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A. G. Pronko, S. K. Syrygina

QUANTUM L-OPERATOR OF THE CRITICAL ISING MODEL

Abstract. We consider the two-dimensional Ising model on the square lattice at the critical temperature. This model can be related to the free-fermion eight-vertex model with the trigonometric dependence of the Boltzmann weights on the spectral variable. We obtain the quantum L-operator as a solution of the RLL-relation assuming the spectral variable dependence similar to that of the R-matrix.

1. Introduction

The two-dimensional Ising model is probably one of the most famous models of statistical mechanics. Historically, this is the first model for which the existence of a phase transition at a non-zero temperature have been demonstrated and an explicit expression for the free energy in the thermodynamic limit is obtained [1]. The model admits exact solution at the zero field for the regular square, triangular, and hexagonal lattices as well as for the case of a non-zero field at the critical temperature in the continuum limit. There exist a vast of literature devoted to the Ising model, see, e.g., monographs [2–4] and references therein.

In the Ising model (and its generalizations, Potts and Ashkin–Teller models) the local variables (discrete "spins") are placed at sites of the lattice. Another way to build a planar statistical mechanics model is to place spins at faces and edges, that correspond to face and vertex models, respectively. The latter acquire their names since they are usually defined by specifying the allowed configurations of states around a vertex (other configurations assigned infinite energies). Ising-type models, face models and vertex models, to one degree or another, admit equivalent (re-)formulations through each other.

All these models are closely related to quantum one-dimensional systems, such as quantum spin chains, which have numerous applications in

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various fields: condensed matter physics, quantum field theory, supersymmetric Yang-Mills field theory, quantum computing algorithms, see, e.g., review [5]. Recent studies show a renewed and sharply increased interest in constructing new quantum systems with specific properties, such as frustration-free quantum spin chains [6–8]. Applications in the context of the quantum information theory stimulate an activity in finding new solutions of the Yang–Baxter equation [9–14].

The standard modern approach in studying of integrable two-dimensional statistical mechanics models and related one-dimensional quantum systems is the Quantum Inverse Scattering method (QISM) [15–17]. In the framework of this method, new systems can be constructed by finding the quantum L-operator satisfying the so-called RLL-relation, which generalizes the Yang-Baxter equation. The RLL-relation allows for searching an L-operator starting with a given R-matrix, which can be specified, for example, as a matrix of the Boltzmann weights of an integrable (usually, vertex) model. A prototypical example here is the R-matrix of the six-vertex model, which leads, through the RLL-relation, to the quantum L-operator of the Heisenberg XXZ spin chain of an arbitrary spin [18].

Motivated by the problem of constructing new integrable models one may wonder which quantum systems can be obtained from the Ising, and more generally, Potts models. It is known that for the free-fermion eight-vertex model, which includes the Ising model as a particular case, this can be done by q-deforming the Clifford algebra [19]. In this case one deals with the free-fermion elliptic R-matrix. At the same time it is known that the trigonometric case may admit a richer set of solutions in comparison with the elliptic one. Hence, it is intriguing to study the special case of that R-matrix in which it becomes trigonometric but remains related to the free-fermion eight-vertex model. This is also a necessary step towards addressing the similar problem in the context of the Potts model [20].

In the present paper, we consider the Ising model with the Bolztmann weights corresponding to the critical temperature. In this case, the related vertex model is a free-fermion eight-vertex model with the trigonometric dependence on the spectral variable. We solve the RLL-relation assuming the spectral parameter dependence of the L-operator similar to that of the R-matrix. Taking the central elements of the induced quadratic algebra proportional to the identity operator, we have obtained two solutions for the L-operator.

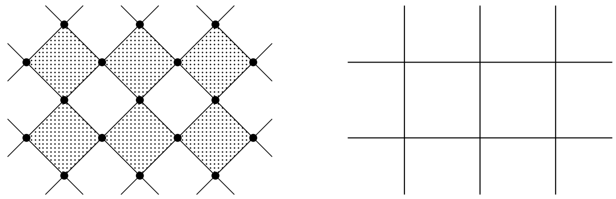

Figure 1. Square lattice with Ising spins sitting at sites with the chessboard coloring of faces (left) and the corresponding square lattice of a vertex model (right).

We organize the paper as follows. In the next section we recall how the Ising model can be formulated as the free-fermion eight-vertex model and consider the case of the critical temperature where the R-matrix become trigonometric. In §3 we propose an ansatz for the L-operator, present the induced quadratic algebra of commutation relations, and derive solutions for these relations.

2. Ising model as a vertex model

In this section we formulate the Ising model as a vertex model and present the R-matrix at the critical temperature. Here we follow mainly the ideas of papers [21–24].

The Ising model can be defined as a special case of the Potts model in which interaction between two spins μ and ν connected by an edge is described by the Boltzmann weight

$$w(x|\mu,\nu) = x^{\delta_{\mu\nu}},$$

where $\delta_{\mu\nu}$ is the Kronecker symbol and $\mu,\nu = 1,\dots,n$. The Ising model corresponds to $n = 2$. Here, $x = \exp\{J/kT\}$ encodes the dependence on the energy of interaction J, the Boltzmann constant k, and temperature T. The partition function is given as the sum over values of all lattice spins of the product of the edge Boltzmann weights.

To map the Ising model on to a vertex model let us first consider the square lattice with the Ising spins and color its faces in the chessboard manner, in "empty" and "dashed" faces, see Fig. 1. Next, consider another square lattice obtained by assigning vertices to the "dashed" faces. The corresponding vertex model follows by identifying the Boltzmann weight

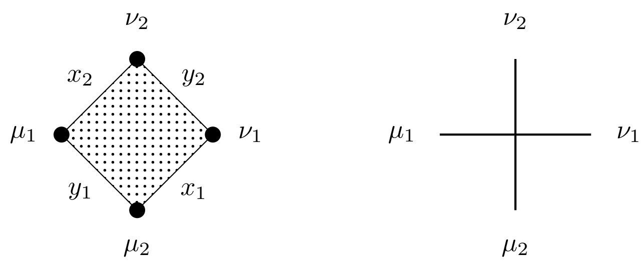

Figure 2. "Dashed" face with spin variables at the sites (left) and the corresponding vertex of the resulting square lattice (right); also shown the variables x_1, x_2, y_1, and y_2 of the edge Boltzmann weights of the Ising model.

of a vertex as the product of the four Boltzmann weights of edges around the "dashed" face.

Namely, let x_1, x_2, y_1, and y_2 be the variables of the Boltzmann weights of the four edges of a "dashed" face of the starting lattice, and μ_1, μ_2, ν_1, and ν_2 are the spins at the sites of this face, see Fig. 2. The corresponding vertex carries at its edges the same spins μ_1, μ_2, ν_1, and ν_2. The Boltzmann weight $W_{\mu_1\mu_2}^{\nu_1\nu_2}$ of the vertex is given as the product of those of edges of the "dashed" face:

$$W_{\mu_1\mu_2}^{\nu_1\nu_2} = w(y_1|\mu_1,\mu_2)w(x_1|\mu_2,\nu_1)w(x_2|\mu_1,\nu_2)w(y_2|\nu_1,\nu_2).$$

The weight $W_{\mu_1\mu_2}^{\nu_1\nu_2}$ can be regarded as a matrix element of an operator acting in $\mathbb{C}^2\otimes\mathbb{C}^2$. We will treat μ's as "out" and ν's as "in" indices, μ_i and ν_i being assigned to the ith copy of $\mathbb{C}^2$. Writing the weight explicitly

$$W_{\mu_1\mu_2}^{\nu_1\nu_2} = y_1^{\delta_{\mu_1\mu_2}} x_1^{\delta_{\mu_2\nu_1}} x_2^{\delta_{\mu_1\nu_2}} y_2^{\delta_{\nu_1\nu_2}} \tag{2.1}$$

one easily obtains

$$W = \begin{pmatrix} y_1 &&& \\ & 1 && \\ && 1 & \\ &&& y_1 \end{pmatrix} P\left[\begin{pmatrix} x_1 & 1 \\ 1 & x_1 \end{pmatrix} \otimes \begin{pmatrix} x_2 & 1 \\ 1 & x_2 \end{pmatrix}\right] \begin{pmatrix} y_2 &&& \\ & 1 && \\ && 1 & \\ &&& y_2 \end{pmatrix}.$$

Here, P stands for the permitation matrix

$$P = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \tag{2.2}$$

and we use the convention to omit in writing an entry when it is zero. We also use the convention that indices of the first (respectively, second) space label blocks (elements of blocks) of a matrix.

The weight matrix W can be represented as

$$W = (\Lambda \otimes \Lambda) R (\Lambda \otimes \Lambda)^{-1}, \qquad \Lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

where R is the matrix of Boltzmann weights of the eight-vertex model

$$R = \begin{pmatrix} w_1 & & & w_7 \\ & w_3 & w_5 & \\ & w_6 & w_4 & \\ w_8 & & & w_2 \end{pmatrix}.$$

Configurations of the eight-vertex model are given in terms of arrows placed at edges of the square lattice; the allowed vertex configurations and their respective Boltzmann weights (we follow conventions of [3]) are shown in Fig. 3. The weights are given by

$$\begin{aligned} &w_1 = a_+, \quad w_2 = a_-, \quad w_3 = b_+, \quad w_4 = b_-, \\ &w_5 = c_+, \quad w_6 = c_-, \quad w_7 = d_+, \quad w_8 = d_- \end{aligned}$$

where

$$\begin{aligned} a_\pm &= \frac{1}{2}\big[(x_1x_2+1)(y_2y_1+1) \pm (x_1+x_2)(y_2+y_1)\big], \\ b_\pm &= \frac{1}{2}\big[(x_1x_2-1)(y_2y_1-1) \pm (x_1-x_2)(y_2-y_1)\big], \\ c_\pm &= \frac{1}{2}\big[(x_1x_2-1)(y_2y_1+1) \pm (x_1-x_2)(y_2+y_1)\big], \\ d_\pm &= \frac{1}{2}\big[(x_1x_2+1)(y_2y_1-1) \pm (x_1+x_2)(y_2-y_1)\big]. \end{aligned}$$

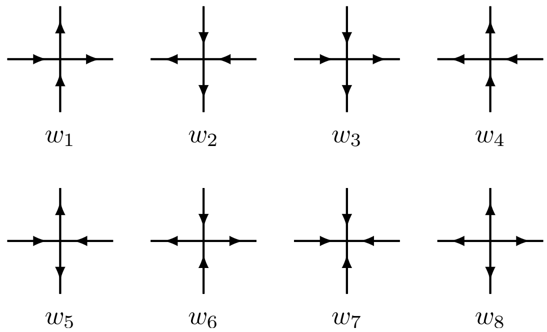

Figure 3. The eight arrow configurations allowed at a vertex, and their Boltzmann weights.

The weights obey the relations

$$w_1w_2 + w_3w_4 - w_5w_6 - w_7w_8 = 0, \tag{2.3a}$$
$$w_1w_3 + w_2w_4 - w_5w_7 - w_6w_8 = 0, \tag{2.3b}$$
$$w_1w_4 + w_2w_3 - w_5w_8 - w_6w_7 = 0. \tag{2.3c}$$

Relation (2.3a) is the so-called free fermion condition. Relations (2.3b) and (2.3c) specify the asymmetry between the pairs of the weights w_5, w_6 and w_7, w_8 of the generic free-fermion vertex model that corresponds to the Ising model with the face weight (2.1).

It is well known (see, e.g., [21,22]) that the free-fermion model is critical when one of the following four quantities vanish

$$w_1-w_2-w_3-w_4, \quad w_1-w_2+w_3+w_4, \quad w_1+w_2-w_3+w_4, \quad w_1+w_2+w_3-w_4.$$

For the first quantity equals zero this amounts to

$$\frac{(x_1x_2-1)(y_1y_2-1)}{(x_1+x_2)(y_1+y_2)} = 1. \tag{2.4}$$

Other cases correspond to the mappings of pairs of the variables $(x_1, x_2) \mapsto (-x_1, -x_2)$, $(x_1, y_1) \mapsto (-x_1, -y_1)$, and $(x_1, y_2) \mapsto (-x_1, -y_2)$, respectively, in (2.4). An exhaustive study of criticality conditions for the Ising model can be found in [25].

In this paper we focus our attention to the case where $x_1 = x_2 \equiv x$ and $y_1 = y_2 \equiv y$, so that (2.4) becomes the Kramers–Wannier duality relation

$$\left(\frac{x - x^{-1}}{2}\right)\left(\frac{y - y^{-1}}{2}\right) = 1.$$

The Baxter's substitution

$$x = \frac{1 + \cos u}{\sin u}, \qquad y = \frac{1 + \sin u}{\cos u}$$

yields

$$R = R(u) = \rho(u)\begin{pmatrix} 2 + \sin 2u & & & 2\sin u \\ & \sin 2u & 2\cos u & \\ & 2\cos u & \sin 2u & \\ 2\sin u & & & 2 - \sin 2u \end{pmatrix}, \tag{2.5}$$

where $\rho(u) = 4\sin^2(u/2)\sin^2(u/2 - \pi/4)$.

Matrix (2.5) satisfies the Yang–Baxter equation

$$\begin{aligned}\left(\check{R}(u-v)\otimes I\right)\left(I\otimes\check{R}(u)\right)\left(\check{R}(v)\otimes I\right)& \\ = \left(I\otimes\check{R}(v)\right)\left(\check{R}(u)\otimes I\right)\left(I\otimes\check{R}(u-v)\right),&\end{aligned}$$

where I denotes 2×2 identity matrix and $\check{R}(u) \equiv PR(u)$ with P being the permutation matrix (2.2).

3. Quantum L-operator

A generalization of the Yang–Baxter relation is the so-called RLL-relation:

$$\check{R}(u-v)\big[L(u)\otimes L(v)\big] = \big[L(v)\otimes L(u)\big]\check{R}(u-v). \tag{3.1}$$

Here, $L(u)$ is a 2×2 matrix with the entries being quantum operators acting in $\mathcal{H}$, a space of quantum states. The RLL-relation can be seen as an equation for $L(u)$, called quantum L-operator. In the QISM, quantum L-operators can be constructed by quantization of classical ones, or, more generally, they can be found by directly solving (3.1) with a given R-matrix [17].

Here we consider the problem of construction of the quantum L-operator of the form

$$L(u) = \begin{pmatrix} \mathrm{e}^{2\mathrm{i}u}a_+ + a_0 + \mathrm{e}^{-2\mathrm{i}u}a_- & \mathrm{e}^{\mathrm{i}u}b_+ + \mathrm{e}^{-\mathrm{i}u}b_- \\ \mathrm{e}^{\mathrm{i}u}c_+ + \mathrm{e}^{-\mathrm{i}u}c_- & \mathrm{e}^{2\mathrm{i}u}d_+ + d_0 + \mathrm{e}^{-2\mathrm{i}u}d_- \end{pmatrix}. \tag{3.2}$$

Expression (3.2) seems to be the simplest possible ansatz to satisfy (3.1) in the case of the R-matrix given by (2.5).

In the critical Ising model $\dim\mathcal{H}=2$ and the operator coefficients are represented by the following 2×2 matrices:

$$
\begin{aligned}
a_\pm &= \mathrm{diag}\left(\pm\frac{1}{2\mathrm{i}},\pm\frac{1}{2\mathrm{i}}\right), \qquad a_0=\mathrm{diag}(2,0),\\
b_\pm &= \begin{pmatrix}0 & \mp\mathrm{i}\\ 1 & 0\end{pmatrix}, \qquad c_\pm=\begin{pmatrix}0 & 1\\ \mp\mathrm{i} & 0\end{pmatrix},\\
d_\pm &= \mathrm{diag}\left(\pm\frac{1}{2\mathrm{i}},\mp\frac{1}{2\mathrm{i}}\right), \qquad d_0=\mathrm{diag}(0,2).
\end{aligned}
\tag{3.3}
$$

In this case $L(u)=R(u)$, where $R(u)$ is given by (2.5).

3.1. Commutation relations. In general, the coefficients are subject to the following algebra of quadratic relations.

The operators a_+ and a_- are central elements, i.e., commute with all operators of the algebra,

$$[a_\pm,e]=0, \qquad e=a_{\pm,0},b_\pm,c_\pm,d_{\pm,0}.$$

The operators d_+ and d_- commute with each other, with a_0 and d_0,

$$[d_+,d_-]=0, \qquad [d_\pm,a_0]=[d_\pm,d_0]=0, \tag{3.4}$$

and anti-commute with b_+, b_-, c_+, and c_-,

$$\{d_\pm,b_+\}=\{d_\pm,b_-\}=\{d_\pm,c_+\}=\{d_\pm,c_-\}=0. \tag{3.5}$$

The remaining relations can be split on two sets, of 'even' and 'odd' type relations, according to the number of b- and c-factors. Both sets consists of 16 independent relations. The 'even' relations are

$$d_\pm^2=a_\pm^2, \tag{3.6a}$$

$$b_\pm^2=c_\pm^2=a_\pm a_0-d_0d_\pm, \tag{3.6b}$$

$$b_\pm b_\mp=c_\mp c_\pm, \tag{3.6c}$$

$$\{b_+,b_-\}=2\left(a_+a_--d_+d_-\right), \tag{3.6d}$$

$$[b_\pm,c_\pm]=\pm 2\mathrm{i}\left(a_\pm d_0-a_0d_\pm\right), \tag{3.6e}$$

$$[b_+,c_-]=[b_-,c_+]=2\mathrm{i}\left(a_+d_--a_-d_+\right), \tag{3.6f}$$

$$[a_0,d_0]=2\mathrm{i}\left(b_+c_--b_-c_+\right)=2\mathrm{i}\left(c_-b_+-c_+b_-\right), \tag{3.6g}$$

$$a_0^2-d_0^2=(1-2\mathrm{i})b_-b_++(1+2\mathrm{i})b_+b_-. \tag{3.6h}$$

The 'odd' relations are

$$b_\pm d_\pm = \pm \mathrm{i} a_\pm c_\pm, \tag{3.7a}$$

$$c_\pm d_\pm = \mp \mathrm{i} a_\pm b_\pm, \tag{3.7b}$$

$$[b_\pm, a_0] = \pm \mathrm{i} \{c_\pm, d_0\} = \pm 2\mathrm{i} a_\pm b_\mp + 2 c_\mp d_\pm, \tag{3.7c}$$

$$b_\pm a_0 \mp \mathrm{i} d_0 c_\pm = a_\pm b_\mp \pm \mathrm{i} c_\mp d_\pm, \tag{3.7d}$$

$$[c_\pm, a_0] = \mp \mathrm{i} \{b_\pm, d_0\} = \mp 2\mathrm{i} a_\pm c_\mp + 2 b_\mp d_\pm, \tag{3.7e}$$

$$c_\pm a_0 \pm \mathrm{i} d_0 b_\pm = a_\pm c_\mp \mp \mathrm{i} b_\mp d_\pm. \tag{3.7f}$$

Since a_+ and a_- are central elements, it is interesting to study how the relations above simplify when a_+ and a_- are proportional to the identity operator, $\mathrm{id} \equiv I$. We will assume that $a_\pm \neq 0$. By shifting the variable u one can always make such that $a_- = -a_+$, and we will assume this choice hereafter. By fixing further a_+, one can fix the overall normalization of the solution; below we use $a_+ = 1/2\mathrm{i}$ as in (3.3).

From (3.4), (3.5), and (3.6a) it follows that

$$d_\pm = \nu_\pm a_\pm X, \qquad \nu_\pm^2 = 1,$$

where X is an operator satisfying

$$X^2 = I, \tag{3.8}$$

and

$$\{b_\pm, X\} = \{c_\pm, X\} = [a_0, X] = [d_0, X] = 0. \tag{3.9}$$

The further analysis depends upon whether $\nu_- = \nu_+$ or $\nu_- = -\nu_+$, where without loosing generality one can put $\nu_+ = 1$ (otherwise the sign can be absorbed into the operator X). We will refer to the case $\nu_- = \nu_+$ as the regular solution, since it includes as a particular case the R-matrix itself, see (3.3). The case $\nu_- = -\nu_+$ will be referred to as the irregular solution.

3.2. Regular solution. Using $d_\pm = a_\pm X$ in (3.7a) and (3.7b) one gets

$$c_\pm = \mp \mathrm{i} b_\pm X. \tag{3.10}$$

Relation (3.6c) and the relation given by the first equality in (3.6b) are fulfilled with (3.8) and (3.10). The relations given by the second equality in (3.6b) imply

$$\frac{b_+^2}{a_+} = \frac{b_-^2}{a_-} = a_0 - d_0 X. \tag{3.11}$$

Relation (3.6d) becomes

$$\{b_+, b_-\} = 0. \tag{3.12}$$

Relations (3.6e) boils down to (3.11) and (3.6f) to (3.12). Relation (3.6g) is just

$$[a_0, d_0] = 0$$

and (3.6h) with (3.12) taken into account gives

$$a_0^2 - d_0^2 = 4\mathrm{i} b_+ b_-. \tag{3.13}$$

Relations (3.7c), (3.7d), (3.7e), and (3.7f) simplify to

$$[b_\pm, a_0] = \{b_\pm, d_0\}X = \pm 4\mathrm{i} a_\pm b_\mp \tag{3.14}$$

and

$$b_\pm a_0 = d_0 b_\pm X. \tag{3.15}$$

Note that from the first equality in (3.14) also follows that

$$a_0 b_\pm = -b_\pm d_0 X. \tag{3.16}$$

To complete constructing the solution, we have to find how the operators a_0 and d_0 can be expressed in terms of b_+, b_-, and X. The following properties of the operators b_+ and b_- are crucial for accomplishing this task.

An important observation which can be made from the obtained relations is that b_+^2 and b_-^2 are central elements. To show this one has just to prove that these operators commute with a_0 and d_0. Indeed, acting with $b_\pm$ on (3.15) from the left, moving operator X through $b_\pm$ and using (3.16) gives

$$b_\pm^2 a_0 = b_\pm d_0 b_\pm X = -b_\pm d_0 X b_\pm = a_0 b_\pm^2$$

and similarly we have $b_\pm^2 d_0 = d_0 b_\pm^2$.

Furthermore, the operators b_+ and b_- are not independent from each other, as it follows from the first equality in (3.11) and from (3.12). To tackle this issue more precisely, we now on set $a_- = -a_+$, so that the first equality in (3.11) becomes

$$b_+^2 = -b_-^2. \tag{3.17}$$

Again using $b_\pm X = -X b_\pm$ one can easily find that (3.12) and (3.17) are fulfilled with

$$b_- = \pm b_+ X. \tag{3.18}$$

It turns out that besides (3.18) there may exist other solutions to (3.12) and (3.17), provided that certain conditions are met. In Appendix A we give an example valid in the case where the operators are represented by

4×4 matrices with 2×2 block structure, which shows that in addition to the "trivial" solutions (3.18) there exist "non-trivial" solutions

$$b_- = \pm \left(\mu b_+ X + \nu b_+^{-1} X \right), \tag{3.19}$$

where μ and ν are numerical coefficients. They are functions of the entries of the matrix b_+, i.e., $\mu = \mu(||b_+||)$ and $\nu = \nu(||b_+||)$, see (A.2). Apparently, for (3.19) to exist, b_+ must be invertible; from (3.17) it follows that if this is true for b_+, then b_- is also invertible.

Now we are ready to compute a_0 and d_0. Let us first consider the case where $b_\pm$ are represented by invertible matrices. Rewriting (3.13) as

$$(a_0 - d_0 X)(a_0 + d_0 X) = 4\mathrm{i} b_+ b_-$$

and using (3.11), we get

$$a_0 + d_0 X = 4\mathrm{i} b_+^{-1} b_-.$$

Hence,

$$a_0 = 2\mathrm{i} a_+ b_+^{-1} b_- + \frac{b_+^2}{2a_+}, \qquad d_0 = 2\mathrm{i} a_+ b_+^{-1} b_- X - \frac{b_+^2}{2a_+} X.$$

Setting $a_+ = 1/2\mathrm{i}$ that fix the overall normalization, we thus arrive at the following expression for the L-operator:

$$L(u) = \begin{pmatrix} \sin 2u \cdot I + b_+^{-1} b_- + \mathrm{i} b_+^2 & \mathrm{e}^{\mathrm{i}u} b_+ + \mathrm{e}^{-\mathrm{i}u} b_- \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}u} b_+ X + \mathrm{i}\mathrm{e}^{-\mathrm{i}u} b_- X & \sin 2u \cdot X + b_+^{-1} b_- X - \mathrm{i} b_+^2 X \end{pmatrix}. \tag{3.20}$$

Here, the operators b_+, b_-, and X satisfy

$$\{b_+, b_-\} = b_+^2 + b_-^2 = \{b_\pm, X\} = 0, \qquad X^2 = I,$$

and we also note that $b_+^{-1} b_- = -b_+ b_-^{-1}$.

Let us now consider the case of relation (3.18), without the assumption that b_+ and b_- are invertible. We introduce new operator B by

$$b_+ \equiv B, \qquad b_- = \pm B X. \tag{3.21}$$

To find a_0 and d_0, we take the following ansatz:

$$a_0 = \alpha I + \beta X, \qquad d_0 = \gamma I + \delta X,$$

where α, β, γ, and δ are some functions of B^2, the central element. From (3.14), (3.15), and (3.16) it follows that

$$\beta = \gamma = \pm 2\mathrm{i} a_+, \qquad \delta = -\alpha,$$

and from the second equality in relation (3.11), or from (3.13), one can find

$$\alpha = \frac{B^2}{2a_+}.$$

Finally setting $a_+ = 1/2\mathrm{i}$, we arrive at

$$L(u) = \begin{pmatrix} \sin 2u \cdot I + \mathrm{i}B^2 \pm X & \mathrm{e}^{\mathrm{i}u} B \pm \mathrm{e}^{-\mathrm{i}u} BX \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}u} BX \pm \mathrm{i}\mathrm{e}^{-\mathrm{i}u} B & \sin 2u \cdot X - \mathrm{i}B^2 X \pm I \end{pmatrix}, \tag{3.22}$$

where

$$XB = -BX, \qquad X^2 = I.$$

Note that (3.22) can be obtained from (3.20) upon formally using the substitution defined by (3.21).

3.3. Irregular solution. Setting $d_+ = a_+ X$ and $d_- = -a_- X$ in (3.7a) and (3.7b), where the operator X is defined as above, see (3.8) and (3.9), we get

$$c_\pm = -\mathrm{i}b_\pm X. \tag{3.23}$$

Relations (3.6c) turn into

$$[b_+, b_-] = 0 \tag{3.24}$$

and (3.6d) into

$$\{b_+, b_-\} = 4a_+ a_-. \tag{3.25}$$

Second equality in (3.6b) gives us two relations

$$\frac{b_\pm^2}{a_\pm} = a_0 \mp d_0 X$$

from which we immediately obtain

$$a_0 = \frac{1}{2}\left(\frac{b_-^2}{a_-} + \frac{b_+^2}{a_+}\right), \qquad d_0 = \frac{1}{2}\left(\frac{b_-^2}{a_-} - \frac{b_+^2}{a_+}\right) X.$$

These expressions for a_0 and d_0 together with (3.23), (3.24), and (3.25) make the remaining relations of the algebra, namely, (3.6e)–(3.6h) and (3.7c)–(3.7f), totally fulfilled.

To fix the expression for the L-operator, we set, as above, $a_- = -a_+$ and $a_+ = 1/2\mathrm{i}$. As result, we obtain

$$L(u) = \begin{pmatrix} \sin 2u \cdot I + \mathrm{i}b_+^2 - \mathrm{i}b_-^2 & \mathrm{e}^{\mathrm{i}u} b_+ + \mathrm{e}^{-\mathrm{i}u} b_- \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}u} b_+ X - \mathrm{i}\mathrm{e}^{-\mathrm{i}u} b_- X & -\mathrm{i}\cos 2u \cdot X - \mathrm{i}b_+^2 X - \mathrm{i}b_-^2 X \end{pmatrix}. \tag{3.26}$$

Here,

$$\{b_+, b_-\} = I, \qquad [b_+, b_-] = \{b_\pm, X\} = 0, \qquad X^2 = I.$$

From these relations follows that $b_+b_- = b_-b_+ = \frac{1}{2}I$, hence both b_+ and b_- must be invertible, and no analogue of (3.22) exists in this case. This also means that in (3.26) one can set $b_- = \frac{1}{2}b_+^{-1}$.

4. Conclusion

In this paper, we have considered the Ising model at the critical temperature and studied solutions of the RLL-relation. The starting object is the R-matrix with trigonometric dependence on the spectral variable. Using the ansatz for the L-operator with the similar spectral variable dependence, we have obtained two expressions for the L-operator valid in the case where two central elements of the algebra of commutation relations are proportional to the identity operator. Note that, since the algebra of operators defining entries of the L-operator contains one more central element, b_+^2, the obtained expressions for the L-operator can be multiplied by an arbitrary function (which can also depend on the spectral variable) of b_+^2.

As a further development of the obtained results one could be interested in construction of local spin-chain Hamiltonians. Unlike, e.g., the case of the five-vertex model where the L-operator has a simple spectral variable dependence such that the expansion of the transfer matrix at infinity generates local interaction Hamiltonians [26], for the L-operator considered here no such a simple construction seems to exist. This means that one has to study the corresponding fundamental R-matrix [18]. We intend to address this problem in the sequel.

Another direction of further study could be an extension of the present results to the cases $n \geqslant 3$ of the critical n-state Potts model. These models are known to be trigonometric with an exception of the $n = 4$ case where a rational R-matrix arises [20]. One more interesting problem to be addressed is the role of the star-triangle relation in constructing of new integrable systems. Indeed, the star-triangle relation is a basic ingredient for integrability of the Ising and Potts models and there are evidences that it could be even more fundamental in general than the Yang-Baxter and RLL-relations [27].

Appendix A. An example of matrices b_+ and b_-

Here we give an example of solutions to the matrix system of equations

$$b_+b_- + b_-b_+ = 0, \qquad b_+^2 + b_-^2 = 0, \tag{A.1}$$

where matrices b_+ and b_- are subject to the relations

$$b_\pm X + X b_\pm = 0, \qquad X^2 = I.$$

Consider the case of 4×4 matrices with 2×2 block structure, induced by the choice of the matrix X in the form:

$$X = \operatorname{diag}(1,1,-1,-1).$$

This means that

$$b_+ = \begin{pmatrix} & b \\ a & \end{pmatrix}, \qquad b_- = \begin{pmatrix} & g \\ f & \end{pmatrix},$$

where a, b, f, and g are some 2×2 matrices.

Solving (A.1) for entries of f and g against those of a and b one can find four solutions. Two solutions have the form

$$f = \pm a, \qquad g = \mp b,$$

which apparently describe relation (3.18). Other two solutions are

$$f = \pm(\mu a + \nu b^{-1}), \qquad g = \mp(\mu b + \nu a^{-1}),$$

where

$$\mu = \frac{\operatorname{tr} ab}{\sqrt{(\operatorname{tr} ab)^2 - 4\det ab}}, \qquad \nu = -\frac{2\det ab}{\sqrt{(\operatorname{tr} ab)^2 - 4\det ab}}.$$

These solutions describe relation (3.19) where μ and ν are the following functions of the entries of b_+:

$$\mu = \frac{\operatorname{tr} b_+^2}{\sqrt{\left(\operatorname{tr} b_+^2\right)^2 - 16\det b_+}}, \qquad \nu = -\frac{4\det b_+}{\sqrt{\left(\operatorname{tr} b_+^2\right)^2 - 16\det b_+}}. \tag{A.2}$$

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Steklov Mathematical Institute,
St. Petersburg, Russia
E-mail: a.g.pronko@gmail.com

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Theoretical Physics Department,
St. Petersburg State University
E-mail: s.k.syrygina@gmail.com