterminate) case for the system (2) .

§2. de Branges spaces.

Here we provide the information on de Branges spaces in accordance with [16, 17]. The entire function $E:\mathbb{C}\mapsto\mathbb{C}$ is called a *Hermite–Biehler function* if $|E(z)|>|E(\overline{z})|$ for $z\in\mathbb{C}_+$. We use the notation $F^{\#}(z)=\overline{F(\overline{z})}$. The *Hardy space* H_2 is defined by: $f\in H_2$ if f is holomorphic in $\mathbb{C}^+$ and $\sup\limits_{y>0}\int\limits_{-\infty}^{\infty}|f(x+iy)|^2\,dx<\infty$. Then the *de Branges space* $B(E)$ consists of entire functions such that:

$$B(E):=\left\{F:\mathbb{C}\mapsto\mathbb{C},\ F\text{ entire},\ \frac{F}{E},\frac{F^{\#}}{E}\in H_2\right\}.$$

The space $B(E)$ with the scalar product

$$[F,G]_{B(E)} = \frac{1}{\pi}\int\limits_{\mathbb{R}} \overline{F(\lambda)}G(\lambda)\frac{d\lambda}{|E(\lambda)|^2}$$

is a Hilbert space. For any $z \in \mathbb{C}$ the *reproducing kernel* is introduced by the relation [8, Theorem 19, p. 50].

$$J_z(\xi) := \frac{\overline{E(z)}E(\xi) - E(\overline{z})\overline{E(\overline{\xi})}}{2i(\overline{z}-\xi)}. \tag{5}$$

Then

$$F(z) = [J_z, F]_{B(E)} = \frac{1}{\pi}\int\limits_{\mathbb{R}} \overline{J_z(\lambda)}F(\lambda)\frac{d\lambda}{|E(\lambda)|^2}.$$

We observe that a Hermite–Biehler function $E(\lambda)$ defines J_z by (5). The converse is also true [8, Theorem 22, p. 55].

Theorem 1. *Let X be a Hilbert space of entire functions with reproducing kernel such that*

1) *if $f \in X$ then $f^{\#} \in X$ and $\|f\|_X = \|f^{\#}\|_X$,*
2) *if $f \in X$ and $\omega \in \mathbb{C}$ such that $f(\omega) = 0$, then $\frac{z-\overline{\omega}}{z-\omega}f(z) \in X$ and $\left\|\frac{z-\overline{\omega}}{z-\omega}f(z)\right\|_X = \|f\|_X$,*

then X is a de Branges space based on the function

$$E(z) = \sqrt{\pi}(1-iz)J_i(z)\|J_i\|_X^{-1},$$

where J_z is a reproducing kernel.

§3. Classical moment problems, Hankel matrices, minimal eigenvalues and closability. Associated Hilbert space of analytic functions.

With the semi-infinite matrix A we associate the symmetric operator A (we keep the same notation) in the space l_2, defined on finite sequences:

$$D = \{\varkappa = (\varkappa_1, \varkappa_2, \ldots) \,|\, \exists\, N \in \mathbb{N}:\ \varkappa_n = 0,\ n \geqslant N\}, \tag{6}$$

and given by the rule

$$(A\theta)_1 = b_1\theta_1 + a_1\theta_2,$$
$$(A\theta)_n = a_n\theta_{n+1} + a_{n-1}\theta_{n-1} + b_n\theta_n, \quad n \geqslant 2.$$

By $[\,\cdot\,,\,\cdot\,]$ we denote the scalar product in l_2. For a given sequence $\varkappa = (\varkappa_1, \varkappa_2, \ldots)$ we define a new sequence

$$(G\varkappa)_1 = b_1\varkappa_1 + a_1\varkappa_2,$$
$$(G\varkappa)_n = a_n\varkappa_{n+1} + a_{n-1}\varkappa_{n-1} + b_n\varkappa_n, \quad n \geqslant 2.$$

The adjoint operator $A^*\varkappa = G\varkappa$ is defined on the domain

$$D(A^*) = \{\varkappa = (\varkappa_1, \varkappa_2, \ldots) \in l_2 \,|\, (G\varkappa) \in l_2\}.$$

In the limit point case (i.e when A has deficiency indices $(0,0)$), A is essentially self-adjoint. In the limit circle case, i.e. when A has deficiency indices $(1,1)$, we denote by $p(\lambda) = (p_1(\lambda), p_2(\lambda), \ldots)$ and by $q(\lambda) = (q_1(\lambda), q_2(\lambda), \ldots)$ two solutions of the difference equation (we set here $a_0{=}1$):

$$a_n\phi_{n+1} + a_{n-1}\phi_{n-1} + b_n\phi_n = \lambda\phi_n, \quad n \geqslant 1, \tag{7}$$

satisfying Cauchy data $p_1(\lambda) = 1$, $p_2(\lambda) = \frac{\lambda - b_1}{a_1}$, $q_1(\lambda) = 0$, $q_2(\lambda) = \frac{1}{a_1}$. Thus $p_n(\lambda)$, $q_n(\lambda)$ are polynomials of orders $n-1$ and $n-2$. Then [19, Lemma 6.22]

$$D(A^*) = D(\overline{A}) \dot{+} \mathbb{C}p(0) \dot{+} \mathbb{C}q(0),$$

where $\dot{+}$ denotes the direct sum and $\overline{A}$ is a closure of A. All self-adjoint extensions of A are parameterized by $h \in \mathbb{R} \cup \{\infty\}$, are denoted by $A_{\infty,\,h}$ and are defined on the domain

$$D(A_{\infty,\,h}) = \begin{cases} D(\overline{A}) \dot{+} \mathbb{C}(q(0) + hp(0)), & h \in \mathbb{R} \\ D(\overline{A}) \dot{+} \mathbb{C}p(0), & h = \infty. \end{cases}$$

All the details the reader can find in [18, 19]. We introduce the measure $d\rho_{\infty,\,h}(\lambda) = \left[dE_\lambda^{A_{\infty,\,h}} e_1, e_1\right]$, where $e_1 = (1, 0, \ldots)$ and $dE_\lambda^{A_{\infty,h}}$ is the projection-valued spectral measure of $A_{\infty,\,h}$ such that $E_{\lambda-0}^{A_{\infty,\,h}} = E_\lambda^{A_{\infty,\,h}}$.

Let $\{s_0, s_1.s_2, \ldots\}$ be a moment sequence, $d\rho \in \mathcal{M}_H$ be a solution to moment problem (3), S, S_N be Hankel matrices (4). The well-known fact on the solvability of the moments problem [1, 18, 19] is given in terms of positivity of these matrices.

Theorem 2. *The moment problem* (3) *has a solution if and only if the following condition holds:*

$$S_N > 0, \quad N = 1, 2, 3, \ldots$$

Note that in this case the matrix A is determined by the moments in a one-to-one manner.

In [5–7, 20, 21] the authors studied properties of matrix S_N and corresponding semi-infinite matrix S. Below we list the number of important results obtained by them. Having fixed N, we denote by λ_N the smallest eigenvalue of S_N:

$$\lambda_N = \min\{l_k \mid l_k \text{ is eigenvalue of } S_N,\ k = 1, \dots, N\}.$$

The following theorem was proved in [5].

Theorem 3. *The moment problem associated with the sequence* $\{s_k\}$ *is determined* (*the matrix* A *is in the limit point case*) *if and only if*

$$\lim_{N\to\infty} \lambda_N \to 0.$$

In the limit circle case

$$\lim_{N\to\infty} \lambda_N \geqslant \left(\int_0^{2\pi} l\left(e^{i\theta}\right) \frac{d\theta}{2\pi}\right)^{-1}, \quad l(z) = \left(\sum_{k=0}^{\infty} |p_k(z)|^2\right)^{-1}.$$

The Hankel matrix S give rise to the formally defined operator Q in l_2:

$$(Qf)_n = \sum_{m=0}^{\infty} s_{n+m} f_m, \quad f \in l_2.$$

Then without any a priory assumptions on the sequence $\{s_k\}$, only the quadratic form of this operator

$$S[f, f] = \sum_{m,n\geqslant 0} s_{m+n} \overline{f_m} f_n \tag{8}$$

is well-defined on D. We always assume the positivity condition (cf. Theorem 2)

$$\sum_{m,n\geqslant 0} s_{m+n} \overline{f_m} f_n \geqslant 0, \quad f \in D.$$

The following result is obtained in [7].

Theorem 4. *If operator* A *in the limit circle case then operator* S *defined via quadratic form* (8) *is closable.*

There exist operator A *in the limit point case such that operator* S *defined via quadratic form* (8) *is closable.*

The solutions of the indeterminate moment problem form an infinite convex set $\mathcal{M}_H$ of measures M with unbounded support, for which moment identities (3) hold. In this set there exist *extremal* measures $\widetilde{M} \in \mathcal{M}_H$ such that the set of polynomials $\mathbb{C}[\lambda]$ is dense in $L_2(\mathbb{R}, \widetilde{M})$. These measures correspond to Neumann extension $A_{\infty,h}$ of operator A were described in the beginning of Section 3, we denoted them $d\rho_{\infty,h}$, these measures are discrete and are of the form

$$\widetilde{M} = \sum_k c_k \delta_{\lambda_k}(\lambda).$$

Then the result of Stieltjes says that if one mass is removed, then the new measure

$$M_1 := \widetilde{M} - c_1 \delta_{\lambda_1}(\lambda)$$

is determinate. The existence of examples of the closability of S in the determinate situation follows from this theorem.

In the indeterminate case the orthonormal with respect to measure M polynomials form an orthonormal basis in $L_2(\mathbb{R}, M)$ if M is extremal. If M is not extremal then they are basis in $\overline{\mathbb{C}}[\lambda]$ where the closure is assumed in $L_2(\mathbb{R}, M)$. In both cases $\overline{\mathbb{C}}[\lambda]$ is isomorphic to the space $\mathcal{E}$ of entire functions of the form

$$u(z) = \sum_{k=1}^{\infty} g_k p_k(z), \quad g \in l^2, \quad p_k \text{ are orthonormal w.r.t. measure } M.$$

Note (see [19, Sec. 7]) that the reproducing kernel in this space has a form

$$J_z^{\infty}(\lambda) = \sum_{n=1}^{\infty} \overline{p_n(z)} p_n(\lambda) \tag{9}$$

and the right hand side converges uniformly on compact subsets of $\mathbb{C}$ to holomorphic function on $\mathbb{C}^2$. For this kernel one has

$$\int_{-\infty}^{\infty} \overline{J_z^{\infty}}(\lambda) f(\lambda) dM(\lambda) = f(z), \tag{10}$$

for all polynomials $f(z)$.

In the determinate (limit point) case $\mathbb{C}[\lambda]$ is dense in $L_2(\mathbb{R}, M)$, but since the quantity (9) diverges, and as a consequence, the reproducing kernel is absent, there are no results on the equivalence of $\mathbb{C}[\lambda]$ to some space of analytic functions.

§4. Dynamical system with discrete time associated with finite and semi-infinite Jacobi matrices.

Proofs of statements of this section can be found in [12, 13].

We consider the dynamical system with discrete time associated with the matrix A^N:

$$\begin{cases} v_{n,t+1}+v_{n,t-1}-a_nv_{n+1,t}-a_{n-1}v_{n-1,t}-b_nv_{n,t}=0,\ t\geqslant 0,\ 1\leqslant n\leqslant N,\\ v_{n,-1}=v_{n,0}=0,\ 1\leqslant n\leqslant N+1,\\ v_{0,t}=f_t,\quad v_{N+1,t}=0,\quad t\geqslant 0, \end{cases} \tag{11}$$

where $f=(f_0,f_1,\ldots)$ is a *boundary control*, $f_i\in\mathbb{C}$, $i=0,1,2,\ldots$. The solution to (11) is denoted by v^f.

The operator corresponding to a finite Jacobi matrix we also denote by $A^N:\mathbb{R}^N\mapsto\mathbb{R}^N$, it is given by

$$\begin{cases} (A\psi)_1=b_1\psi_1+a_1\psi_2,\quad n=1,\\ (A\psi)_n=a_n\psi_{n+1}+a_{n-1}\psi_{n-1}+b_n\psi_n,\quad 2\leqslant n\leqslant N-1,\\ (A\psi)_N=a_{N-1}\psi_{N-1}+b_N\psi_N,\quad n=N, \end{cases}$$

Denote by $\{\lambda_k\}_{k=1}^N$ roots of the equation $p_{N+1}(\lambda)=0$, it is known [1] that they are real and distinct. We introduce the vectors $\phi^n\in\mathbb{R}^N$ by the rule $\phi_i^n:=p_i(\lambda_n)$, $n,i=1,\ldots,N$, and define the numbers ρ_k by

$$(\phi^k,\phi^l)=\delta_{kl}\rho_k,\quad k,l=1,\ldots,N,$$

where $(\cdot,\cdot)$ is a scalar product in $\mathbb{R}^N$, and δ_{kl} is a Kronecker symbol.

Definition 1. *The set of pairs*

$$\{\lambda_k,\rho_k\}_{k=1}^N$$

is called spectral data of the operator A^N.

The spectral function of operator A^N is introduced by the rule

$$\rho_N(\lambda):=\sum_{\{k\,|\,\lambda_k<\lambda\}}\frac{1}{\rho_k}.$$

The results of [2, Sections 4.5, 5.5] imply that in the limit circle case $d\rho_N\to d\rho_{\infty,\,h}$ $*-$weakly as $N\to\infty$, where

$$h=-\lim_{n\to\infty}\frac{q_n(0)}{p_n(0)}. \tag{12}$$

The outer space of dynamical systems (11), (2) is $\mathcal{F}^T := \mathbb{C}^T$, $\mathcal{F}^T \ni g, f = (f_0, f_1, \dots, f_{T-1})$ with the inner product $(f,g)_{\mathcal{F}^T} := \sum_{n=0}^{T-1} \overline{f_i} g_i$.

The input $\longmapsto$ output correspondences in systems (11), (2) are realized by *response operators*: R_N^T, $R^T : \mathcal{F}^T \mapsto \mathbb{C}^T$ defined by rules

$$\left(R_N^T f\right)_t = v_{1,\,t}^f = \left(r^N * f_{\cdot-1}\right)_t = \sum_{s=0}^t r_s^N f_{t-1-s}, \quad t = 1, \dots, T,$$

$$\left(R^T f\right)_t = u_{1,\,t}^f = \left(r * f_{\cdot-1}\right)_t = \sum_{s=0}^t r_s f_{t-1-s}, \quad t = 1, \dots, T,$$

where $r^N = (r_0^N, r_1^N, \dots, r_{T-1}^N)$, $r = (r_0, r_1, \dots, r_{T-1})$ are *response vectors*, convolution kernels of response operators.

Let $\mathcal{T}_k(2\lambda)$ be Chebyshev polynomials of the second kind, i.e. they satisfy

$$\begin{cases} \mathcal{T}_{t+1}(\lambda) + \mathcal{T}_{t-1}(\lambda) - \lambda \mathcal{T}_t(\lambda) = 0, \\ \mathcal{T}_0(\lambda) = 0, \ \mathcal{T}_1(\lambda) = 1. \end{cases}$$

Proposition 1. *The solution* v^f *to system* (11) *and the response vector* r^N *admit representations*

$$v_{n,t}^f = \int_{-\infty}^{\infty} \sum_{k=1}^t \mathcal{T}_k(\lambda) f_{t-k} \phi_n(\lambda)\, d\rho^N(\lambda), \quad t \in \mathbb{N},\, n = 1, \dots N, \tag{13}$$

$$r_{t-1}^N = \int_{-\infty}^{\infty} \mathcal{T}_t(\lambda)\, d\rho^N(\lambda), \quad t \in \mathbb{N}. \tag{14}$$

Remark 1. The solution u^f and response vector r corresponding to system (2) with semi-infinite Jacobi matrix A admit representations (13), (14) with $d\rho^N$ substituted by any $d\rho(\lambda) \in \mathcal{M}_H$ and $n \in \mathbb{N}$.

The *inner space* of dynamical system (11) is $\mathcal{H}^N := \mathbb{C}^N$, $h \in \mathcal{H}^N$, $h = (h_1, \dots, h_N)$, $h, m \in \mathcal{H}^N$, $(h,m)_{\mathcal{H}^N} := \sum_{k=1}^N \overline{h_k} m_k$, $v_{\cdot,\,T}^f \in \mathcal{H}^N$ for all $T \in \mathbb{N}$. For the system (11) the *control operator* $W_N^T : \mathcal{F}^T \mapsto \mathcal{H}^N$ is defined by the rule

$$W_N^T f := \left(v_{1,\,T}^f, \dots, v_{N,\,T}^f\right).$$

The set

$$\mathcal{U}^T := W_N^T \mathcal{F}^T = \left\{ \left(v_{1,\,T}^f, \dots, v_{N,\,T}^f\right) \mid f \in \mathcal{F}^T \right\}$$

is called reachable. For the system (2) the control operator $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$ is introduced by

$$W^T f := \left(u^f_{1,\,T}, \dots, u^f_{T,\,T}\right).$$

This operator admits the representation $W^T = W_T J_T$:

$$W_T = \begin{pmatrix} a_0 & w_{1,1} & w_{1,2} & \dots & w_{1,T-1} \\ 0 & a_0 a_1 & w_{2,2} & \dots & w_{2,T-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & \prod_{j=1}^{k-1} a_j & \dots & w_{k,T-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \prod_{j=1}^{T-1} a_j \end{pmatrix}, \quad J_T = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

Everywhere below we substantially use the finiteness of the speed of wave propagation in systems (11), (2), which implies the following remark.

Remark 2. Solution v^f to system (2) and solution u^f to system (11) satisfy

$$u^f_{n,\,t} = v^f_{n,\,t}, \quad n \leqslant t \leqslant N.$$
$$W^N = W^N_N, \quad r_t = r^N_t, \quad t = 0, \dots, 2N-1.$$

The *connecting operator* for the system (11) $C^T_N : \mathcal{F}^T \mapsto \mathcal{F}^T$ is defined via the quadratic form: for arbitrary $f, g \in \mathcal{F}^T$ we set

$$\left(C^T_N f, g\right)_{\mathcal{F}^T} := \left(W^T_N f, W^T_N g\right)_{\mathcal{H}^N} = \left(v^f_{\cdot,\,T}, v^g_{\cdot,\,T}\right)_{\mathcal{H}^N}.$$

For the system (2) the connecting operator $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is introduced by the rule:

$$\left(C^T f, g\right)_{\mathcal{F}^T} := \left(W^T f, W^T g\right)_{\mathcal{H}^T} = \left(u^f_{\cdot,\,T}, u^g_{\cdot,\,T}\right)_{\mathcal{H}^T}. \tag{16}$$

In [11, 12] the following formulas were obtained.

Proposition 2. *The matrix of the connecting operator C^T_N for systems* (11) *and the matrix of the connecting operator C^T for system* (2) *admit* *spectral representations*

$$\{C_N^T\}_{l+1,\,m+1} = \int_{-\infty}^{\infty} \mathcal{T}_{T-l}(\lambda)\mathcal{T}_{T-m}(\lambda)\,d\rho^N(\lambda), \quad l,m = 0,\dots,T-1,$$

$$\{C^T\}_{l+1,\,m+1} = \int_{-\infty}^{\infty} \mathcal{T}_{T-l}(\lambda)\mathcal{T}_{T-m}(\lambda)\,d\rho(\lambda), \quad l,m = 0,\dots,T-1,$$

in the last equality we can take any $d\rho(\lambda) \in \mathcal{M}_H$*. The following dynamic representation valid if* $T \leqslant N$ *:*

$$C^T = C_N^T, \quad \{C^T\}_{ij} = \sum_{k=0}^{T-\max\{i,j\}} r_{|i-j|+2k},$$

$$C^T = \begin{pmatrix} r_0+r_2+\dots+r_{2T-2} & \cdot & \dots & r_T+r_{T-2} & r_{T-1} \\ r_1+r_3+\dots+r_{2T-3} & \cdot & \dots & r_{T-1}+r_{T-3} & r_{T-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{T-3}+r_{T-1}+r_{T+1} & \cdot & r_0+r_2+r_4 & r_1+r_3 & r_2 \\ r_T+r_{T-2} & \cdot & r_1+r_3 & r_0+r_2 & r_1 \\ r_{T-1} & \cdot & r_2 & r_1 & r_0 \end{pmatrix} \tag{17}$$

We introduce matrices $C_T := J_T C^T J_T$ thus this matrix keeps the structure of C^T but "is filled" from the upper left corner. Then we have

$$C_T = (W_T)^* W_T.$$

§5. De Branges spaces for finite Jacobi matrices

By $d\rho$ we denote the spectral measure of A in the limit point case or $d\rho_{\infty,h}$ with h defined in (12) in the limit circle case. According to [2] this measure give rise to the Fourier transform $F : l^2 \mapsto L_2(\mathbb{R}, d\rho)$, defined by the rule:

$$(Fa)(\lambda) = \sum_{n=0}^{\infty} a_k p_k(\lambda), \quad a = (a_0, a_1, \dots) \in l^2,$$

where $p_k(\lambda)$ is a solution to (7) satisfying Cauchy data $p_1(\lambda)=1$, $p_2(\lambda)=\frac{\lambda-b_1}{a_1}$. The inverse transform and Parseval identity have forms:

$$a_k=\int_{-\infty}^{\infty}(Fa)(\lambda)p_k(\lambda)\,d\rho(\lambda),$$

$$\sum_{k=0}^{\infty}\overline{a_k}b_k=\int_{-\infty}^{\infty}\overline{(Fa)(\lambda)}(Fb)(\lambda)\,d\rho(\lambda). \tag{18}$$

We assume that T is fixed and $f\in\mathcal{F}^T$. Then for such control and for $\lambda\in\mathbb{C}$ we have the following representation [11] for the Fourier transform of the solution to (2) at $t=T$:

$$\left(Fu^f_{\cdot,T}\right)(\lambda)=\sum_{k=1}^{T}\mathcal{T}_k(\lambda)f_{T-k},\quad \lambda\in\mathbb{C}.$$

Now we introduce the linear manifold of Fourier images of states of dynamical system (2) at time $t=T$, i.e., the Fourier image of the reachable set:

$$B^T_A:=F\mathcal{U}^T=\left\{\left(Fu^{J_Tf}_{\cdot,T}\right)(\lambda)\mid J_Tf\in\mathcal{F}^T\right\}=\left\{\sum_{k=1}^{T}\mathcal{T}_k(\lambda)f_k\mid f\in\mathcal{F}^T\right\}. \tag{19}$$

Note that $B^T_A=\mathbb{C}_{T-1}[\lambda]$. It would be preferable for us to use C_T instead of C^T, although that leads to changing of some formulas comparing to [11,14].

We equip B^T_A with the scalar product defined by the rule:

$$[F,G]_{B^T_A}:=(C_Tf,g)_{\mathcal{F}^t},\quad F,G\in B^T_A, \tag{20}$$

$$F(\lambda)=\sum_{k=1}^{T}\mathcal{T}_k(\lambda)f_k,\ G(\lambda)=\sum_{k=1}^{T}\mathcal{T}_k(\lambda)g_k,\ f,g\in\mathcal{F}^T.$$

Evaluating (20) making use of (18) yields:

$$[F,G]_{B^T_A}=(C_Tf,g)_{\mathcal{F}^T}=\left(C^TJ_Tf,J_Tg\right)_{\mathcal{F}^T}=\left(u^{J_Tf}_{\cdot,T},u^{J_Tg}_{\cdot,T}\right)_{\mathcal{H}^T}$$

$$=\int_{-\infty}^{\infty}\overline{(Fu^{J_Nf}_{\cdot,N})(\lambda)}(Fu^{J_Ng}_{\cdot,N})(\lambda)\,d\rho_{\infty,h}(\lambda)=\int_{-\infty}^{\infty}\overline{F(\lambda)}G(\lambda)\,d\rho_{\infty,h}(\lambda). \tag{21}$$

Where the last equality is due to the finite speed of wave propagation in (2), (11). In [12] the authors proved the following theorem.

Theorem 5. *The vector* $(r_0, r_1, r_2, \dots, r_{2N-2})$ *is a response vector for the dynamical system* (11) *if and only if the matrix of connecting operator* C^N *defined by* (16), (17) *with* $T = N$ *is positive definite.*

This theorem shows that (21) is a scalar product in B_A^T. But we can say even more [11].

Theorem 6. *By dynamical system with discrete time* (2) *one can construct the de Branges space by* (19) *As a set of functions it coincides with the space of Fourier images of states of dynamical system* (2) *at time* $t = T$ (*or what is the same, states of* (11) *with* $N = T$ *at the same time*), *i.e. the Fourier image of a reachable set, and is the set of polynomials with real coefficients of the order less or equal to* $T-1$. *The norm in* B_A^T *is defined via the connecting operator:*

$$[F, G]_{B_A^T} := (C_T f, g)_{\mathcal{F}^T}, \quad F, G \in B_A^T,$$

where

$$F(\lambda) = \sum_{k=1}^{T} \mathcal{T}_k(\lambda) f_k,\ G(\lambda) = \sum_{k=1}^{T} \mathcal{T}_k(\lambda) g_k,\ f, g \in \mathcal{F}^T.$$

The reproducing kernel has a form

$$J_z^T(\lambda) = \sum_{k=1}^{T} \mathcal{T}_k(\lambda)(j_T^z)_k,$$

where j_T^z *is a solution to Krein-type equation*

$$C_T j_T^z = \overline{\begin{pmatrix} \mathcal{T}_1(z) \\ \mathcal{T}_2(z) \\ \cdot \\ \mathcal{T}_T(z) \end{pmatrix}}. \tag{22}$$

Note [10,12] that control $J_T j_T^z$ drives the system (2) to the special state at $t = T$:

$$\left(W^T J_T j_T^z\right)_i = \left(W_T^T J_T j_T^z\right)_i = \overline{p_i(z)}, \quad i = 1, \dots, T.$$

Thus the reproducing kernel in B_A^T is given by

$$J_z^T(\lambda) = \sum_{k=1}^{T} \mathcal{T}_k(\lambda)(j_T^z)_k = \left(C_T j_T^\lambda, j_T^z\right)_{\mathcal{F}^T} = \left(W_T^* W_T j_T^\lambda, j_T^z\right)_{\mathcal{F}^T}$$
$$= \left(W_T j_T^\lambda, W_T j_T^z\right)_{\mathcal{H}^T} = \sum_{n=1}^{T} \overline{p_n(z)} p_n(\lambda).$$

Remark 3. Due to the finite speed of wave propagation in (2), (11) (cf. Remark 2) we can use the system (11) with $N = T$ and use operator C_T^T and the measure $d\rho_T(\lambda)$ in formulas for the scalar product (21).

5.1. Remark on canonical systems and Jacobi matrices. Let 2×2 matrix function $0 \leqslant H = H^* \in L_{1,loc}([0,L);\mathbb{R}^{2\times 2})$ be a Hamiltonian, define the matrix $J = \begin{pmatrix} & 0 & 1 \\ - & 1 & 0 \end{pmatrix}$ and the vector $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ be solution to the following Cauchy problem:

$$\begin{cases} -J\frac{dY}{dx} = \lambda H Y, \\ Y(0) = C, \end{cases} \tag{23}$$

for $C \in \mathbb{R}^2$, $C \neq 0$. Without loss of generality we assume that $\operatorname{tr} H(x) = 1$. Then the function $E_x(\lambda) = Y_1(x,\lambda) + iY_2(x,\lambda)$ is a Hermite–Biehler function ($E_L(\lambda)$ makes sense if $L < \infty$), it is called de Branges function of the system (23) since one can construct de Branges space based on this function. On the other hand, E_L serves as an inverse spectral data for the canonical system (23). The main result of the theory [8,17] says that every Hermite-Biehler function satisfying some additional conditions comes from some canonical system.

Remark 4. The Jacobi matrices are particular examples of canonical system (23). The Hamiltonian $H(x)$ defined on the interval $(0, L)$ related to the matrix A is piecewise constant and has a special structure [17]. Moreover, the Jacobi matrix A is in the limit circle case if and only if $L < +\infty$.

This remark leads to the following question: is it possible to introduce the de Branges spaces by the method from Section 5 for semi-infinite Jacobi matrices without passing to canonical systems? If the answer is “yes” then this method should “feel” the difference between limit point and limit circle cases for Jacobi operator A, and the norm in the space can be expressed in the dynamic terms as in finite-dimensional situation.

§6. Properties of connecting operator and Hilbert spaces of functions associated with semi-infinite Jacobi matrices.

6.1. Operator C. We consider the matrix C formally defined by the product of two matrices: $C = (W)^* W$ (cf. (15), (16)), where

$$W = \begin{pmatrix} a_0 & w_{1,1} & w_{1,2} & \dots \\ 0 & a_0 a_1 & w_{2,2} & \dots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & \prod_{j=1}^{k-1} a_j & \dots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots \end{pmatrix}$$

and

$$C = \begin{pmatrix} r_0 & r_1 & r_2 & \dots & \dots \\ r_1 & r_0 + r_2 & r_1 + r_3 & \dots & \dots \\ r_2 & r_1 + r_3 & \cdot & \cdot & \cdot \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \{C\}_{ij} = \sum_{k=0}^{\max\{i,j\}-1} r_{|i-j|+2k}.$$

In our approach to de Branges spaces the finite matrices C_T plays an important role since they are used in the scalar product (20). We suggest that matrix C should play the same role in semi-infinite case.

The matrix C_T is connected with the classical Hankel matrix S_T by the following rule:

$$C_T = \Lambda_T S_T (\Lambda_T)^* .$$

Proposition 3. *The entries of the matrix $\Lambda_T \in \mathbb{R}^{T\times T}$ are given by*

$$\Lambda_T = \{\alpha_{ij}\} = \begin{cases} 0 & \text{if } i < j, \\ 0 & \text{if } i + j \text{ is odd}, \\ D^j_{\frac{i+j}{2}} (-1)^{\frac{i+j}{2}+j} & \text{otherwise} \end{cases}$$

where D^k_n are binomial coefficients. The entries of the response vector are related to moments by the rule:

$$\begin{pmatrix} T_1(\lambda) \\ \dots \\ T_T(\lambda) \end{pmatrix} = \Lambda_T \begin{pmatrix} 1 \\ \lambda \\ \dots \\ \lambda^{T-1} \end{pmatrix}, \quad \begin{pmatrix} r_0 \\ r_1 \\ \dots \\ r_{T-1} \end{pmatrix} = \Lambda_T \begin{pmatrix} s_0 \\ s_1 \\ \dots \\ s_{T-1} \end{pmatrix}. \tag{24}$$

We rewrite the Krein equation (22) in terms of S_T: taking into account (24) we get that

$$S_T\Lambda_T^* j_T^\lambda = \overline{\begin{pmatrix} 1 \\ \lambda \\ \dots \\ \lambda^{T-1} \end{pmatrix}}.$$

And introducing the vector $f_T^\lambda := \Lambda_T^* j_T^\lambda$ we come to the equivalent form of (22):

$$S_T f_T^\lambda = \overline{\begin{pmatrix} 1 \\ \lambda \\ \dots \\ \lambda^{T-1} \end{pmatrix}}. \tag{25}$$

Therefore the reproducing kernel in B_A^T is represented by

$$J_z^T(\lambda) = \left(C_T j_T^\lambda, j_T^z\right)_{\mathcal{F}^T} = \left(\overline{\begin{pmatrix} 1 \\ \dots \\ \lambda^{T-1} \end{pmatrix}}, \Lambda_T^* j_T^z\right)_{\mathcal{F}^T} = \sum_{n=0}^{T-1} f_{T,k}^z \lambda^k.$$

Krein equations in the form (22) and in the form (25) demonstrate the importance of the knowledge of the invertability properties of operators S_T and C_T when T goes to infinity. The Theorem 3 answers this question for S, below we answer the same question for C.

We introduce the notation:

$$\beta_T := \min\left\{\gamma_k \,|\, \gamma_k \text{ is an eigenvalue of } C_T,\ k = 1, \dots, T\right\}.$$

Then we can formulate the “dynamic” analog of Theorem 3.

Theorem 7. *If the moment problem associated with sequence* $\{s_k\}$ *is indeterminate* (*the matrix* A *is in the limit circle case*) *then*

$$\lim_{T\to\infty} \beta_T \geqslant \left(\int_{-1}^{1} l^{-1}(x) \frac{dx}{\sqrt{1-x^2}}\right)^{-1}, \quad l^{-1}(z) = \sum_{k=0}^{\infty} |p_k(z)|^2. \tag{26}$$

Proof. We use the variational principal which says that

$$\beta_T = \min\left\{(C_T f, f)_{\mathcal{F}^T} \mid \|f\|^2 = 1\right\}.$$

Passing to reciprocal gives

$$\frac{1}{\beta_T} = \max\left\{\|f\|^2 \mid (C_T f, f)_{\mathcal{F}^T} = 1\right\}. \tag{27}$$

Take functions $F, G \in B_A^T$, then

$$F(\lambda) = \sum_{k=1}^{T} f_k \mathcal{T}_k(\lambda), \quad G(\lambda) = \sum_{k=1}^{T} g_k \mathcal{T}_k(\lambda)$$

for some $f, g \in \mathcal{F}^T$. Since $T_k(\lambda)$ are Chebyshev polynomials of the second kind, they are orthogonal with respect to the measure

$$d\nu(\lambda) = \frac{\chi_{(-1,1)}(\lambda)}{\sqrt{1-\lambda^2}} d\lambda,$$

then

$$\|f\|^2 = \sum_{k=1}^{T} |f_k|^2 = \int_{-1}^{1} |F(\lambda)|^2 \frac{d\lambda}{\sqrt{1-\lambda^2}},$$

$$(C_T f, g)_{\mathcal{F}^T} = \int_{\mathbb{R}} \overline{F(\lambda)} G(\lambda)\, d\rho(\lambda), \quad d\rho\lambda \in \mathcal{M}_H.$$

Let us take another representation of $F(\lambda)$:

$$F(\lambda) = \sum_{k=1}^{T} c_k p_k(\lambda).$$

Then for such F we have that

$$\int_{\mathbb{R}} |F(\lambda)|^2\, d\rho(\lambda) = (C_T f, f)_{\mathcal{F}^T} = \sum_{k=1}^{T} |c_k|^2.$$

Thus we can rewrite (27) as:

$$\frac{1}{\beta_T} = \max \left\{ \int_{-1}^{1} \overline{\sum_{i=1}^{T} c_i p_i(\lambda)} \sum_{j=1}^{T} c_j p_j(\lambda) \frac{d\lambda}{\sqrt{1-\lambda^2}} \;\Big|\; \sum_{k=1}^{T} |c_k|^2 = 1 \right\}. \tag{28}$$

Introducing the notation $k_{ij} := \int_{-1}^{1} \overline{p_i(\lambda)} p_j(\lambda) \frac{d\lambda}{\sqrt{1-\lambda^2}}$, we rewrite (28) as

$$\frac{1}{\beta_T} = \max \left\{ \sum_{i,j=1}^{T} k_{ij} \overline{c_i} c_j \mid \sum_{k=1}^{T} |c_k|^2 = 1 \right\}. \tag{29}$$

Define the matrix $K := \{k_{ij}\}_{i,j=1}^{T}$, being a Gram matrix, K is positive. The latter implies that the right hand side of (29) is monotonically increasing, and consequently, β_T is monotonically decreasing as $T \to \infty$, which guarantees the existence of the limit in (26). Then we proceed with the estimate:

$$\frac{1}{\beta_T} \leqslant \operatorname{Tr} K = \int_{-1}^{1} \frac{\sum_{k=1}^{T} |p_k(\lambda)|^2 \, d\lambda}{\sqrt{1-\lambda^2}} \leqslant \int_{-1}^{1} \frac{l^{-1}(\lambda)\, d\lambda}{\sqrt{1-\lambda^2}}.$$

The latter inequality yields the statement of the theorem. □

Remark 5. We note that the characterization limit point/limit circle in terms of β_T as in the Theorem 3 does not hold. The simple example of free Jacobi operator, i.e. when $a_k = 1$, $b_k = 0$, $k = 1, 2, \ldots$ confirms this. In this case it is not hard to see that $C_T = I_T$ is an identity operator and consequntly, $\beta_T = 1$ for any N.

We introduce the notation:

$$\gamma_T = \max\bigl\{\gamma_k \,|\, \gamma_k \text{ is an eigenvalue of } C_T,\ k = 1, \ldots, T\bigr\}.$$

Lemma 1. *If there exist such a constant $M \in \mathbb{R}$ that $\gamma_T \leqslant M$ for all $T = 1, 2, \ldots$, then operator A is in the limit point case* (*the moment problem associated with the sequence $\{s_k\}$ is determined*).

Proof. The condition of the statement implies that the spectrum of C_T^{-1} is contained in $[\frac{1}{M}, +\infty)$. Then we can estimate the quadratic form

$$\Bigl((S_T)^{-1}\,\xi, \xi\Bigr) = \Bigl((C_T)^{-1}\,\Lambda_T\xi, \Lambda_T\xi\Bigr) \geqslant \frac{1}{M}\left(\Lambda_T\xi, \Lambda_T\xi\right), \quad \xi \in \mathcal{F}^T. \tag{30}$$

Choosing $\xi = e_1 = (1, 0, \ldots, 0)$ we observe that due to (24) we have that

$$\Lambda_T \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} T_1(0) \\ T_2(0) \\ \cdot \\ T_T(0) \end{pmatrix}. \tag{31}$$

Since it is known that

$$T_{2n-1}(0) = (-1)^{n-1}, \quad T_{2n}(0) = 0, \tag{32}$$

using (30), (31), (32), we come to the estimate

$$\Bigl((S_T)^{-1}\,e_1, e_1\Bigr) \geqslant \frac{1}{M}\,\frac{T}{2}.$$

The latter inequality means that the maximal eigenvalue of S_T^{-1} tends to infinity when T goes to infinity, which in turn implies that the minimal eigenvalue of S_T tends to zero when T goes to infinity. That yields the limit point case due to Theorem 3. □

The matrix $C = \{c_{n,m}\}_{n,m=0}^{\infty}$ give rise to the (formally defined) operator $\mathcal{C}$ in l_2:

$$(\mathcal{C}f)_n := \sum_{m=0}^{\infty} c_{nm} f_m, \quad f \in l_2.$$

Then without any a'priory assumptions on c_{nm}, only the quadratic form

$$C[f,f] := \sum_{m,n\geqslant 0} c_{nm} \overline{f_m} f_n$$

is well-defined on the domain D (6).

We always assume the positivity condition

$$\sum_{m,n\geqslant 0} c_{nm} \overline{f_m} f_n \geqslant 0, \quad f \in D,$$

which guarantees [14] the existence Jacobi matrix A and the (not necessarily uniquely defined) measure $M \in \mathcal{M}_H$ which solves the moment problem (3). The following result is valid.

Theorem 8. *The quadratic form $C[\,,\,]$ is closable in l_2 if one of the following occurs:*

- a) *If the Jacobi operator A is in the limit circle case (the moment problem associated with sequence $\{s_k\}$ is indetermined)*
- b) *If operator A is bounded, i.e., for all $k = 1, 2, \ldots$, $|a_k|, |b_k| \leqslant c$ for some $c \in \mathbb{R}_+$ and its spectral measure is absolutely continuous with respect to Lebesgue measure.*

Proof. Consider the operator $B : l^2 \mapsto L_2(\mathbb{R}, M)$, $M \in \mathcal{M}_H$, acting by the rule

$$Bf := \sum_k f_k \mathcal{T}_k(\lambda),$$

with the domain D. In view of (20) the relationship between B and $C[,]$ is of the form

$$C[f,f] = \|Bf\|^2_{L_2(\mathbb{R},M)}, \quad f \in D.$$

By definition, the form $C[\,,\,]$ is closable in l_2 if and only if the operator B is closable.

We need to show that for the sequence $D \ni f^{(n)} \to 0$ in l_2 such that and $Bf^{(n)} \to F$ in $L_2(\mathbb{R}, M)$ we necessarily have that $F = 0$.

In the limit circle case we use the existence of the holomorphic kernel (9), which gives that the limit function is analytic: $F \in \mathcal{E}$. Indeed the sequence $f^n \in D$, so $\mathcal{E} \ni Bf^{(n)} \to F$ in $L_2(\mathbb{R}, M)$ then (see the end of the Section 3) $F \in \mathcal{E}$. Alternatively, to get the same result, we can pass to the limit in (10). Since $f^{(n)} \to 0$ in l_2, we have that corresponding

$$Bf^{(n)} =: F^{(n)} \to 0 \quad \text{in} \quad L_2\left(-1, 1; \frac{d\lambda}{\sqrt{1-\lambda^2}}\right).$$

The latter immediately yields $F = 0$ in $L_2(-1, 1; d\rho)$, so $F = 0$ on $(-1, 1)$ and thus being analytic in $\mathbb{C}$, $F = 0$ in $\mathbb{C}$. This implies the closability of the operator B and of the quadratic form $C[\,,\,]$.

In the case of bounded Jacobi matrix A we can assume that $\|A\| < 1$ (otherwise considering the operator αA with appropriate α). In this case we have that the solution of the moment problem is unique and denoted by M. The support of theis measure is bounded: $\operatorname{supp} M \subset (-1, 1)$. Repeating the above arguments we get that for the limit function F, which is in this case is not necessarily analytic, we also have that $F = 0$ in $L_2(-1, 1; d\rho)$ and thus $F = 0$ almost everywhere on $(-1, 1)$. But then we immediately obtain that $F = 0$ in $L_2(\mathbb{R}, M)$ since the support of $M \subset (-1, 1)$ and M is absolutely continuous with respect to dx. And thus $C[\,,\,]$ and B are closable. $\square$

Now we assume the sequence $s_0, s_1, \ldots$ is indeterminate (the matrix A is in the limit circle case). By analogy with (19) we introduce the linear manifold

$$B_A^\infty := \left\{ \sum_{k=1}^{\infty} f_k \mathcal{T}_k(\lambda) \;\Big|\; C[f, f] < \infty \right\}.$$

The scalar product in B_A^∞ is given by the rule

$$[F, G]_{B_A^\infty} := C[f, g] = \int_{-\infty}^{\infty} \overline{F(\lambda)} G(\lambda)\, dM, \quad M \in \mathcal{M}_H,$$

$$F, G \in B_A^\infty, \quad F(\lambda) = \sum_{k=1}^{\infty} f_k \mathcal{T}_k(\lambda), \quad G(\lambda) = \sum_{k=1}^{\infty} g_k \mathcal{T}_k(\lambda).$$

The reproducing kernel is given by (9):

$$J_z^\infty(\lambda) = \sum_{n=1}^{\infty} \overline{p_n(z)} p_n(\lambda).$$

The conditions of Theorem 1 verifying B_A^∞ is a de Branges space are trivially checked.

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St. Petersburg Department
of V.A. Steklov Institute of Mathematics
of the Russian Academy of Sciences,
27, Fontanka, 191023 St. Petersburg, Russia;
St.Petersburg State University,
7/9 Universitetskaya nab.,
St.Petersburg, 199034 Russia.
E-mail: mikhaylov@pdmi.ras.ru
E-mail: ftvsm78@gmail.com

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