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BOUNDEDNESS OF THE WEAK SOLUTIONS TO CONORMAL PROBLEMS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH MORREY DATA

Abstract. We consider a conormal problem for a class of quasilinear divergence form elliptic equations modeled on the m-Laplacian. The nonlinearities support controlled growths in the solution and its gradient, while their behaviour with respect to the independent variable is restrained in terms of Morrey spaces.

We show global essential boundedness for the weak solutions, generalizing this way the classical L^p-result of Ladyzhenskaya and Ural'tseva to the settings of the Morrey spaces.

To Nina N. Ural'tseva with profound respect and all the best wishes on the occasion of her 90th anniversary

§1. Introduction

Our goal in the present paper is to derive essential boundedness of the weak solutions u belonging to the Sobolev space $W^{1,m}(\Omega)$ with $m \in (1,n]$, of the conormal derivative problem for second-order, divergence form elliptic equations

$$\begin{cases} \operatorname{div} \mathbf{a}(x,u,Du) = b(x,u,Du) & x \in \Omega, \\ \mathbf{a}(x,u,Du)\cdot \boldsymbol{\nu}(x) = \psi(x,u) & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^n$, $n \geqslant 2$, is a bounded domain with Lipschitz continuous boundary, and $\boldsymbol{\nu}(x) = \big(\nu_1(x),\dots,\nu_n(x)\big)$ stands for the unit outward normal to $\partial\Omega$.

The nonlinear terms $\mathbf{a}(x,z,\xi) = \big(a_1(x,z,\xi),\dots,a_n(x,z,\xi)\big)$, $b(x,z,\xi)$ and $\psi(x,z)$ are Carathéodory functions, i.e., these are measurable in $x \in \Omega$ ($x \in \partial\Omega$ in the case of ψ) for all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and continuous in (z,ξ) for almost all $x \in \Omega$.

Key words and phrases: nonlinear elliptic equations, divergence form, weak solution, conormal problem, coercivity, controlled growths, boundedness, Morrey spaces.

The divergence form nonlinear operator in (1.1) is modeled on the m-Laplacian with arbitrary $m \in (1,n]$, that means we suppose hereafter the validity of the following *coercivity condition* of order m: There exist constants $\gamma, \Lambda > 0$ such that

$$\mathbf{a}(x,z,\xi)\cdot\xi \geqslant \gamma|\xi|^m - \Lambda|z|^{m^*} - \Lambda\varphi_1(x)^{\frac{m}{m-1}} \tag{1.2}$$

for a.a. $x \in \Omega$, all $(z,\xi) \in \mathbb{R}\times\mathbb{R}^n$, $\varphi_1 \in L^{\frac{m}{m-1}}(\Omega)$, and where m^* stands for the Sobolev conjugate of m,

$$m^* = \begin{cases} \frac{nm}{n-m} & \text{if } m < n, \\ \text{any exponent} > n & \text{if } n = m. \end{cases}$$

Apart from the *coercivity* condition (1.2), we suppose that the nonlinearities in (1.1) support *controlled growths* with respect to the unknown function and its gradient. Namely, we assume

$$|\mathbf{a}(x,z,\xi)| \leqslant \Lambda\Big(\varphi_1(x) + |z|^{\frac{m^*(m-1)}{m}} + |\xi|^{m-1}\Big) \tag{1.3}$$

with φ_1 as in (1.2);

$$|b(x,z,\xi)| \leqslant \Lambda\Big(\varphi_2(x) + |z|^{m^*-1} + |\xi|^{\frac{m(m^*-1)}{m^*}}\Big) \tag{1.4}$$

with $\varphi_2 \in L^{\frac{mn}{mn+m-n}}(\Omega)$ and

$$|\psi(x,z)| \leqslant \psi_1(x) + \psi_2(x)|z|^\beta, \tag{1.5}$$

where $\beta \in \Big[0, \frac{n(m-1)}{n-m}\Big)$, $\psi_1 \in L^{\frac{m(n-1)}{n(m-1)}+\varkappa}(\partial\Omega)$, $\psi_2 \in L^{\frac{m(n-1)}{n(m-1)-\beta(n-m)}+\varkappa}(\partial\Omega)$ with a $\varkappa > 0$ and $\frac{n(m-1)}{n-m}$ to be intended as $+\infty$ when $m = n$.

As usual, fixed a real number $m \in (1,n]$, a function $u \in W^{1,m}(\Omega)$ is called a weak solution to (1.1) when it satisfies the standard integral identity

$$\begin{aligned}\int_\Omega \mathbf{a}\big(x,u(x),Du(x)\big)\cdot Dv(x)\,dx + \int_\Omega b\big(x,u(x),Du(x)\big)v(x)\,dx \\ = \int_{\partial\Omega} \psi\big(x,u\big)v(x)\,d\Gamma_x\end{aligned} \tag{1.6}$$

for each $v \in W^{1,m}(\Omega)$. Indeed, the convergence of all the integrals above is ensured by the controlled growths assumptions (1.3), (1.4) and (1.5) and the required integrabilities of the functions governing the x-behaviours therein. It is worth to note that the controlled growths conditions are the minimal ones that guarantee the concept of weak solution makes sense.

In what follows, only for the sake to avoid unessential technicalities in managing the surface integral in (1.6), we will suppose additionally that

$$\psi(x,z)z \leqslant 0 \qquad \text{for a.a. } x \in \partial\Omega, \quad z \in \mathbb{R}. \tag{1.7}$$

Boundedness of the classical solutions ($u \in C^1(\overline{\Omega} \cap C^2(\Omega))$ to quasilinear conormal problems has been studied by Lieberman in [8, 9] (see also [10]) in the *sub-controlled* case when $\mathbf{a}(x,z,\xi)$ and $b(x,z,\xi)$ grow as $|z|^{m-1}$ and $|\xi|^{m-1}$. In the same situation, Winkert proved in [14] boundedness also of the weak solutions to (1.1). Regarding the case of *controlled growths* of the nonlinearities, we dispose of the classical result of Ladyzhenskaya and Ural'tseva [7, Chapter X, §2] where boundedness and Hölder continuity of the weak solutions have been proved when the behaviour in x of nonlinearities in (1.1) is controlled in terms of suitable Lebesgue spaces (see also [6] in the particular case when $m = 2$). We have to note also the deep papers of Arkhipova [2, 3], where reverse Hölder inequalities have been derived for solutions of quasilinear conormal problems. As consequence, improving of gradient integrability follows as well as various regularity results.

Since the controlled growths are *sharp* for what concerns boundedness of the weak solutions (see the counterexamples in [7, Chapter I, §2]), our aim here is to weaken the hypotheses on the functions φ_1 and φ_2 that govern the behaviour of the nonlinear terms of (1.1) with respect to the independent variable x. Precisely, we will suppose these belong to Morrey spaces with suitable exponents. For readers convenience, recall that the Morrey space $L^{s,\theta}(\Omega)$, $s \in [1,\infty)$ and $\theta \in [0,n]$, is the collection of all functions $v \in L^s(\Omega)$ such that

$$\|v\|_{L^{s,\theta}(\Omega)} := \sup_{x_0\in\Omega,\ \rho>0} \left(\rho^{-\theta} \int\limits_{B_\rho(x_0)\cap\Omega} |v(x)|^s\,dx \right)^{1/s} < \infty.$$

The last quantity defines a norm, under which $L^{s,\theta}(\Omega)$ becomes a Banach space, and the limit cases $\theta = 0$ and $\theta = n$ give rise, respectively, to $L^s(\Omega)$ and $L^\infty(\Omega)$.

Turning back to our problem (1.1), regarding the behaviour of the nonlinear terms with respect to x, we suppose

$$\begin{cases} \varphi_1 \in L^{p,\lambda}(\Omega), \ \ p > \dfrac{m}{m-1}, & \lambda \in (0,n), \ \ (m-1)p+\lambda > n; \\ \varphi_2 \in L^{q,\mu}(\Omega), \ \ q > \dfrac{mn}{mn-n+m}, & \mu \in (0,n), \ \ mq+\mu > n. \end{cases} \tag{1.8}$$

Let us recall that $\varphi_1 \in L^{\frac{m}{m-1}}(\Omega)$ and $\varphi_2 \in L^{\frac{mn}{mn+m-n}}(\Omega)$ are necessary conditions, ensuring convergence of the integrals involved in (1.6). We need here the slightly stronger hypotheses $p > \frac{m}{m-1}$ and $q > \frac{mn}{mn-n+m}$ in order to get better integrability of the gradient through the results of Arkhipova [2, 3].

Under the above assumptions, our main result Theorem 3.1 asserts that each $W^{1,m}(\Omega)$-weak solution of the conormal problem (1.1) is essentially bounded in terms of known quantities, of $\|Du\|_{L^m(\Omega)}$ and of the uniform integrability of $|Du|^m$ in Ω. Similar result has been proved in [4] for the weak solutions to quasilinear Dirichlet problems. It is worth noting also the deep paper of Nazarov and Ural'tseva [12] where local properties as strong maximum principle, Harnack inequality and Hölder continuity have been obtained for weak solutions to linear divergence form equations with Morrey lower-order coefficients.

Let us note that, taking $\lambda = \mu = 0$ in (1.8), we recover the boundedness result of Ladyzhenskaya and Ural'tseva ([7, Chapter X, §2]). On the other hand, the restrictions $(m-1)p > n$ and $mq > n$ are sharp when working in the framework of the Lebesgue spaces as known by the counterexamples in [7, Chapter I, §2]. Our boundedness result show that, taking φ_1 and φ_2 in Morrey spaces, the values of p and q could be even decreased at the expense of increase λ and μ, still maintaining the restrictions $(m-1)p+\lambda > n$ and $mq + \mu > n$, respectively.

The technique used is the proof of Theorem 3.1 is that of [4], and it relies on the De Giorgi approach to the boundedness as adapted by Ladyzhenskaya and Ural'tseva (cf. [7, Chapter IV]) to quasilinear equations. Namely, using the controlled growth assumptions, we derive decay estimates for the total mass of the weak solution taken over its level sets. However, unlike the L^p-approach of Ladyzhenskaya and Ural'tseva, the mass we have to do with is taken with respect to a positive Radon measure $\mathfrak{m}$ which depends not only on the Lebesgue measure, but also on $\varphi_1^{\frac{m}{m-1}}$, φ_2 and a suitable power of the solution itself. Thanks to the Morrey hypotheses (1.8), the measure $\mathfrak{m}$ allows to apply a precise inequality of trace type due to D. R. Adams [1], and this leads to a bound of the $\mathfrak{m}$-mass of u in terms of the m-energy of u. At this point we combine the controlled growths with the better-gradient-integrability results of Arkhipova ([2, 3]) in order to estimate the m-energy of u in terms of a small multiplier of the same quantity plus a suitable power of the $\mathfrak{m}$-measure of the solution level set. The global boundedness of the weak solution then follows by a classical result known as *Hartman–Stampacchia maximum principle.*

Throughout the paper the phrase "*known quantities*" means that a given constant depends on the data in the above hypotheses, which include n, m, γ, Λ, p, q, λ, μ, β, $\varkappa$, $\|\varphi_1\|_{L^{p,\lambda}(\Omega)}$, $\|\varphi_2\|_{L^{q,\mu}(\Omega)}$, $\|\psi_1\|_{L^{\frac{m(n-1)}{n(m-1)}+\varkappa}(\partial\Omega)}$, $\|\psi_2\|_{L^{\frac{m(n-1)}{n(m-1)-\beta(n-m)}+\varkappa}(\partial\Omega)}$, $\operatorname{diam}\Omega$ and the Lipschitz regularity of $\partial\Omega$. We will denote by C a generic constant, depending on known quantities, which may vary within the same formula.

§2. Auxiliary results

We list here some auxiliary results to be used in the proof of our main result.

Lemma 2.1. (Embeddings between Morrey spaces, see [13]) *For arbitrary* $s', s'' \in [1,\infty)$ *and* $\theta', \theta'' \in [0,n)$, *one has*

$$L^{s',\theta'}(\Omega) \subseteq L^{s'',\theta''}(\Omega)$$

if and only if

$$s' \geqslant s'' \geqslant 1 \quad \textit{and} \quad \frac{s'}{n-\theta'} \geqslant \frac{s''}{n-\theta''}.$$

In what follows, we will use a fine integral inequality, known as *Adams trace inequality* (see [1]), which regards functions in $W_0^{1,r}(\Omega)$. Precisely, given a positive Radon measure $\mathfrak{m}$, supported in Ω, assume that

$$\mathfrak{m}(B_\rho(x)) \leqslant K\rho^{\alpha_0} \qquad \forall\, x \in \mathbb{R}^n, \quad \forall \rho > 0, \tag{2.1}$$

where $B_\rho(x)$ is the ball centered at x and of radius ρ, K is an absolute constant, and

$$\alpha_0 = \frac{s}{r}(n-r), \quad 1 < r < s < \infty, \quad r < n. \tag{2.2}$$

Then

$$\left(\int_\Omega |v(x)|^s\, d\mathfrak{m}\right)^{1/s} \leqslant C(n,s,r)K^{1/s}\left(\int_\Omega |Dv(x)|^r\, dx\right)^{1/r} \quad \forall v\in W_0^{1,r}(\Omega).$$

We will need a variant of the this inequality, valid for functions $v\in W^{1,r}(\Omega)$ with non necessarily zero boundary trace, where Ω is a bounded and Lipschitz domain. To get the desired result, it suffices to extend v to $V\in W^{1,r}(\mathbb{R}^n)$ in a way that $\|V\|_{W^{1,r}(\mathbb{R}^n)}\leqslant C(\partial\Omega)\|v\|_{W^{1,r}(\Omega)}$ and then multiply by a suitable cut-off function ζ such that $\zeta\equiv 1$ over Ω. Assuming that the measure $\mathfrak{m}$ is extended as zero outside Ω, application of the original Adams trace inequality to ζV yields

$$\left(\int_\Omega |v(x)|^s\, d\mathfrak{m}\right)^{1/s} \leqslant C(n,s,r,K,\partial\Omega)\left(\int_\Omega \big(|Dv(x)|^r+|v(x)|^r\big)\, dx\right)^{1/r}$$

for all $v\in W^{1,r}(\Omega)$. At this point, interpolating $\int_\Omega |v(x)|^r\,dx$ on the right-hand side by means of the Gagliardo–Nirenberg multiplicative inequality (e.g. [11, Theorem 1.4.8/1]) implies

Lemma 2.2. *Let $\mathfrak{m}$ be a positive Radon measure supported in Ω satisfying* (2.1) *and suppose* (2.2)*.*

Then there exists a constant $C=C(n,s,r,\sigma,K,\partial\Omega)$ such that

$$\left(\int_\Omega |v(x)|^s\, d\mathfrak{m}\right)^{1/s} \leqslant C\left[\left(\int_\Omega |Dv(x)|^r\, dx\right)^{1/r} + \left(\int_\Omega |v(x)|^\sigma\, dx\right)^{1/\sigma}\right]$$

for all $v\in W^{1,r}(\Omega)$ and all $\sigma\in(0,r]$.

If, in particular, $d\mathfrak{m}=c(x)\,dx$ with $c\in L^{1,n-r+\varepsilon_0}(\Omega)$ and $\varepsilon_0>0$, then there is a constant $C=C\left(n,\varepsilon_0,r,\sigma,\|c\|_{L^{1,n-r+\varepsilon_0}(\Omega)},\partial\Omega\right)$ such that

$$\left(\int_\Omega |v(x)|^s c(x)\, dx\right)^{1/s} \leqslant C\left[\left(\int_\Omega |Dv(x)|^r\, dx\right)^{1/r} + \left(\int_\Omega |v(x)|^\sigma\, dx\right)^{1/\sigma}\right]$$

for all $v\in W^{1,r}(\Omega)$ and all $\sigma\in(0,r]$, where s is the unique solution of the equation $n-r+\varepsilon_0=\frac{s}{r}(n-r)$.

In the particular case when $\mathfrak{m}$ is the Lebesgue measure supported in Ω, we have $\alpha_0 = n$ whence $s = r^*$ and Lemma 2.2 implies the next version of the Sobolev embedding theorem, valid for functions with non necessary zero boundary trace.

Lemma 2.3. (Sobolev inequality, see also [7, Chapter II, §2]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary and $r < n$.*

Then there exists a constant C depending on n, r, $\operatorname{diam}\Omega$ and the Lipschitz norm of $\partial\Omega$, such that

$$\|v\|_{L^{r^*}(\Omega)} \leqslant C \left[\left(\int_\Omega |Dv(x)|^r \, dx \right)^{1/r} + \left(\int_\Omega |v(x)|^\sigma \, dx \right)^{1/\sigma} \right]$$

for all $v \in W^{1,r}(\Omega)$ and all $\sigma \in (0, r]$.

The next result asserts an *improving of integrability* property of the weak solutions to (1.1) and it requires only coercivity and controlled growths of the nonlinear terms, together with some more integrability of the functions φ_1 and φ_2 than $L^{\frac{m}{m-1}}(\Omega)$ and $L^{\frac{mn}{mn+m-n}}(\Omega)$, respectively. We refer the reader to [3, Theorem 4] and the subsequent remarks, and also to [6, Section 8] in the particular case $m = 2$.

Lemma 2.4. *Assume* (1.2)–(1.5) *together with $\varphi_1 \in L^p(\Omega)$, $p > \frac{m}{m-1}$ and $\varphi_2 \in L^q(\Omega)$, $p > \frac{mn}{mn-n+m}$. Let $u \in W^{1,m}(\Omega)$ be a weak solution of the conormal problem* (1.1).

Then there exists an exponent $m_0 > m$ such that $u \in W^{1,m_0}(\Omega)$ with the estimate

$$\|Du\|_{L^{m_0}(\Omega)} \leqslant C,$$

where the constant C depends on known quantities, on $\|Du\|_{L^m(\Omega)}$ and on the uniform integrability of $|Du|^m$ in Ω.

Lemma 2.5. (Hartman–Stampacchia maximum principle, see [5] and [7, Chapter II, Lemma 5.1]) *Let $\tau \colon \mathbb{R} \to [0, \infty)$ be a non-increasing function and suppose there exist constants $C > 0$, $k_0 \geqslant 0$, $\delta > 0$ and $\alpha \in [0, 1 + \delta]$ such that*

$$\int_k^\infty \tau(t) \, dt \leqslant C k^\alpha \big(\tau(k)\big)^{1+\delta} \quad \forall k \geqslant k_0.$$

Then τ supports the finite time extinction property, *that is, there is a number $k_{\max}$, depending on C, k_0, δ, α and $\int_{k_0}^{\infty} \tau(t)\,dt$, such that*

$$\tau(k) = 0 \quad \forall k \geqslant k_{\max}.$$

§3. Global essential boundedness of the weak solutions to (1.1)

Our main result is the next

Theorem 3.1. *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$ and suppose the hypotheses* (1.2)–(1.5) *and* (1.7) *are satisfied.*

Then each $W^{1,m}(\Omega)$ weak solution to the conormal problem (1.1) *is globally essentially bounded. Precisely, there exists a constant M depending on known quantities, on $\|u\|_{L^m(\Omega)}$ and $\|Du\|_{L^m(\Omega)}$ and on the uniform integrability of $|Du|^m$, such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant M. \tag{3.1}$$

Proof. Let us concentrate to the case $m < n$ firstly, and for this goal consider the measure

$$d\mathfrak{m} := \left(\chi(x) + \varphi_1(x)^{\frac{m}{m-1}} + \varphi_2(x) + |u(x)|^{\frac{m^2}{n-m}}\right) dx,$$

where $\chi(x)$ is the characteristic function of Ω, dx is the Lebesgue measure and, without loss of generality, the functions φ_1 and φ_2 are supposed to be extended as zero outside Ω.

Setting B_ρ for any ball of radius ρ, and using the assumptions (1.8), we have

$$\int_{B_\rho} \varphi_1(x)^{\frac{m}{m-1}}\,dx \leqslant \|\varphi_1\|^{\frac{m}{m-1}}_{L^{p,\lambda}(\Omega)} \rho^{n-\frac{m(n-\lambda)}{p(m-1)}} = \|\varphi_1\|^{\frac{m}{m-1}}_{L^{p,\lambda}(\Omega)} \rho^{n-m+\left(m-\frac{m(n-\lambda)}{p(m-1)}\right)}$$

$$\int_{B_\rho} \varphi_2(x)\,dx \leqslant \|\varphi_2\|_{L^{q,\mu}(\Omega)} \rho^{n-\frac{n-\mu}{q}} = \|\varphi_2\|_{L^{q,\mu}(\Omega)} \rho^{n-m+\left(m-\frac{n-\mu}{q}\right)}$$

where $m - \frac{m(n-\lambda)}{p(m-1)} > 0$ and $m - \frac{n-\mu}{q} > 0$ because of $(m-1)p + \lambda > n$ and $mq + \mu > n$. Moreover, $u \in L^{m_0^*}(\Omega)$ as consequence of Lemmas 2.4 and 2.3, whence the Hölder inequality yields

$$\int_{B_\rho} |u(x)|^{\frac{m^2}{n-m}}\,dx \leqslant \|u\|^{\frac{m^2}{n-m}}_{L^{m_0^*}(\Omega)} \rho^{n-m+\frac{m m_0^*(n-m)-nm^2}{m_0^*(n-m)}} \tag{3.2}$$

with $\frac{mm_0^*(n-m)-nm^2}{m_0^*(n-m)} > 0$ since $m_0^* > m^* = \frac{nm}{n-m}$.

At this point, defining

$$\varepsilon_0 := \min\left\{m - \frac{m(n-\lambda)}{p(m-1)}, m - \frac{n-\mu}{q}, \frac{mm_0^*(n-m)-nm^2}{m_0^*(n-m)}\right\} > 0,$$

we have

$$\mathfrak{m}(B_\rho) \leqslant K\rho^{n-m+\varepsilon_0}$$

with a constant K depending on known quantities, and this means that $\mathfrak{m}$ is a measure for which the Adams trace inequality Lemma 2.2 holds.

Take now any positive constant $k \geqslant 1$ and set

$$u_k(x) = \max\{u(x) - k, 0\}$$

for the truncated function, and

$$A_k = \{x \in \Omega\colon\ u(x) > k\}$$

for the corresponding upper zero-level set.

Let us note that $\mathfrak{m}(A_k) = 0$ for large enough values of k would ensure the desired boundedness from above for the weak solution, and we will reach that by means of the Hartman–Stampacchia maximum principle, Lemma 2.5. For, noting that $u_k \in W^{1,m}(\Omega)$ and $u_k \equiv 0$ on $\Omega \setminus A_k$, we employ first the Hölder inequality to get

$$\int_\Omega u_k(x)\, d\mathfrak{m} = \int_{A_k} u_k(x)\, d\mathfrak{m} \leqslant \left(\int_{A_k} d\mathfrak{m}\right)^{1-1/s} \left(\int_{A_k} |u_k(x)|^s\, d\mathfrak{m}\right)^{1/s}.$$

After that application of the Adams trace inequality (Lemma 2.2) with

$$\alpha_0 = n - m + \varepsilon_0, \quad s = \frac{m(n-m+\varepsilon_0)}{n-m}, \quad r = m, \tag{3.3}$$

and arbitrary $\sigma \in (0, m]$ yields

$$\int_\Omega u_k(x)\, d\mathfrak{m} \leqslant C(\mathfrak{m}(A_k))^{1-\frac{n-m}{m(n-m+\varepsilon_0)}} \left[\left(\int_{A_k} |Du_k(x)|^m\, dx\right)^{1/m} + \left(\int_{A_k} |u_k(x)|^\sigma\, dx\right)^{1/\sigma}\right]. \tag{3.4}$$

To estimate the m-energy of u_k above, we remember that $u_k \in W^{1,m}(\Omega)$, $u_k \equiv 0$ on $\Omega \setminus A_k$, $Du = Du_k$ a.e. A_k, and use u_k as a test function in (1.6) in order to obtain

$$\int_{A_k} \mathbf{a}\big(x, u(x), Du(x)\big) \cdot Du_k(x)\, dx + \int_{A_k} b\big(x, u(x), Du(x)\big) u_k(x)\, dx = \int_{\partial\Omega \cap \overline{A}_k} \psi\big(x, u(x)\big) u_k(x)\, d\Gamma_x. \tag{3.5}$$

Now, (1.2) yields

$$\int_{A_k} \mathbf{a}\big(x, u(x), Du(x)\big) \cdot Du_k(x)\, dx \geqslant \gamma \int_{A_k} |Du_k(x)|^m\, dx - \Lambda \int_{A_k} |u(x)|^{m^*}\, dx - \Lambda \int_{A_k} \varphi_1(x)^{\frac{m}{m-1}}\, dx. \tag{3.6}$$

Further on,

$$0 < \frac{u(x)-k}{u(x)} < 1 \quad \text{in } A_k,$$

whence, for a.a. $x \in A_k$ one has

$$\begin{aligned} \big|b(x, u(x), Du(x)) u_k(x)\big| &= \big|b(x, u(x), Du_k(x)) u(x)\big| \frac{u(x)-k}{u(x)} \\ &\leqslant \Lambda \left(\varphi_2(x)|u(x)| + |u(x)|^{m^*} + |Du_k(x)|^{\frac{m(m^*-1)}{m^*}} |u(x)| \right) \\ &\leqslant \Lambda \left(\varepsilon |Du_k(x)|^m + C(\varepsilon)|u(x)|^{m^*} + \varphi_2(x)|u(x)| \right) \end{aligned}$$

with arbitrary $\varepsilon > 0$ as consequence of (1.4) and the Young inequality

$$|Du_k(x)|^{\frac{m(m^*-1)}{m^*}} |u(x)| \leqslant \varepsilon |Du_k(x)|^m + C(\varepsilon)|u(x)|^{m^*}.$$

Therefore,

$$\int_{A_k} b\big(x, u(x), Du(x)\big) u_k(x)\, dx \leqslant \Lambda\varepsilon \int_{A_k} |Du_k(x)|^m\, dx + \Lambda C(\varepsilon) \int_{A_k} |u(x)|^{m^*}\, dx + \Lambda \int_{A_k} \varphi_2(x)|u(x)|\, dx. \tag{3.7}$$

As for the surface integral in (3.5), we use the hypothesis (1.7) to get

$$\psi\big(x,u(x)\big)u_k(x)=\psi\big(x,u(x)\big)u(x)\frac{u(x)-k}{u(x)}\leqslant 0 \quad \text{for a.a. } x\in\partial\Omega\cap\overline{A}_k$$

whence

$$\int\limits_{\partial\Omega\cap\overline{A}_k}\psi\big(x,u(x)\big)u_k(x)\,d\Gamma_x\leqslant 0. \tag{3.8}$$

Employing (3.6), (3.7) and (3.8) into (3.5), and choosing $\varepsilon>0$ small enough, we obtain the basic energy inequality

$$\int\limits_{A_k}|Du_k(x)|^m\,dx \leqslant C\Bigg(\underbrace{\int\limits_{A_k}\varphi_1(x)^{\frac{m}{m-1}}\,dx}_{I_1}+\underbrace{\int\limits_{A_k}|u(x)|\varphi_2(x)\,dx}_{I_2}+\underbrace{\int\limits_{A_k}|u(x)|^{\frac{nm}{n-m}}\,dx}_{I_3}\Bigg). \tag{3.9}$$

The definition of the measure $\mathfrak{m}$ gives immediately

$$I_1\leqslant\mathfrak{m}(A_k)\leqslant k^m\mathfrak{m}(A_k) \tag{3.10}$$

since $k\geqslant 1$, while

$$\begin{aligned}I_2=\int\limits_{A_k}|u(x)-k+k|\varphi_2(x)\,dx&\leqslant\int\limits_{A_k}|u_k(x)|\varphi_2(x)\,dx+k\int\limits_{A_k}\varphi_2(x)\,dx\\&\leqslant\int\limits_{A_k}|u_k(x)|\,d\mathfrak{m}+k\mathfrak{m}(A_k).\end{aligned}$$

To proceed further, we use (3.4) with (3.3) and the Young inequality to get

$$\int\limits_{A_k}|u_k(x)|d\mathfrak{m}\leqslant C\big(\mathfrak{m}(A_k)\big)^{\frac{s-1}{s}}\times\left[\left(\int\limits_{A_k}|Du_k(x)|^m\,dx\right)^{1/m}+\left(\int\limits_{A_k}|u_k(x)|^\sigma\,dx\right)^{1/\sigma}\right]\leqslant\varepsilon\int\limits_{A_k}|Du_k(x)|^m\,dx$$

$$+ C(\varepsilon)\left(\int_{A_k}|u_k(x)|^\sigma\,dx\right)^{m/\sigma} + C\big(\mathfrak{m}(A_k)\big)^{\frac{s-1}{s}\frac{m}{m-1}}$$

with arbitrary $\varepsilon>0$ and $\sigma\in(0,m]$.

Noting that $\frac{s-1}{s}\frac{m}{m-1}=1+\frac{\varepsilon_0}{(m-1)(n-m+\varepsilon_0)}$ and

$$\begin{aligned}\mathfrak{m}(A_k)\leqslant\mathfrak{m}(\Omega)&=\int_\Omega\left(1+\varphi_1(x)^{\frac{m}{m-1}}+\varphi_2(x)+|u(x)|^{\frac{m^2}{n-m}}\right)dx\\&\leqslant|\Omega|+C\left(\|\varphi_1\|^{\frac{m}{m-1}}_{L^{p,\lambda}(\Omega)}+\|\varphi_2\|_{L^{q,\mu}(\Omega)}+\|u\|^{\frac{m^2}{n-m}}_{L^{m_0^*}(\Omega)}\right),\end{aligned}\tag{3.11}$$

we have

$$\big(\mathfrak{m}(A_k)\big)^{\frac{s-1}{s}\frac{m}{m-1}}=\mathfrak{m}(A_k)\big(\mathfrak{m}(A_k)\big)^{\frac{s-1}{s}\frac{m}{m-1}-1}=\mathfrak{m}(A_k)\big(\mathfrak{m}(\Omega)\big)^{\frac{s-1}{s}\frac{m}{m-1}-1}.$$

This, together with $k\geqslant1$ yields

$$I_2\leqslant\varepsilon\int_{A_k}|Du_k(x)|^m\,dx+Ck^m\mathfrak{m}(A_k)+C(\varepsilon)\left(\int_{A_k}|u_k(x)|^\sigma\,dx\right)^{m/\sigma},\tag{3.12}$$

with arbitrary $\varepsilon>0$ and $\sigma\in(0,m]$.

Regarding I_3 in (3.9), we have

$$\begin{aligned}I_3=\int_{A_k}|u(x)|^{\frac{nm}{n-m}}\,dx&=\int_{A_k}|u(x)-k+k|^m|u(x)|^{\frac{m^2}{n-m}}\,dx\\&\leqslant2^{m-1}\left(\int_{A_k}|u_k(x)|^m|u(x)|^{\frac{m^2}{n-m}}\,dx+k^m\int_{A_k}|u(x)|^{\frac{m^2}{n-m}}\,dx\right)\\&\leqslant2^{m-1}\int_{A_k}|u_k(x)|^m|u(x)|^{\frac{m^2}{n-m}}\,dx+2^{m-1}k^m\mathfrak{m}(A_k).\end{aligned}$$

The first term above will be estimated by means of the Adams trace inequality, observing that $|u|^{\frac{m^2}{n-m}}\in L^{1,\theta}(\Omega)$ as it follows from (3.2) with

$$\theta=n-m+\frac{mm_0^*(n-m)-nm^2}{m_0^*(n-m)}>n-m.$$

We pick now an $r'<m$, close enough to m, and such that

$$n-m<\frac{m}{r'}(n-r')<\theta$$

in order to have

$$n - r' + \frac{(n-r')(m-r')}{r'} < \theta,$$

and then Lemma 2.1 implies $|u|^{\frac{m^2}{n-m}} \in L^{1, n-r'+\frac{(n-r')(m-r')}{r'}}(\Omega)$. Therefore, Lemma 2.2 and the Hölder inequality give

$$\begin{aligned}
&\int\limits_{A_k} |u_k(x)|^m |u(x)|^{\frac{m^2}{n-m}}\,dx \\
&\qquad \leqslant C\left[\left(\int\limits_{A_k} |Du_k(x)|^{r'}\,dx\right)^{m/r'} + \left(\int\limits_{A_k} |u_k(x)|^{\sigma}\,dx\right)^{m/\sigma}\right] \\
&\qquad \leqslant C|A_k|^{\frac{m}{r'}-1}\int\limits_{A_k} |Du_k(x)|^m\,dx + C\left(\int\limits_{A_k} |u_k(x)|^{\sigma}\,dx\right)^{m/\sigma}
\end{aligned}$$

with C depending also on $\left\| |u|^{\frac{m^2}{n-m}} \right\|_{L^{1,\theta}(\Omega)}$ which is anyway bounded in terms of $\|u\|_{L^{m_0^*}(\Omega)}$ as Lemma 2.4 and (3.2) show.

We have this way

$$I_3 \leqslant C\left(|A_k|^{\frac{m}{r'}-1}\int\limits_{A_k} |Du_k(x)|^m\,dx + k^m \mathfrak{m}(A_k) + \left(\int\limits_{A_k} |u_k(x)|^{\sigma}\,dx\right)^{m/\sigma}\right)$$

and using it, together with (3.10) and (3.12), (3.9) takes on the form

$$\begin{aligned}
\int\limits_{A_k} |Du_k(x)|^m\,dx \leqslant\ & C\left[\left(|A_k|^{\frac{m}{r'}-1} + \varepsilon\right)\int\limits_{A_k} |Du_k(x)|^m\,dx \right. \\
& \left. + k^m \mathfrak{m}(A_k) + \left(\int\limits_{A_k} |u_k(x)|^{\sigma}\,dx\right)^{m/\sigma}\right]
\end{aligned} \tag{3.13}$$

Further on, we have

$$
\begin{aligned}
k^{\frac{nm}{n-m}}|A_k| &\leqslant \int\limits_{A_k} |u(x)|^{\frac{nm}{n-m}}\,dx \\
&\leqslant \int\limits_{\Omega} |u(x)|^{\frac{nm}{n-m}}\,dx \leqslant C\left(\|Du\|_{L^m(\Omega)} + \|u\|_{L^m(\Omega)}\right)^{\frac{nm}{n-m}}
\end{aligned}
$$

by Lemma 2.3, that means $C\left(|A_k|^{\frac{m}{r'}-1} + \varepsilon\right)$ on the right-hand side of (3.13) can be made less than $1/2$ if $k \geqslant k_0$ for large enough k_0, depending on known quantities, on $\|u\|_{L^m(\Omega)}$ and on $\|Du\|_{L^m(\Omega)}$, and if $\varepsilon > 0$ is small enough.

Thus (3.13) becomes

$$
\int\limits_{A_k} |Du_k(x)|^m\,dx \leqslant C\left[k^m \mathfrak{m}(A_k) + \left(\int\limits_{A_k} |u_k(x)|^\sigma\,dx\right)^{m/\sigma}\right] \qquad \forall\ k \geqslant k_0
$$

and then (3.4) rewrites as

$$
\begin{aligned}
&\int\limits_{\Omega} u_k(x)\,d\mathfrak{m} \\
&\leqslant C\big(\mathfrak{m}(A_k)\big)^{1-\frac{n-m}{m(n-m+\varepsilon_0)}}\left[k\big(\mathfrak{m}(A_k)\big)^{\frac{1}{m}} + \left(\int\limits_{A_k} |u_k(x)|^\sigma\,dx\right)^{1/\sigma}\right] \qquad (3.14)
\end{aligned}
$$

valid for all $k \geqslant k_0$ and all $\sigma \in (0, m]$.

Since $\sigma \in (0, m]$ is at our disposal, we will show now that, choosing appropriately σ, we can reach

$$
\left(\int\limits_{A_k} |u_k(x)|^\sigma\,dx\right)^{1/\sigma} \leqslant Ck\big(\mathfrak{m}(A_k)\big)^{\frac{1}{m}} \qquad (3.15)
$$

with a constant C depending on known quantities. We will distinguish between two cases:

Case 1: $m < n/2$. Taking $\sigma = \frac{m^2}{n-m}$, we have

$$
\int\limits_{A_k} |u_k(x)|^{\frac{m^2}{n-m}}\,dx = \int\limits_{A_k} |u(x) - k|^{\frac{m^2}{n-m}}\,dx
$$

$$\leqslant C\left(\int\limits_{A_k} |u(x)|^{\frac{m^2}{n-m}}\, dx + k^{\frac{m^2}{n-m}}|A_k|\right)$$
$$\leqslant C\left(\mathfrak{m}(A_k) + k^{\frac{m^2}{n-m}}|A_k|\right) \leqslant Ck^{\frac{m^2}{n-m}}\mathfrak{m}(A_k)$$

since $|A_k| \leqslant \mathfrak{m}(A_k)$ and $k \geqslant 1$. Therefore,

$$\left(\int\limits_{A_k} |u_k(x)|^{\frac{m^2}{n-m}}\, dx\right)^{\frac{n-m}{m^2}} \leqslant Ck\big(\mathfrak{m}(A_k)\big)^{\frac{n-m}{m^2}} = Ck\big(\mathfrak{m}(A_k)\big)^{\frac{1}{m}}\big(\mathfrak{m}(A_k)\big)^{\frac{n-m}{m^2}-\frac{1}{m}}$$
$$\leqslant Ck\big(\mathfrak{m}(A_k)\big)^{\frac{1}{m}}\big(\mathfrak{m}(\Omega)\big)^{\frac{n-m}{m^2}-\frac{1}{m}}$$

since $\frac{n-m}{m^2} - \frac{1}{m} > 0$. Bearing in mind (3.11), we get (3.15) in this case.

Case 2: $m \geqslant n/2$. We choose now $\sigma = 1$ and note that $\frac{m^2}{n-m} \geqslant 1$. Then

$$\int\limits_{A_k} |u_k(x)|\, dx = \int\limits_{A_k} |u(x) - k|\, dx$$
$$\leqslant \int\limits_{A_k} |u(x)|\, dx + k|A_k| \leqslant \int\limits_{A_k} |u(x)|\, dx + k\mathfrak{m}(A_k),$$

while the Hölder inequality and $\frac{m^2}{n-m} \geqslant 1$ yield

$$\int\limits_{A_k} |u(x)|\, dx \leqslant \left(\int\limits_{A_k} |u_k(x)|^{\frac{m^2}{n-m}}\, dx\right)^{\frac{n-m}{m^2}} |A_k|^{1-\frac{n-m}{m^2}} \leqslant \mathfrak{m}(A_k).$$

Remembering once again $k \geqslant 1$, we obtain

$$\int\limits_{A_k} |u_k(x)|\, dx \leqslant 2k\mathfrak{m}(A_k) = 2k\big(\mathfrak{m}(A_k)\big)^{\frac{1}{m}}\big(\mathfrak{m}(A_k)\big)^{1-\frac{1}{m}}$$
$$\leqslant 2k\big(\mathfrak{m}(A_k)\big)^{\frac{1}{m}}\big(\mathfrak{m}(\Omega)\big)^{1-\frac{1}{m}} \leqslant Ck\big(\mathfrak{m}(A_k)\big)^{\frac{1}{m}}$$

by (3.11). So, we have (3.15) also in the second case.

With (3.15) at hand, (3.14) becomes

$$\int\limits_{A_k} u_k(x)\, d\mathfrak{m} \leqslant Ck\big(\mathfrak{m}(A_k)\big)^{1+\frac{\varepsilon_0}{m(n-m+\varepsilon_0)}} \qquad \forall\ k \geqslant k_0 \tag{3.16}$$

with $\varepsilon_0 > 0$.

Now, the Cavalieri principle yields

$$\int_{A_k} u_k(x)\, d\mathfrak{m} = \int_{A_k} (u(x) - k)\, d\mathfrak{m} = \int_k^\infty \mathfrak{m}(A_t)\, dt,$$

whence, setting $\tau(t) := \mathfrak{m}(A_t)$, (3.16) takes on the form

$$\int_k^\infty \tau(t)\, dt \leqslant Ck\tau(k)^{1+\delta} \qquad \forall k \geqslant k_0,\ \delta = \frac{\varepsilon_0}{m(n-m+\varepsilon_0)} > 0.$$

At this point, the Hartman–Stampacchia maximum principle Lemma 2.5 asserts

$$u(x) \leqslant k_{\max} \qquad \text{a.e. } \Omega$$

where $k_{\max}$ depends on known quantities, on $\|u\|_{L^m(\Omega)}$ and on $\|Du\|_{L^m(\Omega)}$ in addition.

To get a bound from below for $u(x)$, it suffices to repeat the above procedure with $-u(x)$ instead of $u(x)$, and this gives the claim (3.1) when $m < n$.

The limit case $m = n$ can be treated easily by slightly changing the approach already adopted. In fact, the coercivity condition (1.2) and the controlled growth assumption (1.4) for the term $b(x, z, \xi)$ have now the form

$$\mathbf{a}(x, z, \xi) \cdot \xi \geqslant \gamma|\xi|^n - \Lambda|z|^{m^*} - \Lambda\varphi_1(x)^{\frac{n}{n-1}}, \tag{3.17}$$

$$|b(x, z, \xi)| \leqslant \Lambda\left(\varphi_2(x) + |z|^{m^*-1} + |\xi|^{\frac{n(m^*-1)}{m^*}}\right), \tag{3.18}$$

respectively, where $m^* > n$ is an *arbitrary* exponent, $\varphi_1 \in L^{p,\lambda}(\Omega)$ with $p > \frac{n}{n-1}$, $\lambda \in (0, n)$ and $(n-1)p + \lambda > n$, and $\varphi_2 \in L^{q,\mu}(\Omega)$ with $q > 1$ and $\mu \in (0, n)$.

We choose now, without loss of generality, a number $m' < n$, close enough to n, such that $m^* = \frac{n^2}{(n-m')(n+1)}$ whence

$$m^* < (m')^* = \frac{nm'}{n-m'}, \quad \frac{n}{n-1} < \frac{m'}{m'-1}, \quad \frac{n(m^*-1)}{m^*} = \frac{m'((m')^*-1)}{(m')^*}.$$

This way (3.17) takes on the form

$$\begin{aligned}\mathbf{a}(x,z,\xi)\cdot\xi &\geqslant \gamma|\xi|^n - \Lambda|z|^{m^*} - \Lambda\varphi_1(x)^{\frac{n}{n-1}} \\ &\geqslant \gamma|\xi|^{m'} - \Lambda|z|^{(m')^*} - \Lambda\varphi_1(x)^{\frac{m'}{m'-1}}\end{aligned} \tag{3.19}$$

when $|z|\geqslant 1$ and $|\xi|\geqslant 1$ and where, without loss of generality, we have supposed $\varphi_1(x)\geqslant 1$, while (3.18) becomes

$$|b(x,z,\xi)| \leqslant \Lambda\left(\varphi_2(x) + |z|^{(m')^*-1} + |\xi|^{\frac{m'((m')^*-1)}{(m')^*}}\right) \tag{3.20}$$

for $|z|\geqslant 1$ and $|\xi|\geqslant 1$.

Considering the measure

$$d\mathfrak{m}' = \left(\chi(x) + \varphi_1(x)^{\frac{m'}{m'-1}} + \varphi_2(x) + |u(x)|^{\frac{m'^2}{n-m'}}\right)dx,$$

we may increase eventually the value of m', maintaining it anyway strictly less than n, in order to have $p > \frac{m'}{m'-1}$, $(m'-1)p+\lambda > n$ and $m'q+\mu > n$. This leads to

$$\mathfrak{m}'(B_\rho) \leqslant K\rho^{n-m'+\varepsilon_0}$$

as above, with a suitable $\varepsilon_0 > 0$.

Defining the functions $u_k(x)$ and the sets A_k as before, it is clear that

$$\int_{\{x\in A_k:\ |Du_k(x)|<1\}} |Du_k(x)|^{m'}\,dx \leqslant |A_k| \leqslant k^{m'}\mathfrak{m}'(A_k),$$

while

$$\int_{\{x\in A_k:\ |Du_k(x)|\geqslant 1\}} |Du_k(x)|^{m'}\,dx$$

can be estimated with the help of (3.19) and (3.20), as already done before when $m<n$. That leads to the bound (3.16) with m' instead of m and the Hartman–Stampacchia maximum principle Lemma 2.5 leads to the desired estimate (3.1) also in the case $m=n$ and this completes the proof of Theorem 3.1. □

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References

1. D. R. Adams, *Traces of potentials arising from translation invariant operators.* — Ann. Scuola Norm. Sup. Pisa, No. 3, **25** (1971), 203–217.
2. A. A. Arkhipova, *Some applications of reverse Hölder inequalities with a boundary integral.* — J. Math. Sci., **72** (1994), No. 6, 3370–3378.
3. A. A. Arkhipova, *Reverse Hölder inequalities with boundary integrals and L^p-estimates for solutions of nonlinear elliptic and parabolic boundary-value problems. Nonlinear evolution equations.* — Amer. Math. Soc. Transl. Ser., No. 2, **164** (1995), 15–42.
4. S.-S. Byun, D.K. Palagachev, P. Shin, *Global Hölder continuity of solutions to quasilinear equations with Morrey data.* — Commun. Contemp. Math., **24**(2022), Art. No. 2150062.
5. Ph. Hartman, G. Stampacchia, *On some non-linear elliptic differential-functional equations.* — Acta Math. **115** (1966), 271–310.
6. D. Kim, *Global regularity of solutions to quasilinear conormal derivative problem with controlled growth.* — J. Korean Math. Soc., No. 6, **49** (2012), 1273–1299.
7. O. A. Ladyzhenskaya, N. N. Ural'tseva, *Linear and quasilinear equations of elliptic type.* — Second Edition Revised, Nauka, Moscow, 1973.
8. G. M. Lieberman, *The conormal derivative problem for elliptic equations of variational type.* — J. Differential Equations, No. 2, **49** (1983), 218–257.
9. G. M. Lieberman, *The conormal derivative problem for equations of variational type in nonsmooth domains.* — Trans. Amer. Math. Soc., No. 1, **330** (1992), 41–67.
10. G. M. Lieberman, *Oblique derivative problems for elliptic equations.* — World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

11. V.G. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations.* — Second, Revised and Augmented Edition, Grundlehren der Mathematischen Wissenschaften, 342, Springer, Heidelberg, 2011.
12. A. I. Nazarov, N. N. Ural'tseva, *The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients.* — Algebra i Analiz, No. 1, **23** (2011), 136–168;
13. L. C. Piccinini, *Inclusioni tra spazi di Morrey.* — Boll. Un. Mat. Ital., No. 4, **2** (1969), 95–99.
14. P. Winkert, *L^∞-estimates for nonlinear elliptic Neumann boundary value problems.* — NoDEA Nonlinear Differential Equations Appl., No. 3, **17** (2010), 289–302.

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