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ON FINDING BIFURCATIONS FOR NONVARIATIONAL ELLIPTIC SYSTEMS BY THE EXTENDED QUOTIENTS METHOD

Abstract. We develop a new method for finding bifurcations for nonlinear systems of equations based on a direct finding of bifurcations through saddle points of extended quotients. The method is applied to find the saddle-node bifurcation point for system of elliptic equations with the nonlinearity of the general convex-concave type. The main result justifies the variational formula for the detection of the maximal saddle-node type bifurcation point of stable positive solutions. As a consequence, a precise threshold value separating the interval of the existence of stable positive solutions is established.

Dedicated to 90th birthday of Nina Nikolaevna Uraltseva, with respect and honor

§1. Introduction

This paper develops a method of detecting bifurcation introduced in [20, 25], which provides a direct way of finding bifurcations by identifying saddle points of the corresponding extended Rayleigh quotient. We develop the method by finding saddle-node bifurcation point for the following system of equations:

$$\begin{cases} -\Delta u_i = a_i(x)u_i^{q_i} + \lambda g_i(x,u), & x \in \Omega, \\ u_i \geqslant 0, & x \in \Omega, \\ u_i|_{\partial\Omega} = 0, \quad i = 1,\dots,m. \end{cases} \tag{1.1}$$

Here $q_i \in (0,1)$, $i = 1,\dots,m$, Ω is a bounded domain in $\mathbb{R}^d$ with $\partial\Omega \in C^2$, $d \geqslant 1$, $\lambda \in \mathbb{R}$, $u := (u_1,\dots,u_m)$. For $i = 1,\dots,m$, $a_i \in L^\infty(\Omega)$, $a_i > 0$ in Ω, $a_i(\cdot)$, $g_i(\cdot,u)$, $\forall u \in \mathbb{R}^m$ are Hölder continuous functions in Ω, and $g_i(x,\cdot) \in C^1(\mathbb{R}^m,\mathbb{R})$. Furthermore,

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$(g_1):\ \exists c_0, c_1 > 0$ and $\exists \gamma \in (1, +\infty)$ such that

$0 \leqslant g_i(x,u) \leqslant c_0|u| + c_1|u|^\gamma,\ g_i(x,u) \not\equiv 0,\ x \in \Omega,\quad u \in \mathbb{R}^m_+,\ i = 1,\dots,m;$

$(g_2):\ \exists c_2, c_3 > 0$ and $\exists \gamma_0 \in (1, \gamma]$ such that

$(g_{i,u_i}(x,u)u_i^2 - g_i(x,u)u_i) \geqslant c_2|u|^{\gamma_0+1} + c_3|u|^{\gamma+1},\quad x \in \Omega,\quad u \in \mathbb{R}^m_+;$

$(g_3):\ \exists c_4, c_5 > 0$ such that

$$|g_{i,u_i}(x,u)| \leqslant c_4 + c_5|u|^{\gamma-1},\quad x \in \Omega,\quad u \in \mathbb{R}^m_+.$$

Throughout this paper the summation convention is in place: we sum over any index that appears twice. A particular example of functions g_i, $i = 1,\dots,m$ that meet condition $(g_1) - (g_3)$ is as follows: $g_i = b_i(x)u_i + b(x)\sum_{j=1}^m u_j^\gamma$, $x \in \Omega$, $u \in \mathbb{R}^m_+$, $i = 1,\dots,m$ with $\gamma_0 = \gamma > m$, and $b, b_i \in L^\infty(\Omega)$, $b_i(x) \geqslant 0$, $i = 1,\dots,m$, $b(x) \geqslant c$ in Ω for some constant $c > 0$.

Hereafter, we denote $\mathcal{W} := (\overset{\circ}{W}{}^1_2(\Omega) \cap L^{\gamma+1}(\Omega))^m$, $F_i(u,\lambda) := -\Delta u_i - a_i|u_i|^{q_i-1}u_i - \lambda g_i(x,u)$, $i = 1,\dots,m$, $F(u,\lambda) = (F_1(u,\lambda),\dots,F_m(u,\lambda))^T$, $u \in \mathcal{W}$, $\lambda \in \mathbb{R}$. $(\mathcal{W})^*$ means the dual space of $\mathcal{W}$. A point $(u,\lambda) \in \mathcal{W} \times \mathbb{R}$, $u_i \geqslant 0$ in Ω, $i = 1,\dots,m$ is called a weak solution of (1.1) if $F(u,\lambda) = 0$ holds true in $(\mathcal{W})^*$.

Define

$$S := \{u \in C^1(\overline{\Omega}) \mid \exists c(u) > 0,\ u > c(u)\ \mathrm{dist}(x, \partial\Omega)\ \text{ in } \Omega,\ u|_{\partial\Omega} = 0\}.$$

We show below that the map $F(\cdot,\lambda) : \mathcal{W} \to \mathcal{W}^*$ is Fréchet differentiable on S^m, $\forall\lambda \in \mathbb{R}$ (see Proposition 2.1 below). We denote by $F_u(u,\lambda)$ the Fréchet derivative of $F(u,\lambda)$ at $u \in S^m$. A solution $u_\lambda \in S^m$ of (1.1) is said to be *stable* if $\lambda_1(F_u(u_\lambda,\lambda)) \geqslant 0$, and *asymptotically stable* if $\lambda_1(F_u(u_\lambda,\lambda)) > 0$, cf. [8, 11]. Hereafter, $\lambda_1(F_u(u,\lambda))$, $u \in S^m$ denotes the first eigenvalue of the operator $F_u(u,\lambda)(\cdot)$, that is,

$$\lambda_1(F_u(u,\lambda)) := \inf_{\phi \in \mathcal{W}} \frac{\int |\nabla\phi|^2 - q_i \int a_i(u_i)^{q_i-1}|\phi_i|^2 - \lambda \int g_{i,u_j}(x,u)\phi_j\phi_i}{\int |\phi|^2}.$$

Introduce

$$\mathcal{W}_s := \{u \in \mathcal{W} \cap S^m :\quad \lambda_1(F_u(u,\tau)) \geqslant 0,\ \tau = \mathcal{R}(u,u)\}.$$

For a definition of the functional $\mathcal{R}(u,u)$ see below (1.4). We call a solution $(\widehat{u},\widehat{\lambda}) \in \mathcal{W}_s \times \mathbb{R}$ of (1.1) the *saddle-node bifurcation point* in $\mathcal{W}_s$ (or, equivalently, fold, turning point) (cf. [26, 27]) if the following is fulfilled: (i) the nullspace $N(F_u(\widehat{u},\widehat{\lambda}))$ of the Fréchet derivative $F_u(\widehat{u},\widehat{\lambda})$ is not empty; (ii) there exists $\varepsilon > 0$ and a neighborhood $U_1 \subset \mathcal{W}$ of $\widehat{u}$ such that for any $\lambda \in (\widehat{\lambda}, \widehat{\lambda}+\varepsilon)$ equation (1.1) has no solutions in $\mathcal{W}_s \cap U$; (iii) for each $\lambda \in (\widehat{\lambda}-\varepsilon, \widehat{\lambda})$, the equation has precisely two distinct solutions in $\mathcal{W}_s \cap U$. This definition corresponds to the solution's curve turning back at the bifurcation value $\widehat{\lambda}$. The solution's curve turning forward is defined similar. In the case only (i)–(ii) are satisfied, we call $(\widehat{u}, \widehat{\lambda})$ the *quasi-saddle-node bifurcation point* (or saddle-node type bifurcation point, cf. [25]) of (1.1) in $\mathcal{W}_s$. A quasi-saddle-node bifurcation point (u^*, λ^*) is said to be *maximal* in $\mathcal{W}_s$ if $\widehat{\lambda} \leqslant \lambda^*$ for any other quasi-saddle-node bifurcation point $(\widehat{u}, \widehat{\lambda})$ of (1.1) in $\mathcal{W}_s$.

A model example for (1.1) in the scalar case, i.e., $m = 1$, is the Ambrosetti–Brezis–Cerami problem [2] with concaveвЂ“convex nonlinearity

$$-\Delta u = u^q + \lambda u^\gamma, \quad u \geqslant 0 \ \text{ in } \ \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.2}$$

where $0 < q < 1 < \gamma$. It is why the nonlinearity in (1.1) can be considered to be of the convex-concave type. From [2] it follows that there exists an extremal value $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*]$, (1.2) has a stable positive solution u_λ, while for $\lambda > \lambda^*$, (1.2) does not admit weak positive solutions. According to [2], the solution u_λ for $\lambda \in (0, \lambda^*)$ is obtained by super-subsolution methods, while u_{λ^*} is shown to exist as a limit point of (u_λ). Unfortunately, this method is not easily adaptable to systems of equations like (1.1). Indeed, the super-sub solution method for a system of equations differs considerably from that which is used for a scalar equation.

In general cases, system (1.1) is not a variational or Hamiltonian. It should be noted that in contrast to the extensive literature concerning the existence of solutions for variational and Hamiltonian systems (see the survey [15]), relatively little research is devoted to nonvariational and non-Hamiltonian systems of equations (see, e.g., [1, 5, 7, 10, 35, 36] and references therein).

The finding of bifurcations of solutions to equations poses a more complex challenge, requiring a comprehensive approach that considers both the finding of solutions themselves and the analysis of the structure of the family of solutions. This problem is still quite challenging even when dealing with scalar equations. The complete answer to the question on the existence of the saddle-node bifurcation point and the exact shape of the positive solution curves for instance of the scalar equation (1.2) was obtained only in radially symmetric solutions [28, 31, 34]. An additional obstacle encountered when studying problems (1.1) and (1.2) is the presence of singular derivatives of the right-hand sides. Specifically, standard methods (see [8, 9, 26, 27]) are not readily applicable for verifying that the solution's curve turns back at $(\widehat{u}, \widehat{\lambda})$, due to the difficulties in testing conditions for the second derivative of $F(u, \lambda)$.

The existence of positive solutions and multiplicity results were studied only in some special cases of system (1.1) in the variational form (see, e.g., [18, 19, 25, 36] and references therein). A recent study [25] answered the question of whether positive solutions of system (1.1) in the variational form have a quasi-saddle-node bifurcation point. However, in the general cases of system (1.1), to the best of our knowledge, no studies have been conducted on the existence of non-negative solutions and saddle-node bifurcation points.

Let us state our main results. Observe that by the definition the quasi-saddle-node bifurcation point $(u, \lambda) \in \mathcal{W}_s \times \mathbb{R}$ of (1.1) should satisfy the system of equations

$$\begin{cases} F(u, \lambda) = 0, \\ F_u(u, \lambda)(v) = 0, \end{cases} \tag{1.3}$$

with some $v \in N(F_u(u, v))$. To analyze this system, following [25] we introduce the *extended Rayleigh quotient* (*extended quotient* for short) associated with (1.1)

$$\mathcal{R}(u, v) := \frac{\int (\nabla u_i, \nabla v_i) - \int a_i u_i^{q_i} v_i}{\int g_i(x, u) v_i}, \quad u \in \mathcal{W}_s, \ \ v \in \Sigma(u). \tag{1.4}$$

Here $\Sigma(u) := \{v \in \mathcal{W} : \int g_i(x, u) v_i \neq 0\}$ for $u \in \mathcal{W}_s$. Clearly, (g_1) entails $S^m \subset \Sigma(u)$, and thus, $\mathcal{W}_s \subset \Sigma(u)$. Hence, $\mathcal{R}(u, u)$ is well-defined for $u \in \mathcal{W}_s$. Observe, for $u \in \mathcal{W}_s$, $v \in \Sigma(u)$,

$$\begin{cases} \lambda = \mathcal{R}(u, v), \\ \mathcal{R}_v(u, v) = 0, \\ \mathcal{R}_u(u, v) = 0 \end{cases} \Leftrightarrow \begin{cases} F(u, \lambda) = 0, \\ F_u(u, \lambda)(v) = 0, \end{cases}$$

that is, the set of quasi-saddle-node bifurcation points of (1.1) is contained in the set of critical points of $\mathcal{R}(u, v)$ on $\mathcal{W}_s \times \Sigma(u)$.

In our approach, the following minimax formula plays a major role (cf. [25])

$$\lambda_s^* := \sup_{u \in \mathcal{W}_s} \inf_{v \in \Sigma(u)} \mathcal{R}(u, v). \tag{1.5}$$

The main result of the work is as follows

Theorem 1.1. *Assume* (g_1)–(g_2), $q_i \in (0, 1)$, $i = 1, \ldots, m$.

(1^o)**:** *Then* $0 \leqslant \lambda_s^* < +\infty$.

(a) : *For* $\lambda = \lambda_s^*$, *there exists a weak positive solution* $u_s^* \in (C^{1,\alpha}(\overline{\Omega}))^m \cap \mathcal{W}$ *of system* (1.1).

(b) : *For any* $\lambda > \lambda_s^*$, *system* (1.1) *has no stable weak positive solutions.*

(2^o)**:** *Assume in addition that* (g_3) *is satisfied,* $\gamma < 2^* - 1$, *and* $q_i < \frac{(d+2)}{2(d-2)}$, $i = 1, \dots, m$ *if* $d > 6$.

Then $0 < \lambda_s^* < +\infty$, (u_s^*, λ_s^*) *is a maximal quasi-saddle-node bifurcation point of* (1.1) *in* $\mathcal{W}_s$. *Moreover,* u_s^* *is a stable solution of* (1.1).

Here $2^* = 2d/(d-2)$ if $d \geqslant 3$, and $2^* = +\infty$ if $d = 1, 2$.

Remark 1.1. Statement (1^o) can be supplemented as follows. There exists $\lambda \in [0, \lambda_s^*]$ such that (1.1) has a stable positive weak solution $u_\lambda \in (C^{1,\alpha}(\overline{\Omega}))^m \cap \mathcal{W}$. Indeed, we will see below that (1.1) has a stable positive weak solution at least for $\lambda = 0$.

The following can also be considered in conjunction with the value (1.5)

$$\lambda_{as}^* := \sup_{u \in \mathcal{W}_{as}} \inf_{v \in \Sigma(u)} \mathcal{R}(u, v). \tag{1.6}$$

Here $\mathcal{W}_{as} := \{u \in \mathcal{W} \cap S^m : \ \lambda_1(F_u(u,\tau)) > 0, \ \tau = \mathcal{R}(u,u)\}$. It easily see that $\lambda_{as}^* \leqslant \lambda_s^*$. For (1.6), it can be obtained a result similar to Theorem 1.1. In particular, we have the following

Theorem 1.2. *Assume* (g_1)–(g_3), $q_i \in (0,1)$, $i = 1, \dots, m$, *and* $q_i < \frac{(d+2)}{2(d-2)}$, $i = 1, \dots, m$ *if* $d > 6$. *Then* $0 < \lambda_{as}^* < +\infty$, *and*

(1) : *For* $\lambda = \lambda_{as}^*$, *system* (1.1) *has a stable weak positive solution* $u_{as}^* \in (C^{1,\alpha}(\overline{\Omega}))^m \cap \mathcal{W}$. *Furthermore,* $(u_{as}^*, \lambda_{as}^*)$ *is a maximal quasi-saddle-node bifurcation point of* (1.1) *in* $\mathcal{W}_{as}$.

(2) : *For any* $\lambda > \lambda_{as}^*$, (1.1) *has no asymptotically stable weak positive solutions.*

(3): *There exists a sequence of asymptotically stable weak positive solutions* $u_{\lambda_n} \in (C^{1,\alpha}(\overline{\Omega}))^m \cap \mathcal{W}$ *of* (1.1) *with* $\lambda = \lambda_n \in (0, \lambda_{as}^*)$, $n = 1, \dots$ *such that* $u_{\lambda_n} \to u_{as}^*$ *in* $\mathcal{W}$ *and* $\lambda_n \to \lambda_{as}^*$ *as* $n \to +\infty$.

Remark 1.2. It is natural to expect that $u_s^* = u_{as}^*$, $\lambda_s^* = \lambda_{as}^*$ and (u_s^*, λ_s^*) is indeed a saddle-node bifurcation point of (1.1) in $\mathcal{W}_s$. It should be noted that assertions (1^o), (b) and (2) of Theorems 1.1 and 1.2 do not necessarily mean that (1.1) has no positive solutions for $\lambda > \lambda_s^*$. Furthermore, such a behavior is possible if (1.1) has an S-shaped bifurcation curve (see [4,6,17]).

Remark 1.3. The literature on the existence of bifurcations usually deals with finding saddle-node bifurcation points for equations rather than quasi-saddle-node bifurcations (see, e.g., [26,27]). However, in numerical methods and applications, bifurcations are typically detected by finding a point on the solutions curve where operator $F_u(u_\lambda, \lambda)$ is singular, thus finding actual quasi-saddle-node bifurcation (see, e.g., [29]).

Remark 1.4. We believe that the variational formula (1.5) has a potential to provide a useful tool in further analysis of saddle-node bifurcation points and in construction of numerical methods for finding them (cf. [21–24,32]).

The rest of the paper is organised as follows. Section 2 presents preliminaries. In Section 3, we prove Theorems 1.1, 1.2. In Appendix, we present a proof of a version of Ekeland's principal for smooth functional.

§2. Preliminaries

We use the standard notation $L^p := L^p(\Omega)$ for the Lebesgue spaces, $1 \leqslant p \leqslant +\infty$, and denote by $\|\cdot\|_p$ the associated norm. By $\overset{\circ}{W}{}^1_2 := \overset{\circ}{W}{}^1_2(\Omega)$ we denote the standard Sobolev space, endowed with the norm $\|u\|_{1,2} = (\int |\nabla u|^2)^{1/2}$. Hereafter, we denote $W := \overset{\circ}{W}{}^1_2 \cap L^{\gamma+1}$, $d(x) :=$ dist$(x, \partial\Omega)$.

For $\delta > 0$, define $S(\delta) := \{u \in S \mid \; u(x) > \delta d(x) \text{ in } \Omega\}$, $S(0) := S$. Clearly, $S(\delta)$, $\forall\delta \geqslant 0$ is an open subset in $C^1 := C^1(\overline{\Omega})$. Let Y be a topological space such that $S^m(\delta) \subset Y$. The set $S^m(\delta)$ endowed with topology of Y we denote by $S^m_Y(\delta)$. $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ denotes the Banach space of bounded linear operators from $\mathcal{W}$ into $\mathcal{W}^*$.

Proposition 2.1. *Assume* (g_1)–(g_3). *Let* $\lambda \in \mathbb{R}$, $0 < q_i < 1$, $i = 1, \dots, m$. *Then* $F_i(\,\cdot\,, \lambda) : \mathcal{W} \to \mathcal{W}^*$ *is Fréchet differentiable at any* $u \in S^m$, *and* $F_i(\,\cdot\,, \lambda) \in C^1(S^m_{C^1}, \mathcal{L}(\mathcal{W}, \mathcal{W}^*))$. *Furthermore, if in addition* $q_i \leqslant (\bar\gamma - 1)/2$, *where* $\bar\gamma = \max\{\gamma, 2^* - 1\}$, *then* $F_{i,u}(\,\cdot\,, \lambda) \in C(S^m_{\mathcal{W}}(\delta), \mathcal{L}(\mathcal{W}, \mathcal{W}^*))$, $\forall\delta > 0$.

Proof. We develop an approach proposed in [2]. First, we verify the assertion for the map $Q(u) := a_i u^{q_i}$, $u \in S$. Using the estimate $u(x) \geqslant c(u)d(x)$ in Ω and Hölder's inequalities we derive

$$\Big|\int a_i u^{q_i-1}\phi\psi\, dx\Big| = \Big|\int a_i u^{q_i}\left(\frac{\phi}{u}\right)\psi\, dx\Big| \leqslant \frac{\|a_i\|_\infty}{c(u)}\|u\|_p^q \cdot \|\frac{\phi}{d(\cdot)}\|_2 \cdot \|\psi\|_{\gamma+1},$$
$$u \in S,\ \forall\phi, \psi \in C_0^\infty(\Omega),$$

where $p = 2q_i(\gamma+1)/(\gamma-1)$. By the Hardy inequality, $\|\phi/d(\cdot)\|_2 \leqslant C\|\phi\|_{1,2}$, $\forall\phi \in C_0^\infty(\Omega)$, and thus, we derive

$$\Big|\int a_i u^{q_i-1}\phi\psi\,dx\Big| \leqslant \frac{C\|a_i\|_\infty}{c(u)}\|u\|_p^q\|\phi\|_{1,2}\|\psi\|_{\gamma+1}, \qquad u \in S,\ \forall\phi,\psi \in C_0^\infty(\Omega), \tag{2.1}$$

where $C \in (0,+\infty)$ does not depend on u,ϕ,ψ. This implies that $Q(\,\cdot\,) : W \to W^*$ is Fréchet differentiable at any $u \in S$. In the same manner we can see that $Q_u(\cdot) \in C(S_{C^1}, \mathcal{L}(W,W^*))$.

Let us prove the second part for $Q(\cdot)$. For simplicity we assume that $\bar\gamma = \gamma$. Let $\delta > 0$. Suppose $u_n, u \in S(\delta)$, $n = 1,\ldots$, $u_n \to u$ in $L^{\gamma+1}$ as $n \to +\infty$. This implies that there is $\bar u \in L^{\gamma+1}$ and a subsequence $(u_{n_k})_{k=1}^\infty$ such that $|u_{n_k}|, |u| \leqslant \bar u$ in Ω. Indeed, since $\|u_n - u\|_{\gamma+1} \to 0$, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\|u_{n_k} - u\|_{\gamma+1} \leqslant \frac{1}{2^k}$. Consider the pointwise limit $\bar u := |u| + \sum_{k=1}^\infty |u_{n_k} - u|$. By Beppo Levi's lemma $\bar u \in L^{\gamma+1}$, and thus, Minnowski's inequality yields

$$\|\bar u\|_{\gamma+1} \leqslant \|u\|_{\gamma+1} + \sum_{k=1}^\infty \|u_{n_k} - u\|_{\gamma+1} \leqslant \|u\|_{\gamma+1} + 1.$$

Moreover, $\bar u \geqslant |u| + |u_{n_k} - u| \geqslant |u_{n_k}|$, $\forall k \in \mathbb{N}$.

Hence, $u_{n_k}^{q_i-1}d(x) \leqslant \delta^{-1}\bar u^{q_i}$ in Ω, $k = 1,\ldots$, and therefore, Lebesgue's dominated convergence theorem yields $u_{n_k}^{q_i-1}d(x) \to u^{q_i-1}d(x)$ in $L^{(\gamma+1)/q_i}$ as $k \to +\infty$. Similar to (2.1) we have

$$\Big|\int a_i(u_{n_k}^{q_i-1} - u^{q_i-1})\phi\psi\,dx\Big| \leqslant C\|(u_{n_k}^{q_i-1} - u^{q_i-1})d(\cdot)\|_{p/q_i}^{q_i}\|\phi\|_{1,2}\|\psi\|_{\gamma+1}, \qquad \forall\phi,\psi \in W, \tag{2.2}$$

for some $C < +\infty$ which does not depend on $u, u_{n_k} \in S$, $\phi,\psi \in (W)^*$. Observe, the assumption $q_i \leqslant (\gamma-1)/2$ implies $p \leqslant \gamma+1$. Hence,

$$\|(u_{n_k}^{q_i-1} - u^{q_i-1})d(\,\cdot\,)\|_{p/q_i} \to 0$$

as $k \to +\infty$, and thus, $Q_u(\,\cdot\,) \in C(S_W(\delta), \mathcal{L}(W,W^*))$.

The proof of the assertions of the proposition for the remaining terms in $F_i(\,\cdot\,,\lambda)$ is similar. The only remark we wish to make is that condition (g_3) must be considered when proving the assertions for $G_i(u) := g_i(x,u)$, $u \in S^m$. □

Proposition 2.2. *If $u \in \mathcal{W}$ is a weak solution to* (1.1)*, then $u \in S^m$.*

Proof. Note that (1.1) implies that $-\Delta u_i \geqslant 0$, $i = 1,\dots,m$, and thus, by the maximum principles for the elliptic problems, $u_i > 0$ in Ω, $i = 1,\dots,m$. The standard bootstrap argument and Sobolev's embedding theorem entail that $u_i \in L^\infty(\Omega)$, $i = 1,\dots,m$. This by the regularity results for elliptic problems [16,30] implies that $u_i \in C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$, $i = 1,\dots,m$. Moreover, since $a_i(\cdot)$ and $g_i(x,u_\lambda(x))$, $i = 1,\dots,m$ are Hölder continuous functions in Ω, the Schauder estimates and the Hopf boundary maximum principle [16] imply that $u_\lambda \in (C^2(\Omega))^m$ and $\min_{x' \in \partial\Omega} |\partial u_i(x')/\partial\nu(x')| > 0$, $i = 1,\dots,m$. Hence we get that $u > c(u)\,\mathrm{dist}(x,\partial\Omega)$ in Ω for some $c(u) > 0$, and thus, $u \in S^m$. □

Let $i = 1,\dots,m$. By Brezis–Oswald's result [3] there exists a unique solution $w_i \in \overset{\circ}{W}{}^1_2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$ of

$$\begin{cases} -\Delta w = a_i w^{q_i} \quad \text{in} \quad \Omega, \\ \quad w|_{\partial\Omega} = 0. \end{cases} \tag{2.3}$$

By the assumption $a_i(\cdot)$ is a Hölder continuous function in Ω, and hence, by the Schauder estimates (see, e.g., [16]), $w_i \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Furthermore, the strong maximum principles for the elliptic problems imply that $\min_{x' \in \partial\Omega} \dfrac{\partial w_i(x')}{\partial\nu(x')} > 0$, where $\nu(x')$ denotes the interior unit normal at $x' \in \partial\Omega$, see, e.g., Lemma 3.4 in [16]. Thus, $w_i \in S$.

Moreover, $\overline{w} := (w_1,\dots,w_m)$ is a stable solution of (1.1) with $\lambda = 0$, i.e., $\overline{w} \in \mathcal{W}_s$. Indeed, from Proposition 2.1 it follows that $F(\overline{w},0) \in C^1(S^m_{C^1}; \mathcal{L}(\mathcal{W},\mathcal{W}^*))$, and therefore, $\lambda_1(F_u(\overline{w},0))$ is well defined. It is not hard to show that $\overline{w}$ is a minimizer of

$$E(v) := \left(\frac{1}{2} \sum_{i=1}^m \int |\nabla v_i|^2 - \frac{1}{q+1} \int a_i |v_i|^{q+1} \right) \quad \text{on} \quad (\overset{\circ}{W}{}^1_2)^m,$$

that is, $E(\overline{w}) = \inf_{v \in (\overset{\circ}{W}{}^1_2)^m} E(v)$. In view of that $E(v)$ is a strong convex functional, this means that

$$\lambda_1(F_u(\overline{w},0)) > 0. \tag{2.4}$$

Lemma 2.1. *Assume that $u^0 \in \mathcal{W}_s$ is such that*

$$-\infty < \lambda_0 := \inf_{v \in \Sigma(u^0)} \mathcal{R}(u^0,v) < +\infty.$$

Then u^0 is a weak solution of (1.1) for $\lambda = \lambda_0$.

Proof. Let $v^k \in \Sigma(u^0)$, $k = 1, \ldots$, and

$$\lambda_k \equiv \mathcal{R}(u^0, v^k) \to \inf_{v \in \Sigma(u^0)} \mathcal{R}(u^0, v) \equiv \lambda_0 \quad \text{as} \quad k \to +\infty.$$

Since $\mathcal{R}(u, v) = \mathcal{R}(u, sv)$, $\forall s \in \mathbb{R} \setminus 0$, $\forall v \in \Sigma(u)$, $\forall u \in \mathcal{W}_s$, we may assume that

$$\int g_i(x, u^0) v_i^k = 1, \quad k = 1, \ldots. \tag{2.5}$$

Calculate

$$\mathcal{R}_v(u^0, v^k)(\xi) = \frac{\int (\nabla u_i^0, \nabla \xi_i) - \int a_i (u_i^0)^{q_i} \xi_i - \mathcal{R}(u^0, v^k) \int g_i(x, u^0) \xi_i}{\int g_i(x, u^0) v_i^k},$$

$$\mathcal{R}_{vv}(u^0, v^k)(\xi, \zeta) =$$

$$- \frac{\left(\int (\nabla u_i^0, \nabla \xi_i) - \int a_i (u_i^0)^{q_i} \xi_i - \lambda_k \int g_i(x, u^0) \xi_i\right) \cdot \int g_i(x, u^0) \zeta_i}{(\int g_i(x, u^0) v_i^k)^2}$$

$$\frac{\left(\int (\nabla u_i^0, \nabla \zeta_i) - \int a_i (u_i^0)^{q_i} \zeta_i - \lambda_k \int g_i(x, u^0) \zeta_i\right) \cdot \int g_i(x, u^0) \xi_i}{(\int g_i(x, u^0) v_i^k)^2}, \ \zeta, \xi \in \mathcal{W}.$$

Let $\phi \in \mathcal{W}$, $\|\phi\|_{\mathcal{W}} = 1$. Using (2.5) and the Hölder and Sobolev inequalities one can see that

$$\Big| \int (g_i(x, u^0)(v_i^k + \tau \phi_i) \Big| = \Big| 1 + \tau \int g_i(x, u^0) \phi_i \Big| \geqslant 1 - a_0 |\tau|, \tag{2.6}$$

where $a_0 \in (0, \infty)$ does not depend on ϕ and $k = 1, \ldots$. Hence $v^k + \tau \phi \in \Sigma(u^0)$ for any $k = 1, \ldots$ and τ such that $|\tau| < \tau_0 := 1/a_0$.

By (2.6) and (g_1), we have

$$\begin{aligned} \|\mathcal{R}_{vv}(u^0, v^k + \tau\phi)\|_{(\mathcal{W} \times \mathcal{W})^*} &= \frac{2}{|\int g_i(x, u^0)(v_i^k + \tau \phi_i)|^2} \\ &\times \sup_{\xi, \zeta \in \mathcal{W}} \frac{\left| \left(\int (\nabla u_i^0, \nabla \xi_i) - \int a_i (u_i^0)^{q_i} \xi_i - \tilde{\lambda}_k \int g_i(x, u^0) \xi_i \right) \cdot \int g_i(x, u^0) \zeta_i \right|}{\|\xi\|_{\mathcal{W}} \|\zeta\|_{\mathcal{W}}} \\ &\leqslant \frac{2}{(1 - a_0|\tau|)^2} \left(\sum_{i=1}^m \| - \Delta u_i^0 - a_i (u_i^0)^{q_i} - \tilde{\lambda}_k g_i(x, u^0) \|_{\mathcal{W}^*} \right) \|u^0\|_{\mathcal{W}} \\ &\leqslant \frac{C_0}{(1 - a_0|\tau|)^2}, \end{aligned} \tag{2.7}$$

where $C_0 \in (0,\infty)$ does not depend on $k=1,\dots$. Here $\tilde\lambda_k := \mathcal{R}(u^0, v^k+\tau\phi)$, $k = 1,\dots$ which as it is easy to see are bounded. We thus may apply Theorem 4.1 to the functional $G(v) := \mathcal{R}(u^0, v)$ defined in the open set $V := \Sigma(u^0) \subset \mathcal{W}$. Indeed, it is easily seen that $G \in C^2(\Sigma(u^0))$, and (2.7) implies (4.2), while by (2.6) there holds (4.3). Thus, we have

$$\epsilon_k := \|\mathcal{R}_v(u^0, v^k)\|_{\mathcal{W}^*} \to 0 \quad \text{as} \quad k \to +\infty,$$

which by (2.5) yields:

$$\Big|\int (\nabla u_i^0, \nabla\xi) - \int a_i (u_i^0)^{q_i}\xi - \lambda_k \int g_i(x, u^0)\xi\Big| \leqslant \epsilon_k \|\xi\|_{\mathcal{W}}, \quad \forall \xi \in W.$$

$i = 1,\dots,m$. Now passing to the limit as $k \to +\infty$ we obtain (1.1). □

§3. Proof of main results

Proof of Theorem 1.1. Let us prove (1^o). Let $\overline{w} := (w_1,\dots,w_m)$, where w_i, $i = 1,\dots,m$ be a solution of (2.3). By the above $\overline{w} \in \mathcal{W}_s$, and thus, we have

$$\lambda_s^* := \sup_{u\in\mathcal{W}_s} \inf_{v\in\Sigma(u)} \mathcal{R}(u,v) \geqslant \inf_{v\in\Sigma(\overline{w})} \frac{\int (\nabla w_i, \nabla v_i) - \int a_i w_i^{q_i} v_i}{\int g_i(x, \overline{w}) v_i} = 0.$$

Since $0 \leqslant \lambda_s^* \leqslant +\infty$, there exists a maximizing sequence $u^n \in \mathcal{W}_s$, $n = 1,\dots$, such that

$$\lambda_n := \lambda(u^n) := \inf_{v\in\Sigma(u^n)} \mathcal{R}(u^n, v) \to \lambda_s^* \quad \text{as} \quad n \to +\infty.$$

By Lemma 2.1,

$$\begin{cases} -\Delta u_i^n = a_i (u_i^n)^{q_i} + \lambda_n g_i(x, u^n), & x \in \Omega, \\ u_i^n|_{\partial\Omega} = 0, \quad i = 1,\dots,m. \end{cases} \tag{3.1}$$

Testing (3.1) by u_i, $i = 1,\dots,m$ and integrating by parts we derive

$$\|u^n\|_{1,2}^2 = \int a_i |u_i^n|^{q_i+1} + \lambda_n \int g_i(x, u^n) u_i^n, \quad n = 1,\dots. \tag{3.2}$$

Since $u^n \in \mathcal{W}_s$, $n = 1,\dots$,

$$\lambda_1(F_u(u^n,\lambda_n)) \\ := \inf_{\phi\in\mathcal{W}} \frac{\int |\nabla\phi|^2 - q_i \int a_i (u_i^n)^{q_i-1}|\phi_i|^2 - \lambda_n \int g_{i,u_j}(x, u^n)\phi_j\phi_i}{\int |\phi|^2} \geqslant 0. \tag{3.3}$$

Hence, taking $\phi = u^n$ in (3.3) we obtain

$$\|u^n\|_{1,2}^2 \geqslant q_i \int a_i |u_i^n|^{q_i+1} + \lambda_n \int g_{i,u_i}(x,u^n)(u_i^n)^2 \quad n = 1,\dots. \tag{3.4}$$

Subtracting (3.2) from (3.4), and using (g_2) we obtain

$$\begin{aligned}(1-q_i)\int a_i|u_i^n|^{q_i+1} &\geqslant \lambda_n \int \big(g_{i,u_i}(x,u^n)(u_i^n)^2 - g_i(x,u^n)u_i^n\big)\\ &\geqslant \lambda_n(c_2\|u^n\|_{\gamma_0+1}^{\gamma_0+1} + c_3\|u^n\|_{\gamma+1}^{\gamma+1}), \quad n = 1,\dots.\end{aligned}$$

Applying Hölder's inequality we derive

$$C_1 \geqslant \lambda_n(c_2\|u^n\|_{\gamma_0+1}^{\gamma_0-q} + c_3\|u^n\|_{\gamma+1}^{\gamma-q}), \quad n = 1,\dots. \tag{3.5}$$

Let us show that there exists $C_2 \in (0,+\infty)$ which does not depends on $n = 1,\dots,$ such that

$$\|u^n\|_{\gamma_0+1}, \|u^n\|_{\gamma+1} \geqslant C_2, \quad n = 1,\dots,. \tag{3.6}$$

We need the following assertion that is derived by the same method as Lemma 3.3 in [2]

Lemma 3.1. *Assume that $f(t)$ is a function such that $t^{-1}f(t)$ is decreasing for $t > 0$, $a \in L^\infty$, $a > 0$ in Ω. Let v and w satisfy: $u > 0$, $w > 0$ in Ω, $v = w = 0$ on $\partial\Omega$, and*

$$-\Delta w \leqslant af(w), \quad -\Delta u \geqslant af(u), \quad in \ \ \Omega.$$

Then $u \geqslant w$.

By the assumption $g_i(x,u) \geqslant 0$, $x \in \Omega$, $i = 1,\dots,m$, $u \in \mathbb{R}_+^m$, and therefore,

$$-\Delta u_i^n \geqslant a_i(u_i^n)^{q_i} \ \text{ in } \ \Omega, \quad i = 1,\dots,m, \quad n = 1\dots.$$

Hence, (2.3) and Lemma 3.1 yield

$$u_i^n \geqslant w_i, \quad i = 1,\dots,m, \quad n = 1\dots, \tag{3.7}$$

and as a result, we get (3.6). Clearly, (3.5), (3.6) imply that $\lambda_s^* < +\infty$. From this it is easily follows by (3.2), (g_1), and Sobolev's inequalities that $\|u^n\|_{1,2} \leqslant C_2$, $n = 1,\dots$, where $C_2 \in (0,+\infty)$ does not depend on $n = 1,\dots$. Thus, (u^n) is bounded in $\mathcal{W}$, and therefore, by the Banach–Alaoglu and the Sobolev theorems there exists a subsequence (again denoted by (u^n)) such that

$$u^n \rightharpoonup u_s^* \quad \text{weakly in } \mathcal{W}, \tag{3.8}$$

$$u^n \to u_s^* \quad \text{strongly in } \ (L^r)^m, \ \ 1 \leqslant r < 2^*, \tag{3.9}$$

as $n \to +\infty$ for some $u_s^* \in \mathcal{W}$. From (3.7) it follows tha $u_{s,i}^* \geqslant w_i > 0$, $i = 1, \ldots, m$, $n = 1 \ldots$.

Passing to the limit in (3.1) as $n \to +\infty$ we obtain

$$-\Delta u_{s,i}^* = a_i (u_i^*)^{q_i} + \lambda_s^* g_i(x, u_s^*), \quad x \in \Omega, \quad i = 1, \ldots, m. \tag{3.10}$$

Using Proposition 2.2 we conclude that $u_s^* \in S^m$. Thus, we obtain (1^o), (a).

To show (1^o), (b), suppose conversely that for $\lambda > \lambda_s^*$ there exists a stable weak non-negative solution u_λ of (1.1). Then by Proposition 2.2, $u_\lambda \in \mathcal{W}_s$, and consequently, (1.5) yields $\inf_{v \in \Sigma(u_\lambda)} \mathcal{R}(u_\lambda, v) < \lambda$. Hence, there exists $v \in \Sigma(u_\lambda)$ such that $\mathcal{R}(u_\lambda, v) < \lambda$. Assume that $\int g_i(x, u_\lambda) v_i > 0$. Then

$$\int (\nabla u_{\lambda,i}, \nabla v_i) - \int a_i u_{\lambda,i}^{q_i} v_i - \lambda \int g_i(x, u_\lambda) v_i < 0,$$

which contradicts (1.1), and as a result we get (1^o), (b).

Let us prove (2^o). For simplicity we assume that $d > 6$. Using $\gamma < 2^* - 1$, it is not hard to show that (3.8), (3.9), (3.10), and the condition $u_s^* \neq 0$ imply

$$u^n \to u_s^* \quad \text{strongly in } \ \mathcal{W} \text{ as } n \to +\infty. \tag{3.11}$$

Clearly, by the maximum principle $w_i \in S(\delta)$, $i = 1, \ldots, m$ with some $\delta > 0$. Hence, (3.7) imply that $u_n \in S^m(\delta)$, $n = 1, \ldots$. The assumption $q_i < \frac{(d+2)}{2(d-2)}$, $i = 1, \ldots, m$ and $\gamma < 2^* - 1$ implies by Proposition 2.1 that $F_u(u, \lambda) \in C(S_{\mathcal{W}}^m(\delta); \mathcal{L}(\mathcal{W}, \mathcal{W}^*))$. Hence,

$$\langle F_u(u_n, \lambda_n)(\phi), \psi \rangle \to \langle F_u(u_s^*, \lambda_s^*)(\phi), \psi \rangle \text{ as } \ n \to +\infty, \qquad \forall \phi, \psi \in \mathcal{W}, \tag{3.12}$$

and consequently, $\lambda_1(F_u(u_n, \lambda_n)) \to \lambda_1(F_u(u_s^*, \lambda_s^*)) \geqslant 0$ as $n \to +\infty$. Thus, we get that $u_s^* \in \mathcal{W}_s$.

Let us show that $\lambda_1(F_u(u_s^*, \lambda_s^*)) = 0$. Suppose, contrary to our claim, that $\lambda_1(F_u(u_s^*, \lambda_s^*)) > 0$. Then $F_u(u_s^*, \lambda_s^*)(\cdot) : \mathcal{W} \to \mathcal{W}^*$ is nonsingular linear operator. From Proposition 2.1 we have $F(\cdot, \lambda) \in C^1(S_{C^1}^m; \mathcal{L}(\mathcal{W}, \mathcal{W}^*))$. Hence, by the Implicit Functional Theorem (see, e.g., Theorem 10.2.1 in [12]) there is a neighbourhood $V \times U \subset \mathbb{R} \times S_{C^1}^m$ of (λ_s^*, u_s^*) and a mapping

$V \ni \lambda \mapsto u_\lambda \in U$ such that $u_\lambda|_{\lambda=\lambda_s^*} = u_s^*$ and $F(u_\lambda, \lambda) = 0$, $\forall \lambda \in V$. Furthermore, the map $u_{(\cdot)} : V \to U$ is continuous. Since $\lambda_1(F_u(u_s^*, \lambda_s^*)) > 0$, there exists a neighbourhood $V_1 \subset V$ of λ_s^* such that $\lambda_1(F_u(u_\lambda, \lambda)) > 0$ for every $\lambda \in V_1$. However, this contradicts assertion (1^o), (b) of the theorem. Thus, $\lambda_1(F_u(u_s^*, \lambda_s^*)) = 0$, and (u_s^*, λ_s^*) is a quasi-saddle-node bifurcation point of (1.1) in $\mathcal{W}_s$. Since (1^o), (b), (u_s^*, λ_s^*) is a maximal quasi-saddle-node bifurcation point of (1.1) in $\mathcal{W}_s$.

Finally, let us show that $0 < \lambda_s^*$. Suppose the converse $\lambda_s^* = 0$. Then by the above $0 = \lambda_1(F_u(u_s^*, \lambda_s^*)) = \lambda_1(-\Delta - q|u_s^*|^{q-1})$. However, this contradicts (2.4). □

Proof of Theorem 1.2. is similar to the proof of Theorem 1.1. We only need to show assertion (3) of Theorem. Indeed, by the construction there are sequences λ_n, $u_{as}^n \in \mathcal{W}_{as}$, $n = 1, \dots$ such that $F_u(u_{as}^n, \lambda_n) = 0$, $n = 1, \dots$, $\lambda_n \to \lambda_{as}^*$, and $u_{as}^n \to u_{as}^*$ strongly in $\mathcal{W}$ as $n \to +\infty$. Moreover, $u_{as}^* \in \overline{\mathcal{W}}_{as}$, and $\lambda_1(F_u(u_{as}^*, \lambda_{as}^*)) = 0$. Thus, $u_{as}^* \in \overline{\mathcal{W}}_{as} \setminus \mathcal{W}_{as}$. On the other hand, $u_{as}^n \in \mathcal{W}_{as}$, $n = 1, \dots$. Hence, $u_{as}^n \neq u_{as}^*$, $n = 1, \dots$, and we thus obtain the proof of (3). □

§4. Appendix A

Let X be a Banach space and $V \subset X$ be an open set. Denote $B_r := \{\phi \in X : \ \|\phi\|_X \leqslant r\}$, $r > 0$. Assume that $G : V \to \mathbb{R}$, $G \in C^2(V)$. Consider

$$\widehat{G} = \inf_{v \in V} G(v). \tag{4.1}$$

Theorem 4.1. *Assume that $|\widehat{G}| < +\infty$. Suppose that there exist τ_0, a_0, $C_0 \in (0, +\infty)$, and a minimizing sequence $(v_k) \subset V$ of (4.1) such that*

$$\|G_{vv}(v_k + \tau\phi)\|_{(X \times X)^*} < \frac{C_0}{(1 - |\tau| a_0)^2} < +\infty, \tag{4.2}$$

$$v_k + \tau\phi \in V, \quad \forall \tau \in (-\tau_0, \tau_0), \quad \forall \phi \in B_1, \quad \forall k = 1, \dots. \tag{4.3}$$

Then

$$\|G_v(v_k)\|_{X^*} := \sup_{\xi \in X \setminus 0} \frac{|G_v(v_k)(\xi)|}{\|\xi\|_X} \to 0 \quad \text{as} \quad k \to +\infty.$$

Proof. Suppose, contrary to our claim, that there exists $\alpha > 0$ such that

$$\|G_v(v_k)\|_{X^*} > \alpha, \quad \forall k = 1, \dots.$$

This means that for every $k = 1, \ldots$, there exists $\phi_k \in V$, $\|\phi_k\|_X = 1$ such that $|G_v(v_k)(\phi_k)| > \alpha$. By the Taylor expansion

$$G(v_k + \tau\phi_k) = G(v_k) + \tau G_v(v_k)(\phi_k) + \frac{\tau^2}{2} G_{vv}(v_k + \theta_k\tau\phi_k)(\phi_k, \phi_k),$$

for sufficiently small $|\tau|$, and some $\theta_k \in (0,1)$, $k = 1, \ldots$. Suppose, for definiteness, that $G_v(v_k)(\phi_k) > \alpha$. Then for $\tau \in (-\tau_0, 0)$, by (4.2), we have

$$G(v_k + \tau\phi_k) \leqslant G(v_k) + \tau\alpha + \frac{\tau^2}{2}\frac{C_0}{(1+\tau a_0)^2}, \quad k = 1, \ldots.$$

It is easily seen that there exists $\tau_1 \in (0, \tau_0)$ such that

$$\kappa(\tau) := \tau\left(\alpha + \frac{\tau}{2}\frac{C_0}{(1+\tau a_0)^2}\right) < 0, \quad \forall\tau \in (-\tau_1, 0).$$

Since (v_k) is a minimizing sequence, for any $\varepsilon > 0$ there exists $k(\varepsilon)$ such that

$$G(v_k) < \widehat{G} + \varepsilon, \quad \forall k > k(\varepsilon).$$

Take $\tau \in (-\tau_1, 0)$ and $\varepsilon_0 = -\kappa(\tau)/2$. Then by the above

$$G(v_k + \tau\phi_k) < \widehat{G} + \varepsilon_0 + \kappa(\tau) = \widehat{G} + \kappa(\tau)/2 < \widehat{G}, \quad \forall k > k(\varepsilon_0),$$

and thus, in view of (4.3) we get a contradiction. □

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