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ON A DISCRETE MAX-PLUS TRANSPORTATION PROBLEM

Abstract. We provide an explicit algorithm to solve the idempotent analogue of the discrete Monge–Kantorovich optimal mass transportation problem with the usual real number field replaced by the tropical (max-plus) semiring, in which addition is defined as the maximum and product is defined as usual addition, with $-\infty$ and 0 playing the roles of additive and multiplicative identities. Such a problem may be naturally called tropical or "max-plus" optimal transportation problem. We show that the solutions to the latter, called the optimal tropical plans, may not correspond to perfect matchings even if the data (max-plus probability measures) have all weights equal to zero, in contrast with its classical discrete optimal transportation analogue, where perfect matching optimal plans in similar situations always exist. Nevertheless, in some randomized situation the existence of perfect matching optimal tropical plans may occur rather frequently. At last, we prove that the uniqueness of solutions of the optimal tropical transportation problem is quite rare.

**Dedicated to N. N. Uraltseva
on the occasion of her 90th birthday**

§1. Introduction

In this paper we consider a discrete optimization problem that looks quite similar to the classical Monge–Kantorovich optimal mass transportation problem and in fact, as we argue later, is nothing else but the idempotent version of the latter. We begin with a short motivational introduction.

1.1. Motivation of the problem. Suppose we have m signal sources and n receivers regularly exchanging information between them. Each source $i \in \{1, \ldots, m\}$ may transmit an amount $h_{i,j}$ of information to

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$j \in \{1, \ldots, n\}$. The maximum amount of information the source i may send at one time is given by a number k_i, that is,

$$\max_{j \in \{1,\ldots,n\}} h_{i,j} = k_i. \tag{1}$$

Analogously, the maximum amount of information the receiver j may get at one time is given by a number l_j, that is,

$$\max_{i \in \{1,\ldots,m\}} h_{i,j} = l_j. \tag{2}$$

Of course, (1) and (2) may only be simultaneously valid if

$$\max_{i \in \{1,\ldots,n\}} k_i = \max_{j \in \{1,\ldots,n\}} l_j. \tag{3}$$

The cost $C_{i,j}$ of transmitting between the source i and the receiver j depends affinely on the amount of transmitted information and takes into account the known fixed cost $g_{i,j}$ of using the communication channel between them, that is,

$$C_{i,j} = g_{i,j} + \gamma h_{i,j}$$

for some given coefficient $\gamma > 0$. The goal is to find the values $h_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ (the respective matrix being further called the optimal tropical transportation plan, the explanation of the terminology being given in the sequel) minimizing the maximum of $C_{i,j}$ over all i and j, that is, finding the

$$\inf\{\max_{i,j}(g_{i,j} + \gamma h_{i,j})\colon h_{i,j} \text{ satisfying (1) and (2)}\}.$$

Denoting $c_{i,j} := g_{i,j}/\gamma$, this amounts to solving

$$\inf\{\max_{i,j}(c_{i,j} + h_{i,j})\colon h_{i,j} \text{ satisfying (1) and (2)}\}. \tag{4}$$

1.2. Idempotent (max-plus or tropical) interpretation. Let us now completely change the point of view and look at the above problem as a version of the classical optimal mass transportation problem in the context of *idempotent analysis*: more precisely, analysis over the tropical (max-plus) semiring $\bar{\mathbb{R}}_- := \mathbb{R} \cup \{-\infty\}$ endowed with the operations

$$a \oplus b := \max\{a, b\}, \quad a \otimes b := a + b,$$

which substitute the usual addition and multiplication of real numbers respectively. The value $-\infty$ is an identity with respect to $\oplus$ and 0 is an identity with respect to $\otimes$. Both operations are commutative, associative and $a \otimes (b \oplus c) = a \otimes b + a \otimes c$. Thus the roles of 0 and 1 on the usual real line are played here by $-\infty$ and 0 respectively. For a general overview of idempotent analysis we refer the reader to the classic book [5].

The classical discrete Monge–Kantorovich optimal mass transportation problem (see, e.g. [6] for a comprehensive introduction to the subject) is that of finding the optimal plan of transportation in the following sense: solve the minimization problem

$$\inf\left\{ \sum_{i,j=1}^{m,n} c_{i,j}\pi_{i,j} \;:\; [\pi_{i,j}]_{i,j=1}^{m,n} \right\} \tag{5}$$

where the infimimum is performed over m-by-n matrices $[\pi_{i,j}]_{i,j=1}^{m,n}$ which satisfy the constraints

$$\sum_{j=1}^{n} \pi_{i,j} = k_i, \tag{6}$$

$$\sum_{i=1}^{m} \pi_{i,j} = l_j, \tag{7}$$

with the numbers k_i, l_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$ all fixed. This is usually interpreted as finding the way of optimally transporting the discrete measure

$$\mu := \sum_{i=1}^{m} k_i \delta_{x_i}$$

to another discrete measure

$$\nu := \sum_{j=1}^{n} l_j \delta_{y_i},$$

for some $x_i \in X$, $y_j \in Y$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, with X and Y some sets and δ_z standing for the Dirac point mass at z.

The value $\pi_{i,j}$ is, then, interpreted as the amount of mass transported from x_i to y_j. The matrix $[\pi_{i,j}]_{i,j=1}^{m,n}$ is identified with the discrete measure $\pi = \sum_{i,j=1}^{m,n} \pi_{i,j}\delta_{(x_i,y_j)}$ over $X \times Y$; constraints (6) and (7) now mean that the marginals (or projections) of the measure π along X and Y are μ and ν respectively. The quantity $\sum_{i,j=1}^{m,n} c_{i,j}\pi_{i,j}$ is the total transportation cost, targeted for minimization.

In the idempotent max-plus setting the role of the Dirac measure δ_z over an arbitrary set Z concentrated at a point $z \in Z$ is played by the characteristic function (for which we retain the same notation as for the Dirac measure) δ_z defined by

$$\delta_z(z') := \begin{cases} 0, & z' = z, \\ -\infty, & z' \neq z. \end{cases}$$

The analogues of sums of Dirac masses on sets X and Y are the functions on these sets respectively defined by

$$\mu(x) := \max_{i=1,\ldots,m} (k_i + \delta_{x_i}(x)), \qquad \nu(y) = \max_{j=1,\ldots,n} (l_j + \delta_{y_j}(y)), \tag{8}$$

i.e. μ is the function taking the value k_i at each x_i and $-\infty$ elsewhere, and ν is the function taking the value l_j at each y_i and $-\infty$ elsewhere; the analogue of a discrete measure represented by a sum of Dirac masses with weights h_{ij} at points $(x_i, y_j) \in X \times Y$ is the function

$$\pi(x,y) = \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left(h_{ij} + \delta_{(x_i,y_j)}(x,y) \right). \tag{9}$$

We will be referring to the coefficients k_i as the *weights* of μ and to the coefficients l_j as the weights of ν. The total mass of a discrete measure, which in the traditional setting is the sum of its weights, corresponds, in the max-plus setting, to the maximum of its weights, i.e.

$$|\mu| := \max_{i=1,\ldots,m} k_i, \qquad |\nu| := \max_{j=1,\ldots,n} l_j.$$

We will assume, in complete analogy with the classical mass transportation theory, that $|\mu| = |\nu|$, which is exactly the condition (3), and for purely aesthetical reasons, which imply no loss of generality, we also assume that both total masses are zero, i. e. $|\mu| = |\nu| = 0$, so that μ and ν can be considered tropical versions of discrete *probability* measures. We will call, therefore, functions such as μ, ν, π above *discrete max-plus probability measures*, the set of such functions over a given set Z being denoted $\mathcal{M}(Z)$, so that $\mu \in \mathcal{M}(X)$ and $\nu \in \mathcal{M}(Y)$, $\pi \in \mathcal{M}(X \times Y)$.

Suppose now that $\{x_i\}_{i=1}^m \subset X$, $\{y_j\}_{j=1}^n \subset Y$ are given, and μ, ν are defined by (8) and (9) respectively. The max-plus, or tropical, analogue of a transport plan between μ and ν is a function π defined as in (9) and satisfying the constraints

$$\max_{x\in X} \pi(x,y) = \nu(y), \quad \max_{y\in Y} \pi(x,y) = \nu(x), \tag{10}$$

The Monge–Kantorovich optimal transportation problem (5) with the given cost function $c\colon X \times Y \to \bar{\mathbb{R}}$ then becomes, in the max-plus setting, the problem of solving

$$\inf \{\max(c(x, y) + \pi(x, y))\colon \pi \in \mathcal{M}(X \times Y) \text{ satisfies (10)}\}. \tag{11}$$

It is worth mentioning that the problem just stated is not the unique example of a meaningful idempotent (max-plus or tropical) version of a classical optimization problem; similar tropical formulations have arisen elsewhere in the literature. For instance, this is the case of the so-called bottleneck traveling salesman problem (see e.g. section 8 of [4] or [3]), which can be considered a max-plus version of the classical traveling salesman problem.

We will further identify, whenever convenient, max-plus discrete probability measures with the sequences of their weights, and the transport plan π (given by (9)) with the matrix of coefficients $[h_{i,j}]$, and refer to this object in either interpretation as a *tropical transport plan* for the discrete max-plus probability measures μ and ν (or, equivalently, for the sequences of their weights) whenever (10) holds, which in terms of the matrix $[h_{i,j}]$ amounts precisely to (1) and (2), namely, that maximum of the i-th row of the matrix must be k_i and the maximum of the j-th row of the matrix must be l_j. If we write $c_{i,j} := c(x_i, y_j)$, then, in view of (8) and (9), the problem (11) becomes exactly (4), which is the reason why it may be considered as the max-plus version of the Monge–Kantorovich problem (5). Such an identification of measures with weights, plans and cost functions with matrices is quite natural in the discrete setting we are considering here, especially when the points x_i and y_j themselves are of no practical importance.

1.3. Our contribution. In this paper we provide an explicit algorithm to solve the optimal tropical transportation problem (4) and find an explicit formula for the optimal tropical cost, i.e., the value of (4). As a consequence, we obtain some curious results on the optimal tropical plans and values. In particular:

- In the case $m = n$ optimal tropical plans corresponding to perfect matchings (those given by permutation matrices) may not exist even if the max-plus probability measures μ and ν have all the weights equal to zero (we henceforth call this case *fundamental*), see Example 4.10 below. This is a stark contrast with classical optimal mass transportation theory, where (again with $m = n$) perfect matching optimal transport plans between sums of Dirac masses with equal weights always exist. Nevertheless, it turns out that, at least in the fundamental case, the existence of perfect matching optimal tropical plans occurs rather frequently as the number of weights of both μ and ν becomes large, the respective statement being made precise by introducing randomness in the cost. More precisely, under a concrete randomization of the cost matrix, the existence of a perfect matching optimal plan is "asymptotically almost sure" as the number of weights of the measures approaches infinity. This is Theorem 5.5 below.

- In the fundamental case, under the same type of randomization of the cost matrix, the optimal tropical cost is, asymptotically almost surely, the lowest value among all the entries of the cost matrix. This is the content of Theorem 5.1 and Remark 5.3 below.
- We also prove that uniqueness of an optimal tropical plan asymptotically almost surely fails to occur (in the fundamental case), when the cost matrix entries are sampled uniformly. This is Theorem 5.7 below.

§2. Notation and preliminaries

In complete analogy with the classical optimal transportation theory, the matrix $[h_{i,j}[_{i,j=1}^{m,n}$ with each $h_{i,j} \in [-\infty, 0]$ satisfying (1) and (2) will be called discrete *max-plus (or tropical) plan* (or just a *plan* for brevity) for max-plus discrete probability measures $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$. Equivalently, as remarked earlier, it can be seen as a max-plus discrete probability measure in the sense given by (9). We denote by $\Pi(\mu, \nu)$ the set of all such plans (which is always nonempty, since $\mu \otimes \nu \in \Pi(\mu, \nu)$, where $(\mu \otimes \nu)_{i,j} := k_i + l_j$).

For the given cost matrix $[c_{i,j}]_{i,j=1}^{m,n}$ we define

$$d_c(\mu, \nu) := \inf \left\{ \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} (c_{i,j} + h_{i,j}) \colon h \in \Pi(\mu, \nu) \right\}.$$

If we interpret h as an element of $h \in \mathcal{M}(X \times Y)$, i.e. as in (9), then we may write $h(x_i, y_j)$ and $c(x_i, y_j)$ instead of $h_{i,j}$ and $c_{i,j}$ respectively, since the points x_i and y_j can be assumed fixed in every discussion. Again for purely aesthetical reasons, and to allow for the interpretation of the numbers $c_{i,j}$ as representing a cost, it is convenient to assume $c_{i,j} \geqslant 0$, which can always be done without loss of generality. The minimizer $h \in \Pi(\mu,\nu)$ in the above problem will be called the *minimizing (or optimal) tropical plan*, the set of such minimizing plans being denoted by $\Pi^c(\mu,\nu)$. The number $d_c(\mu,\nu)$ will be called the *optimal tropical cost* between μ and ν. We must say that, despite our choice of notation, the function $d_c(\cdot,\cdot)$ is not a metric.

In the sequel we assume the sequences of weights k_j and l_j to be ordered in decreasing order $k_1 = l_1 = 0$, i.e.

$$k_n \leqslant k_{n-1} \leqslant \cdots \leqslant k_1 = 0, \quad l_n \leqslant l_{n-1} \leqslant \cdots \leqslant l_1 = 0. \tag{12}$$

We denote by $\Lambda(\mu)$ and $\Lambda(\nu)$ the sets of weights of μ and ν respectively. If we wish to retain the interpretation of μ and ν as elements of $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ respectively, then (12) is achieved simply by a relabeling of the fixed points x_i and y_j, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$.

For any $h \in \Pi(\mu,\nu)$, by the *support* of h, denoted $\operatorname{supp}(h)$, we will mean the subset of $X \times Y$ of points (x,y) where $h(x,y) > -\infty$, or (again, equivalently, since any such point must be one of the pairs (x_i,y_j)) with the set of pairs $(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$ such that $h_{i,j} > -\infty$. In the latter case, we may also write $h(i,j)$ rather than h_{ij} (for instance, if we wish to free up the subindex place for another purpose, as in section 4.2 below).

For a set X we denote by $\#X$ its cardinality. We also write sometimes $a \vee b$ for the maximum of the numbers a and b.

§3. Reduced transportation plans and existence of minimizers

We start with the following definition.

Definition 3.1. Given fixed discrete max-plus probability measures μ and ν, we will call a tropical plan $h \in \Pi(\mu,\nu)$ *reduced* if for each i,j such that $h_{i,j} > -\infty$, the element $h_{i,j}$ is a *strict* maximum in its row *or* in its column, and denote by $\Pi_R(\mu,\nu)$ the set of reduced plans for discrete μ and ν.

Without loss of generality for the optimal tropical transportation problem, all the weights of μ and ν can be taken to be finite (i. e. $> -\infty$). In fact, if, say, $k_i = -\infty$ for some $i \in \{1,\dots,m\}$, then the i-th row of h, for any $h \in \Pi(\mu,\nu)$ must consist only of $-\infty$. In this case, in the expression that defines $d_c(\mu,\nu)$, each of the elements over which the minimum is taken is

$$\max_{(i,j)}(h_{i,j}+c_{i,j}) = \max\{\ldots, h_{i,1}+c_{i,1}, h_{i,2}+c_{i,2}, \ldots, h_{i,n}+c_{i,n}, \ldots\}$$
$$= \max\{\ldots, -\infty, -\infty, \ldots -\infty, \ldots\},$$

but the maximum is non-negative, so the the numbers $-\infty$ can be changed to sufficiently small negative numbers (negative but with large absolute value) without affecting the maximum and then the weight $k_j = -\infty$ can be changed to $\max_i h_{i,j}$ where $h_{i,j}$ are the new numbers just mentioned.

The following assertion holds true.

Lemma 3.2. *For all discrete $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$ one has*

$$d_c(\mu,\nu) = \inf\{\max_{(i,j)}(h_{i,j}+c_{i,j}) \colon\; h \in \Pi_R(\mu,\nu)\}.$$

Moreover, for every minimizing plan h there is a reduced minimizing plan $\tilde{h}$ with $\operatorname{supp}\tilde{h} \subset \operatorname{supp} h$ and $\tilde{h} = h$ on the support of $\tilde{h}$.

Proof. If $h_{i,j}$ is not a strict maximum neither in its column nor in its row for some $i, j \in \{1, \ldots, n\}$, then changing $h_{i,j}$ to $-\infty$ (or to any number less than $h_{i,j}$) does not affect $\max_{(i,j)}(h_{i,j}+c_{i,j})$. Changing all such entries of the matrix $[h_{i,j}]$ will transform the plan to a reduced one, and thus

$$d_c(\mu,\nu) = \inf\{\max_{(i,j)}(h_{i,j}+c_{i,j}) \colon\; h \in \Pi(\mu,\nu)\}$$
$$= \inf\{\max_{(i,j)}(h_{i,j}+c_{i,j}) \colon\; h \in \Pi_R(\mu,\nu)\}$$

as claimed. □

As a consequence, the following existence result holds.

Theorem 3.3. *The discrete max-plus transportation problem admits a solution, namely,* inf *is actually a* min*.*

Proof. It is enough to refer to Lemma 3.2 and observe that the set of reduced plans $\Pi_R(\mu,\nu)$ has finitely many elements (indeed, each entry of a reduced plan must be either $-\infty$ or one of the weights of μ and ν). □

§4. Algorithm to solve the discrete max-plus transportation problem

4.1. Partition of the support of a plan. Given discrete μ and ν, for each $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$, let

$$p_i = \max\{j\colon\ l_j \geqslant k_i\}, \quad q_j = \max\{i\colon\ k_i \geqslant l_j\},$$
$$S_i = \{(i,1),\dots,(i,p_i)\}, \quad T_j = \{(1,j),\dots,(q_j,j)\}.$$

The following statement gives some information on the general structure of reduced plans, as long as we adhere to the convention that the weights are sorted as in (12), which we agreed to hold throughout.

Lemma 4.1. *Let $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$ be discrete max-plus probability measures as in* (8) *and let $h \in \Pi_R(\mu,\nu)$. Assume, without loss of generality, that the weights of the measures satisfy* (12)*. The following assertions hold true.*

(1) *For each $i \in \{1,\dots,m\}$, at least one of the numbers $h_{i,1},\dots,h_{i,p_i}$ must be k_i, and the numbers $h_{i,p_i+1},\dots,h_{i,n}$ are all strictly less than k_i. Likewise, for each $j \in \{1,\dots,n\}$, at least one of the numbers $h_{1,j},\dots,h_{q_i,j}$ must be l_j, and the numbers $h_{q_{i+1},j},\dots,q_{n,j}$ are all strictly less than l_j.*
(2) *If the weights k_i and l_j are all distinct, with the exception of $k_1 = l_1 = 0$, then $S_i \cap T_j = \emptyset$ whenever $(i,j) \neq (1,1)$.*
(3) *One has $k_i = l_j$ for some $(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$, if and only if $(i,j) \in S_i \cap T_j$.*

Proof. (1) Fix $i \in \{1,\dots,m\}$. The maximum among $h_{i,1},\dots,h_{i,n}$ must be k_i. If $h_{i,p_i+\bar m} = k_i$ for some $\bar m > 0$, then the maximum among $h_{1,p_i+\bar m},\dots$, $h_{n,p_i+\bar m}$ is at least k_i. The maximum among $h_{1,p_i+m},\dots,h_{n,p_i+\bar m}$ must be $l_{p_i+\bar m}$, which, by definition of p_i is, strictly less than k_i. This contradiction proves that the maximum of $h_{i,1},\dots,h_{i,n}$, equal to k_i, occurs among $h_{i,1},\dots,h_{i,p_i}$, and not among $h_{i,p_i+1},\dots,h_n$, which proves the first part of the assertion. The second part, i.e. the claim about the numbers $h_{1,j},\dots,h_{n,j}$ is proven completely symmetrically.

(2) Suppose $(i,j) \neq (1,1)$ and $(q,p) \in S_i \cap T_j$. Since the pair (q,p) is in S_i, its first component must be i, i.e. $q = i$. Similarly, since it is in T_j, we must have $p = j$. Thus $(q,p) = (i,j)$. Moreover, the definition of p_i and q_j now contains only strict inequalities because we are assuming all the weights distinct with the exception of $k_1 = l_1 = 0$. Having $(i,j) \in S_i$,

then, implies that $l_j > k_i$, while having $(i, j) \in T_j$ implies that $k_i > l_j$, and we have obtained a contradiction.

(3) Suppose $k_i = l_j$ for some pair $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$. Since $l_j \geqslant k_i$, we must have $(i, j) \in S_i$. Likewise, since $k_i \geqslant l_j$, then $(i, j) \in T_j$, so that necessity is proven. Now suppose $(i, j) \in S_i \cap T_j$. Since $(i, j) \in S_i$, $j \leqslant p_i$ so $l_j \geqslant k_i$, and $(i, j) \in T_j$ gives $i \in T_i$, so $k_i \geqslant l_j$. This completes the proof. $\square$

Given discrete max-plus probability measures μ, ν and a real number λ, let

$$R_\lambda := \Bigg(\bigcup_{\{i:\ k_i = \lambda\}} S_i \Bigg) \cup \Bigg(\bigcup_{\{j:\ l_j = \lambda\}} T_j \Bigg), \tag{13}$$

which is a subset of $\{1, \ldots, m\} \times \{1, \ldots, n\}$. We call R_λ a *region* or λ-region to emphasize the dependence on λ. A region can look like an L written backwards (like the one in pink in Figure 1 below), with the ends resting on the top and left edges of the grid, or a rectangle with its left side lying on the left edge of the grid, or a rectangle with its top side on the top edge of the grid, or a rectangle with both its left and top sides lying on the left and top sides of the grid, respectively. We remark that that our notion of region exists only when the measures μ and ν have been fixed. Also, for the description of our algorithm, it is essential that the weights of these measures are labeled as in 12.

Example 4.2. For $m = n = 6$ and the max-plus probability measures

$$\begin{aligned}
\mu &= \max\{0 + \delta_{x_1}, 0 + \delta_{x_2}, -2 + \delta_{x_3}, -3 + \delta_{x_4}, -4 + \delta_{x_5}, -4 + \delta_{x_6}\} \\
&= \begin{cases} 0, & x \in \{x_1 . x_2\}, \\ -2, & x = x_3, \\ -3, & x = x_4, \\ -4, & x \in \{x_5 . x_6\} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\nu &= \max\{0 + \delta_{y_1}, 0 + \delta_{y_2}, 0 + \delta_{y_3}, -1 + \delta_{y_4}, -2 + \delta_{y_5}, -2 + \delta_{y_6}\} \\
&= \begin{cases} 0, & y \in \{y_1 . y_2 . y_3\}, \\ -1, & y = y_4, \\ -2. & y \in \{y_5 . y_6\} \end{cases}
\end{aligned}$$

with x_j, $j = 1, \ldots, m$ as well as y_i, $i = 1, \ldots, n$ all distinct, the regions (each in a different color) and a plan are shown in Figure 1.

	0	0	0	-1	-2	-2
0	-∞	-∞	0	-1	-∞	-2
0	0	0	-∞	-∞	-2	-∞
-2	-∞	-∞	-2	-∞	-∞	-∞
-3	-∞	-∞	-∞	-∞	-3	-∞
-4	-∞	-∞	-∞	-4	-∞	-∞
-4	-4	-∞	-∞	-∞	-∞	-∞

Fig. 1. Regions for the pair (μ, ν) of Example 4.2.

It is convenient to extend the notions of plan and reduced plan as follows. Fix discrete max-plus probability measures μ, ν, with their weights arranged as in (12); suppose λ is one of these weights and consider the corresponding region R_λ. By a *plan* of R_λ we will mean a function $h : R_\lambda \to [-\infty, 0]$ such that the maximum of h on each row and on each column of R_λ is λ. In Figure 1 we see plans of each of the five regions, determined by the numbers in the cells.

Let $\Pi(R_\lambda)$ be the set of plans of R_λ. Like above, a plan

$$h = \{h_{i,j}\}_{(i,j)\in R_\lambda} \in \Pi(R_\lambda)$$

is called *reduced* whenever $h_{i,j}$ is a strict maximum of its row or a strict maximum of its column, as long as $h_{i,j} > -\infty$. Thus, a reduced plan of a λ-region has no numbers other than $-\infty$ and λ. The plans of the regions in Figure 1 are all reduced. We will denote by $\Pi_R(R_\lambda)$ the set of reduced plans of R_λ.

Given discrete max-plus probability measures μ, ν, a region R_λ, and a cost function c, we will use the notation d_c to also mean the following:

$$d_c(R_\lambda) := \min_{h\in\Pi(R_\lambda)} \max_{(i,j)\in R_\lambda} (h_{i,j} + c_{i,j}).$$

A plan $h \in \Pi(R_\lambda)$ at which the min in the preceding formula is attained will be called a *minimizing (or optimal) plan for the region* R_λ. The following assertion holds true.

Proposition 4.3. *Let $\mu \in \mathcal{M}(X), \nu \in \mathcal{M}(Y)$ be arbitrary discrete max-plus probability measures and a cost function $c : X \times Y \to [0,\infty)$ be given.*

Then

$$d_c(\mu,\nu) = \max_{\lambda\in\Lambda(\mu)\cup\Lambda(\nu)} d_c(R_\lambda).$$

Proof. By definition,

$$d_c(\mu,\nu) = \min_{h\in\Pi(\mu,\nu)} \max_{(i,j)}(h_{i,j} + c(x_i,y_j)).$$

Let us look at

$$M = \max_{\lambda} \min_{h\in\Pi(R_\lambda)} \max_{(i,j)\in R_\lambda} (h_{i,j} + c(x_i,y_j)),$$

which is the right hand side of the inequality we wish to prove. For each one of the distinct λ's, we pick $h^\lambda \in R_\lambda$ for which $\max_{(i,j)\in R_\lambda}(h_{i,j} + c(x_i,y_j))$ takes the least possible value, i.e. we pick an optimal plan h^λ of the region R_λ for each λ. Further, let $\bar\lambda$ be the value of λ at which M is attained. Let h^* be the element of $\Pi(\mu,\nu)$ such that its restriction to R_λ is h^λ, for each $\lambda\in\Lambda(\mu)\cup\Lambda(\nu)$. We claim that h^* is optimal for $d_c(\mu,\nu)$. In fact, if it is not, then there is another $h^0\in\Pi(\mu,\nu)$ such that

$$\max_{(i,j)}(h^0_{i,j} + c(x_i,y_j)) \leqslant \max_{(i,j)}(h_{i,j} + c(x_i,y_j)) \quad \forall h\in\Pi(\mu,\nu).$$

In particular, if $h = h^*$, then, by the assumption just made, the inequality must be strict, and

$$\max_{(i,j)\in R_{h^*}}(h^0_{i,j} + c(x_i,y_j)) \leqslant \max_{(i,j)}(h^0_{i,j} + c(x_i,y_j)) < \max_{(i,j)}(h^*_{i,j} + c(x_i,y_j)).$$

But the maximum value of the function $\lambda\mapsto\max_{(i,j)\in R_\lambda}(h^*_{i,j} + c(x_i,y_j))$ is M and is attained at $\lambda = h^*$. Thus, it follows that

$$\begin{aligned}\max_{(i,j)\in R_{\lambda^\star}}(h^0_{i,j} + c(x_i,y_j)) &< \max_{(i,j)\in R_{\lambda^\star}}(\lambda^\star_{i,j} + c(x_i,y_j)) \\ &= \max_{(i,j)\in R_{\lambda^\star}}(h^{\bar\lambda}_{i,j} + c(x_i,y_j)),\end{aligned}$$

which contradicts the definition of $h^{\bar\lambda}$. Therefore, $\lambda^\star$ is optimal for $d_c(\mu,\nu)$, so $d_c(\mu,\nu) = M$. □

4.2. Finding the optimal cost on a region. By Proposition 4.3, to solve the original problem, it is enough to find the optimal plan for each λ-region R_λ, hence also finding the respective optimal costs $d_c(R_\lambda)$; the optimal plan for the original problem will then coincide over each R_λ with the optimal plan for this region.

To find the optimal plan for the given region R_λ, suppose the cost function c be given; let us number the values that c takes over R_λ in an increasing order. Namely, suppose that $s \in \mathbb{Z}^+$ be the number of distinct values that c takes on over the region R_λ and denote these values, in increasing order, by

$$\beta_1 < \cdots < \beta_s. \tag{14}$$

For each $m \in \{1, 2, \ldots, s\}$ we define the function $h_c^m \colon R_\lambda \to \{-\infty, \lambda\}$ by the formula

$$h_c^m(i,j) := \begin{cases} \lambda & \text{if } c(x_i, y_j) \leqslant \beta_m, \\ -\infty, & \text{otherwise.} \end{cases}$$

That is, h_c^m is a plan for the region R_λ such that λ appears in the cells that host one of the smallest m values of c on the region, while $-\infty$ appears in all the other cells. In particular, for $m = s$, h_c^s fills all the cells in the region R_λ with λ, and hence is a plan for R_λ, that is, $h_c^s \in \Pi(R_\lambda)$. This motivates the following definition.

Definition 4.4. Given λ, a λ-region R_λ, and a cost function c, let m be the smallest integer for which the function h_c^m on the region R_λ constitutes a plan for R_λ, i. e.

$$m_c(\lambda) = \min\{m\colon\ h_c^m \in \Pi(R_\lambda)\}.$$

It is convenient to assign to each $(i,j) \in R_\lambda$ the number (from 1 to s) that the value $c(x_i, y_j)$ occupies in the list (14). Such an assignment is given by the function $f : R_\lambda \to \{1, 2, \ldots, s\}$ determined by the condition:

$$f(i_1, j_1) < f(i_2, j_2) \quad \text{if and only if} \quad c(x_{i_1}, y_{j_1}) < c(x_{i_2}, y_{j_2}) \quad \text{for } (i_1, j_1), (i_2, j_2) \in R_\lambda. \tag{15}$$

We illustrate the above definitions with the following example.

Example 4.5. Suppose the region is $\{1, 2, 3\}^2$ and the cost function (restricted to this region) is, in matrix form,

$$[c(x_i, y_j)]_{i,j=1}^3 = \begin{pmatrix} 2 & 4 & 8 \\ 8 & 2 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

Then $f(2,3) = f(3,2) = 1$, $f(1,1) = f(2,2) = f(3,1) = 2$, $f(1,2) = 3$, $f(1,3) = f(2,1) = 4$, and

$$h_c^1 = \begin{pmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & \lambda \\ -\infty & \lambda & -\infty \end{pmatrix}, \quad h_c^2 = \begin{pmatrix} \lambda & -\infty & -\infty \\ -\infty & \lambda & \lambda \\ \lambda & \lambda & -\infty \end{pmatrix},$$
$$h_c^3 = \begin{pmatrix} \lambda & -\infty & -\infty \\ -\infty & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{pmatrix}, \quad h_c^4 = \begin{pmatrix} \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{pmatrix}.$$

Here $m_c(\lambda) = 2$, $h_c^{m_c(\lambda)} = h_c^2$.

Lemma 4.6. *Let R_λ be a λ-region, c be a cost function. Let $h \in \Pi(R_\lambda)$ be a minimizer for $d_c(R_\lambda)$. Then the support of h is included in the support of $h_c^{m_c(\lambda)}$ and, with the notation of* (14),

$$d_c(R_\lambda) = \lambda + \beta_{m_c(\lambda)}.$$

Moreover, $h_c^{m_c(\lambda)}$ is itself a minimizing plan.

Proof. Let $\{(x_{i_1}, y_{j_1}), \ldots, (x_{i_p}, y_{j_p})\}$ be the support of h. Then

$$d_c(R_\lambda) = \max_{1 \leqslant k \leqslant p} \{c(x_{i_k}, y_{j_k}) + \lambda\}.$$

With the notation of (14), let β_m be the largest of the $c(x_{i_k}, y_{j_k})$; then $d_c(R_\lambda) = \lambda + \beta_m$. But then the function h_c^m, by definition, must place a λ in every cell (i,j) such that $c(x_i, y_j) \in \{\beta_1, \ldots, \beta_m\}$. Thus, the support of h is included in the support of h_c^m, and h_c^m is a plan, so $m_c(\lambda) \leqslant m$ and

$$d_c(R_\lambda) = \lambda + \beta_{m_c(\lambda)} \leqslant \lambda + \beta_m = d_c(R_\lambda).$$

On the other hand, since $h_c^{m_c(\lambda)}$ is a plan, we must have

$$d_c(R_\lambda) \leqslant \lambda + \beta_{m_c(\lambda)}.$$

Combining the last two inequalities, we obtain that $d_c(R_\lambda) = \lambda + \beta_{m_c(\lambda)}$, as desired, and $m = m_c(\lambda)$, so the support of h is included in the support of $h_c^{m_c(\lambda)}$. This means $h_c^{m_c(\lambda)}$ is itself a minimizing plan, and the last assertion follows. □

We collect the preceding conclusions in the following:

Theorem 4.7. *Let $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$, that is, discrete max-plus probability measures on X and Y respectively, namely: $\mu = \max_{i=1}^m (k_i + \delta_{x_i})$,*

$\nu = \max_{j=1}^n (k_j + \delta_{y_j})$, and let $c : X \times Y \to [0, \infty)$ be a given cost function. To obtain an optimal tropical plan h between μ and ν, one considers for every $\lambda \in \Lambda(\mu) \cup \Lambda(\nu)$ (i. e. for each distinct weight of either μ and ν) the respective region R_λ and a minimizing plan h_λ for each R_λ (e.g. $h_\lambda := h_c^{m_c(\lambda)}$), setting then $h \in \Pi(\mu, \nu)$ to be the plan whose restriction over each R_λ coincides with h_λ. Furthermore,

$$d_c(\mu, \nu) = \max_{\lambda \in \Lambda(\mu) \cup \Lambda(\nu)} (\lambda + c(x_{i_\lambda}, y_{j_\lambda})),$$

where each $(i_\lambda, j_\lambda) \in f^{-1}(m_c(\lambda))$, f standing for the numbering function defined by condition (15). *In particular, if all the weights k_i and l_j are distinct, except $k_1 = l_1 = 0$, then*

$$d_c(\mu, \nu) = \max_{1 \leqslant i \leqslant m} \min_{j \leqslant p_i} (k_i + c(x_i, y_j)) \vee \max_{1 \leqslant j \leqslant n} \min_{i \leqslant q_j} (l_j + c(x_i, y_j)).$$

Proof. It is a direct consequence of combining Lemma 4.6 with Proposition 4.3. □

4.3. Remarks on uniqueness of plans on a region. As we see fom Example 4.5,, the function $h_c^{m_c(\lambda)}$ (i.e. the first function on R_λ, as we go from $m = 1$ to $m = s$, that happens to be a plan) is *not* necessarily a *reduced* plan. Another, simpler, example of such a situation is

$$[c(x_i, y_j)]_{i,j=1}^2 = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix};$$

indeed, supposing $\{(1,1), (1,2), (2,1), (2,2)\}$ is a region R_λ, then, here, $m_c(\lambda) = 2$, and $h^{m_c(\lambda)}$ is the 2×2 matrix with λ in every entry.

We can state the following about reduced minimized plans and uniqueness of minimizing plans of a region.

Proposition 4.8. *Let R_λ be a λ-region (corresponding to some discrete max-plus probability measures μ and ν), $c : X \times Y \to [0, \infty)$ be a cost function. If $h_c^{m_c(\lambda)}$ is a reduced plan, then it is the* unique *reduced minimizing plan for $d_c(R_\lambda)$. Vice versa, if a minimizing plan for $d_c(R_\lambda)$ contains only $-\infty$ and λ and is unique among minimizing plans with this property, then it is reduced and must coincide with $h_c^{m_c(\lambda)}$.*

Proof. To prove the first assertion, suppose that $h_c^{m_c(\lambda)}$ is a reduced plan for $d_c(R_\lambda)$. It is minimizing by Lemma 4.6. If there is another reduced minimizing plan h for $d_c(R_\lambda)$, then by Lemma 4.6 its support is a subset of the support of $h_c^{m_c(\lambda)}$. Hence if $h \neq h_c^{m_c(\lambda)}$, then, for some (x_i, y_j) one has $h(x_i, y_j) = -\infty$ and $h_c^{m_c(\lambda)}(x_i, y_j) = \lambda$. But, $h_c^{m_c(\lambda)}$ being a reduced plan (by assumption), either the i-th row of the matrix $[h_c^{m_c(\lambda)}(x_k, y_l)]_{k,l}$, or its j-th column, contain only $-\infty$, except at (i, j) where λ is. Therefore, the matrix $[h(x_k, y_l)]_{k,l}$ has either all the i-th column or all the j-th row full of $-\infty$, contradicting the fact that h is a plan for R_λ, hence proving the assertion.

To prove the second assertion, let h be the unique minimizing plan for $d_c(R_\lambda)$ among minimizing plans containing only $-\infty$ and λ. It has to be reduced by Lemma 3.2. On the other hand, also $h_c^{m_c(\lambda)}$ contains only $-\infty$ and λ and is a minimizing plan for $d_c(R_\lambda)$, by Lemma 4.6. Thus $h = h_c^{m_c(\lambda)}$ as claimed. □

We remark that the latter Proposition 4.8 asserts that having a unique plan (among all plans containing only $-\infty$ and λ) is equivalent to $h_c^{m_c(\lambda)}$ being reduced, but this is *not* equivalent to the existence of a unique reduced minimizing plan as the following example shows.

Example 4.9. Suppose $\lambda = 0$.

(1) If the cost function is

$$[c(x_i, y_j)]_{i,j=1}^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},$$

then $h_c^{m_c(\lambda)}$ is not reduced; there are two minimizing plans (containing only 0 and $-\infty$), with one of them the only reduced minimizing plan:

$$h_c^{m_c(\lambda)} = \begin{pmatrix} 0 & 0 \\ -\infty & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}.$$

(2) If the cost function is

$$[c(x_i, y_j)]_{i,j=1}^3 = \begin{pmatrix} 1 & 4 & 2 \\ 6 & 7 & 8 \\ 5 & 9 & 3 \end{pmatrix},$$

then $h_c^{m_c(\lambda)}$ is not reduced, and there are at least two reduced minimizing plans:

$$h_c^{m_c(\lambda)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\infty & -\infty \\ 0 & -\infty & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} -\infty & 0 & 0 \\ 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \end{pmatrix},$$

$$h_2 = \begin{pmatrix} -\infty & 0 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}.$$

4.4. A remark on perfect matchings. Of particular interest, as in the classical mass transportation problem, are minimizing plans supported on subsets of the type $\{(x_1, y_{\sigma(1)}), \ldots, (x_n, y_{\sigma(n)})\}$, where $\sigma\colon \{1.\ldots,n\} \to \{1.\ldots,n\}$. We will call them *perfect matching* plans. The plan h_1 in Example 4.9(1) and the plan h_3 in Example 4.9(2) are perfect matchings, while the other plans in these examples are not. The example below shows that for some data one might have no perfect matching minimizing plans.

Example 4.10. Consider the cost matrix

$$[c(x_i, y_j)]_{i,j=1}^3 = \begin{pmatrix} 5 & 1 & 5 \\ 5 & 2 & 5 \\ 3 & 5 & 4 \end{pmatrix}.$$

If $k_3 = k_2 = k_1 = l_3 = l_2 = l_1 = 0$, then

$$h = \begin{pmatrix} -\infty & 0 & -\infty \\ -\infty & 0 & -\infty \\ 0 & -\infty & 0 \end{pmatrix}$$

is the unique minimizing plan (among plans containing only 0 and $-\infty$), but is not a perfect matching.

We stress that the nonexistence of the optimal tropical plans even when the max-plus probability measures μ and ν have all the weights equal to zero (as we said earlier, we call this case *fundamental*) is in striking contrast with the classical optimal mass transportation. The latter always admits an optimal transport plan corresponding to a perfect matching (i. e. a permutation matrix) between discrete measures which are sums of Dirac masses with equal weights, by virtue of the Birkhoff-von Neumann theorem which states that the set of extreme points of the Birkhoff polytope of bistochastic matrices in $\mathbb{R}^{n^2}$ is exactly the set of permutation matrices (and hence a linear functional on this polytope always attains its minimum on a permutation matrix).

The following assertion holds true.

Proposition 4.11. *Let* $\mu = \max_{j=1}^n (k_j + \delta_{x_j})$, $\nu = \max_{j=1}^n (l_j + \delta_{y_j})$, *with the elements arranged as in* (12) *as usual. If there is* $j \in \{1, \ldots, n\}$ *such that* $k_j \neq l_j$, *then there can be no plan that would correspond to a perfect matching.*

Proof. If $h \in \Pi(\mu, \nu)$ is not reduced, then it does not correspond to a perfect matching, so assume that $h \in \Pi_R(\mu, \nu)$. Recall the definition 13 and consider the disjoint regions R_{λ_k}, $k = 1, \dots, r$ determined by the plan h, where λ_k, $k = 1, \dots, r$ are all the distinct weights of the max-plus probability measures μ and ν. Suppose that the set $\{i\colon k_i = \lambda_k\}$ has $m_{k,1}$ elements, and the set $\{j\colon\ l_j = \lambda_k\}$ has $m_{k,2}$ elements; at least one of these two numbers must be positive. Observe that the plan h must have at least $\max\{m_{k,1}, m_{k,2}\}$ finite (i.e. different from $-\infty$) entries on the region R_{λ_k}. Thus, the plan h has at least

$$m = \max\{m_{1,1}, m_{1,2}\} + \cdots + \max\{m_{r,1}, m_{r,2}\}$$

finite entries in total. Keep in mind that

$$\sum_{k=1}^{r} m_{k,1} = \sum_{k=1}^{r} m_{k,2} = n.$$

The plan will correspond to a perfect matching only if there are n finite entries in total. The only way to have $m = n$ is if $m_{k,1} = m_{k,2}$ for every $k = 1, \dots, r$. Given that the weights are arranged as in (12) as usual, the conclusion follows. ☐

§5. Uniqueness of solution and perfect matchings for random costs

In this section, we will try to elucidate some questions regarding the optimal cost, perfect matchings and uniqueness when we introduce some randomness in the cost function. We will limit ourselves to the fundamental case (i.e. when all the weights of the discrete max-plus probability measures are zero) and with $m = n$, i. e.:

$$\mu_{\mathbf{0}}^n = \max\{0 + \delta_{x_1}, \dots 0 + \delta_{x_m}\},$$
$$\nu_{\mathbf{0}}^n = \max\{0 + \delta_{y_1}, \dots, 0 + \delta_{y_n}\}.$$

with x_j, $j = 1, \dots, n$ as well as y_i, $i = 1, \dots, m$ all distinct. In what follows the sequences of max-plus probability measures $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$ as above are fixed, while the cost function is random, i. e. is represented by a Bernoulli random matrix, i. e. each entry in the $n \times n$ cost matrix is independent from the others and takes the value β_1 with probability p and β_2 with probability $q = 1 - p$, where $\beta_1 < \beta_2$.

5.1. Optimal tropical cost for random cost matrices. The following statement holds true.

Theorem 5.1. *Let β_1, β_2 be nonnegative numbers, with $\beta_1 < \beta_2$, and suppose that for each n, $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$ are discrete max-plus probability measures with all their weights equal to zero, and c^n is a Bernoulli cost matrix: $\mathbb{P}(c^n(x_i, y_j) = \beta_1) = p$, $\mathbb{P}(c^n(x_i, y_j) = \beta_2) = q = 1 - p$ for $i, j \in \{1, \ldots, n\}$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the points of the support of $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$. If $q < 1$, then*

$$\mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = \beta_1) \to 1 \quad \text{as} \quad n \to \infty.$$

Proof. Even though a very short argument can be provided, we will derive a formula for the probability under question. Referring to Lemma 4.6 (and recall definition 4.4) the optimal tropical cost d_{c^n} between $\mu_{\mathbf{0}}^n = \max_{i=1}^n (0 + \delta_{x_i})$ and $\nu_{\mathbf{0}}^n = \max_{j=1}^n (0 + \delta_{y_j})$ will be β_1 or β_2 depending on whether $m_{c^n}(0)$ is 1 or 2 respectively. It is 1 if and only if in the matrix for c^n there is at least one β_1 in every row *and* in every column. Denote by F_i the event that there is at least one β_1 in the i-th row of the matrix, and by C_j the event that there is at least one β_1 in the j-th column of the matrix. In the calculation that follows we retain, for the sake of clarity, the notation m for the number of rows and n for the number of columns in the cost matrix, although one really has $m = n$. Therefore for the indices i and j one has $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. Thus

$$\begin{aligned} \mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = \beta_1) &= \mathbb{P}((\cap_{i=1}^m F_i) \cap (\cap_{j=1} C_j)) \\ &= 1 - \mathbb{P}((\cup_{i=1}^m F_i^c) \cup (\cup_{j=1} C_j^c)), \end{aligned}$$

where the upper index c denotes the complement of the event. We have

$$\begin{aligned} &\mathbb{P}((\cup_{i=1}^m F_i^c) \cup (\cup_{j=1}^n C_j^c)) \\ &= \sum_{s=1}^{m+n} (-1)^{s+1} \sum_{\substack{a+b=s \\ (a,b)\neq(0,0)}} \binom{m}{a}\binom{n}{b} \mathbb{P}(F_1^c \cap \cdots \cap F_a^c \cap C_1^c \cap \cdots \cap C_b^c) \\ &= \sum_{s=1}^{m+n} (-1)^{s+1} \sum_{a+b=s} \binom{m}{a}\binom{n}{b} q^{mn-(m-a)(m-b)} \\ &= -q^{mn} \sum_{\substack{0\leqslant a\leqslant m \\ 0\leqslant b\leqslant n \\ (a,b)\neq(0,0)}} (-1)^{a+b} \binom{m}{a}\binom{n}{b} q^{-(m-a)(m-b)}. \end{aligned}$$

Assuming that $p < 1$ (otherwise $\mathbb{P}(d_c(\mu_{\mathbf{0}}, \nu_{\mathbf{0}}) = \beta_1) = 1$ for any n so that there is nothing to prove). Then

$$\begin{aligned}
&\mathbb{P}((\cup_{i=1}^m F_i^c) \cup (\cup_{j=1}^n C_j^c)) \\
&= -q^{mn}\Bigg(\sum_{\substack{0\leqslant a\leqslant m\\0\leqslant b\leqslant n}} (-1)^{a+b}\binom{m}{a}\binom{n}{b} q^{-(m-a)(m-b)} - q^{-mn}\Bigg) \\
&= -q^{mn}(-1)^n \sum_{a=0}^{m}\binom{m}{a}(-1)^a \sum_{b=0}^{n}(-1)^{n-b}\binom{n}{b}(q^{-(m-a)})^{n-b} + 1 \\
&= -q^{mn}(-1)^{m+n}\sum_{a=0}^{b}\binom{m}{a}(-1)^{m-a}(1-q^{-(m-a)})^n + 1.
\end{aligned}$$

Recalling that $m = n$, we get

$$\mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = \beta_1) = q^{n^2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(1-q^{-j})^n. \tag{16}$$

Thus,

$$\mathbb{P}(d_c^n(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = \beta_1) \to 1 \quad \text{as} \quad n \to \infty,$$

if $q < 1$, proving the claim. ☐

For the following remark, let us introduce a special notation for the expression in the right hand side of (16), namely, set

$$\mathfrak{s}(n;p) := \begin{cases} (1-p)^{n^2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(1-(1-p)^{-j})^n. & \text{if } p \in [0,1), \\ 1, & \text{if } p = 1. \end{cases}$$

Remark 5.2. The relationship (16) reads

$$\lim_n \mathfrak{s}(n;p) = 1, \quad 0 < p \leqslant 1.$$

It is also easy to show that

$$\lim_{p\to 0}\mathfrak{s}(n;p) = 0, \quad \lim_{p\to 1}\mathfrak{s}(n;p) = 1, \qquad n \in \mathbb{N},$$

so that $p \mapsto \mathfrak{s}(n;p)$ is continuous over $[0,1]$. The asymptotics of $\mathfrak{s}$, hence that of the probability of the optimal tropical cost equaling the minimum value of the cost function, may be interesting also for the more general cases when p is not constant but depends on n. For instance, one has $\lim_{n\to\infty}\mathfrak{s}(n, 1/n^\gamma) = 0$ for all $\gamma \geqslant 1$ and $\lim_{n\to\infty}\mathfrak{s}\big(n, 1/n^{1/2}\big) = 1$.

Remark 5.3. A quite similar situation occurs not only when the cost is given not by a Bernoulli random matrix, but, say, by a binomial one. Namely, suppose now that $s \in \mathbb{N}$ is fixed, and each entry in the cost matrix c^n can take one of the values $\beta_1 < \cdots < \beta_s$ (as in (14)), with β_1 appearing with probability p_1. Let $q := 1 - p_1$. Then the lower bound for $\mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = \beta_1)$ can be obtained in the same way as in the proof of the Theorem 5.1. Therefore

$$\lim_{n\to\infty} \mathbb{P}(d_c(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = \beta_1) = 1.$$

Thus, even if the available choices for the entries of the cost matrix for c^n is a large but fixed number, the optimal tropical cost between $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$ is equal to the the smallest value β_1 of the cost with large probability for large n (with probability of this event tending to one as $n \to \infty$). Moreover, if p_j is the probability of β_j appearing in any given entry of the cost matrix, then it follows from the calculation above that

$$\mathbb{P}\left(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = \beta_j\right) = \mathfrak{s}\left(n, \sum_{p=1}^{j} p_k\right) - \mathfrak{s}\left(n, \sum_{p=1}^{j-1} p_k\right), \tag{17}$$

which tends to zero as $n \to \infty$, the above equality (17) giving the rate of convergence.

5.2. Presence of perfect matching optimal plans. We consider the following definition.

Definition 5.4. Let μ and ν be discrete max-plus probability measures and let h be a plan for a *square* region R_λ. We will say that h *contains a perfect matching*, if there is a perfect matching plan $\tilde{h}$ for the same region with support contained in the support of h.

In other words, h is a perfect matching plan for a region R_λ if it can be "simplified" by substituting some of its λ entries by $-\infty$ to get a perfect matching plan for a R_λ. We will again discuss the case of a random cost provided by a Bernoulli cost matrix, and restrict ourselves to the fundamental case. To simplify the discussion, let $\beta_1 = 0$ and $\beta_2 = 1$. If there is a zero in every row and every column of the matrix, then, as we know, the optimal tropical cost is 0, but if we look at the corresponding plan (represented by the matrix h), it may be impossible to "simplify" it (change some of the entries equal to 0 to $-\infty$) so as to produce a perfect matching plan (see Example 4.10), that is, it does not contain a perfect matching. In the opposite direction, if the corresponding optimal plan contains a perfect matching, then the optimal tropical cost is 0. Summing up, there are the following possibilities.

- The optimal tropical cost is 1. This occurs exactly when some row or column of the cost matrix fails to have a 0. Then there is always a perfect matching plan. In fact, the absence of a 0 in some row or column of the cost matrix means that $h_c^{m_c(0)}$ is the matrix with 1 in all the entries, which contains any perfect matching plan. For instance, if the cost matrix is
$$[c_{i,j}]_{i,j=1}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$
then a possible perfect matching minimizing plan is
$$[h_{i,j}]_{i,j=1}^2 = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}.$$
- The optimal tropical cost is 0, but the optimal plan does not contain a perfect matching.
- The optimal tropical cost is 0, and the optimal plan contains a perfect matching.

For the following theorem we give here a random graph argument based on the strong and remarkable result of Bollobás and Thomason (see [2, Theorem 7.11]) that will also be used in the proof of Theorem 5.7 below.

Theorem 5.5. *Let $\beta_1 < \beta_2$, and suppose that for each natural number n, $\mu_{\boldsymbol{0}}^n$ and $\nu_{\boldsymbol{0}}^n$ are discrete max-plus probability measures with all their weights equal to zero, and c^n is a Bernoulli cost matrix: $\mathbb{P}(c^n(x_i, y_j) = \beta_1) = p_n$, $\mathbb{P}(c^n(x_i, y_j) = \beta_2) = q_n = 1 - p_n$ for $i, j \in \{1, \dots, n\}$, where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are the points of the support of $\mu_{\boldsymbol{0}}^n$ and $\nu_{\boldsymbol{0}}^n$. If $p_n \geqslant (\log n)/n$ for all but finitely many n, then*
$$\lim_{n\to\infty} \mathbb{P}(\exists h \in \Pi^{c^n}(\mu_{\boldsymbol{0}}^n, \nu_{\boldsymbol{0}}^n) : h \textit{ contains a perfect matching}) = 1.$$

Proof. We associate c^n with one and only one random bipartite (undirected) graph, denoted by $G_n(c^n)$, with the sets $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$ as the two disjoint sets of vertices in the following way: $c^n(x_i, y_j) = \beta_1$ if $x_i y_j$ is an edge, and $c^n(x_i, y_j) = \beta_2$ otherwise. The plan $h_{c^n}^{m_{c^n}(0)}$ (recall the definitions of Section 4.2) contains a perfect matching plan if and only if the bipartite graph $G_n(c^n)$ contains a perfect matching. In the proof of [2, Theorem 7.11], it is shown that the probability that the random bipartite graph contains a perfect matching approaches 1 as $n \to \infty$. Thus, the probability that $h_{c^n}^{m_{c^n}(0)}$ contains a perfect matching also approaches 1 as $n \to \infty$. Since $h_{c^n}^{m_{c^n}(0)}$ is always an optimal plan, the result follows. □

Remark 5.6. An alternative proof of Theorem 5.5 can be offered as follows. Regardless of whether the optimal tropical cost is β_1 or β_2, for the plan $h_c^{m_c(0)}$ (which is always minimizing), the property of containing a perfect matching plan is characterized by the fact that, for some permutation $\sigma \in S_n$, the product

$$\Pi_{j=1}^{n} |\beta_2 - c(x_j, y_{\sigma(j)})|$$

is different from zero (necessarily then it is equal to $(\beta_2 - \beta_1)^n$). The latter is guaranteed, for instance, when the matrix $[\beta_2 - c(x_i, y_j)]_{i,j=1}^{n}$ is not singular (i.e. has nonzero determinant). By a theorem of Basak and Rudelson [1], this probability approaches 1, for every $0 < p < 1$.

5.3. Uniqueness of minimizing plans. We show now that in the fundamental case (when all the weights of the discrete max-plus masure are zero), when the uniform probability is put on the space of the cost matrices, the uniqueness of a minimizing plan containing only 0 and $-\infty$ is an asymptotically rare event in the sense that its probability tends to zero as the number of weights approaches infinity. Namely, the following result is valid.

Theorem 5.7. *Fix any positive real number $M > 0$ and let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be sequences of subsets of X and Y respectively, with $\#X_n = \#Y_n = n$ for all n. For each $n \in \mathbb{N}$, let*

$$\mu_{\mathbf{0}}^n := \max\{0 + \delta_{x_1}, \dots, 0 + \delta_{x_n}\}, \quad \nu_{\mathbf{0}}^n := \max\{0 + \delta_{y_1}, \dots, 0 + \delta_{y_n}\},$$

where $x_1, \dots, x_n$ are the elements of X_n and $y_1, \dots, y_n$ those of Y_n. For each n let P_n be the uniform probability measure over $[0, M]^{X_n \times Y_n}$. Define $C_n \subset [0, M]^{X_n \times Y_n}$ as the set of functions c such that there is a unique, among plans containing only 0 and $-\infty$, minimizing plan for $d_c(\mu_n, \nu_n)$. Then

$$\lim_{n \to \infty} P_n(C_n) = 0.$$

Proof. In order to apply the theory from [2], let us introduce the notion of *bipartite graph process*, specifically, on the set of vertices $X_n \cup Y_n$. Any given bijective function $f\colon \{1, \dots, n^2\} \to \{1, \dots, n\}^2$ we define determines a sequence of $n^2 + 1$ graphs in the following way: at time step $t = 0$ there are no edges and at step $t \in \{1, \dots, n^2\}$ the edge $(i, j) := f(t)$ is added. At the n^2-th time step we obtain the complete bipartite graph. Note that the set of bijective functions $f\colon \{1,\ldots,n^2\} \to \{1,\ldots,n\}^2$ is in one-to-one correspondence with the set of permutations of $\{1,\ldots,n^2\}$, i. e. with the symmetric group S_{n^2} of order n^2; in fact, each f^{-1} is an enumeration of the cells of an $n \times n$ matrix. If the function f (or equivalently the respective permutation $\sigma \in S_{n^2}$) is chosen randomly, with uniform probability, then we have a *random bipartite graph process*, which coincides with the one described in [2] (see pp. 42 and 171 therein). Let

$$\Omega_n := \{\omega : X_n \times Y_n \to [0, M] \ : \ \omega \text{ takes } n^2 \text{ distinct values } \},$$

and for each $\omega \in \Omega_n$ define the mapping $f_\omega\colon \{1,\ldots,n^2\} \to \{1,\ldots,n\}^2$ by setting $f_\omega(t) := (i, j)$, where (i, j) is the unique pair of indices such that $\omega(x_i, y_j)$ is the t-th largest value among the n^2 distinct values $\omega(x_1, y_1), \ldots,$ $\omega(x_n, y_n)$. Thus, each $\omega \in \Omega_n$ determines an ordering of the matrix cells f_ω which, in turn, gives the above described graph process with $f := f_\omega$.

Since P_n is the uniform measure on $[0, M]^{X_n \times Y_n}$, we have $P_n(\Omega_n) = 1$. Moreover, since P_n is uniform, for each bijective $g\colon \{1, ..., n^2\} \to \{1, ..., n\}^2$, the set $\{\omega \in \Omega_n \colon f_\omega = g\}$ has the same P_n-measure, namely, $1/(n^2)!$. Hence, these sets form a partition of the probability space

$$([0, M]^{X_n \times Y_n}, \mathcal{B}([0, M]^{X_n \times Y_n}), P_n)$$

into $(n^2)!$ equiprobable events, where $\mathcal{B}([0, M]^{X_n \times Y_n})$ stands for the Borel σ-algebra of $[0, M]^{X_n \times Y_n}$. Thus, the bipartite random graph process can be equivalently sampled from this probability space, rather than directly from the set of bijective $g\colon \{1, ..., n^2\} \to \{1, ..., n\}^2$ (or equivalently, from S_{n^2}) endowed with the uniform probability. Let us denote by $\{G_t\}_{t=0}^{n^2}$ a generic realization of our bipartite random graph process on $X_n \cup Y_n$, and let τ be the stopping time $\tau := \min\{t \ : \ G_t \text{ has degree } 1\}$. That is, τ is the first instance t such that every x_i belongs to an edge and also every y_j belongs to an edge. Recalling now Definition 4.4 and the algorithm of Section 4.2, we have:

$$\tau(\omega) = m_\omega(0) \tag{18}$$

for P_n-a.e. $\omega \in \Omega_n$. Denote by D_n the event that G_τ contains a perfect matching. By [2, theorem. 7.11],

$$\lim_{n\to\infty} \mathbb{P}_n(D_n) = 1, \tag{19}$$

which means, in words, that by the time the bipartite graph achieves degree 1 (this is exactly the time when the minimizing plan $h_\omega^{m_\omega(0)}$ is formed, by (18)), the graph contains a perfect matching. Let

$$H_n := \{\omega \in \Omega_n \ : \ h_\omega^{m_\omega(0)} \text{ is not reduced}\}.$$

By Proposition 4.8, we will be done if we show that $P_n(H_n) \to 1$ as $n \to \infty$. Now, the event D_n is the disjoint union of F_n and E_n, where F_n is the event that G_τ is *exactly* a perfect matching, and E_n is the event that G_τ has a perfect matching and at least one more edge. As can easily be argued, $P_n(F_n) \to 0$ as $n \to \infty$ (in fact, for F_n to hold, at the last step of forming G_τ only one possibility of forming an edge, or equivalently only one way of placing a zero in the respective row of the matrix, results in a perfect matching). Thus, by (19), $P_n(E_n) \to 1$ as $n \to \infty$. On the other hand, the event E_n is included in H_n: indeed, a graph in E_n corresponds to a plan in the support of which there is triple of indices, two of which are in the same column and two of which are in the same row, thereby violating Definition 3.1. Therefore, $\lim_{n\to\infty} P_n(H_n) = 1$ hence concluding the proof. □

References

1. A. Basak and M. Rudelson, *Invertibility of sparse non-Hermitian matrices.* — Adv. Math., **310** (2017), 426–483.
2. B. Bollobás, *Random Graphs*, edition 2, Cambridge University Press (2010).
3. R. S. Garfinkel, K. C.Gilbert, *The bottleneck traveling salesman problem: algorithms and probabilistic analysis.* — J. Assoc. Comput. Mach. **25**, No. 5 (1978), 435–448.
4. P. C. Gilmore, R. E. Gomory, *Sequencing a one state-variable machine: A solvable case of the traveling salesman problem.* — Operations Res. **12**, (1964), 655–679.
5. V. Kolokoltsov, V. P. Maslov, *Idempotent analysis and its applications*, Mathematics and Its Applications, Springer Netherlands, (1997).
6. C. Villani, *Optimal transport: old and new*, Grundlehren der Mathematischen Wissenschaften, **338**, Springer-Verlag, Berlin (2008).

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