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ROOT VECTORS IN QUANTUM GROUPS

Abstract. We propose a definition of root vectors in a finite dimensional quantum group which are compatible with the adjoint action of every quantum Levi subgroup (deliver highest and lowest vectors of finite dimensional submodules). We assign for that role certain entries of reduced quantum Lax matrices associated with the fundamental adjoint module of the quantum group. This study is motivated by the theory of Mickelsson algebras.

1. Introduction

A principal difference between the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ [1] and its Drinfeld-Jimbo quantum counterpart $U_q(\mathfrak{g})$ [2] is the absence of a generating finite dimensional vector subspace $\mathfrak{g}$ that is closed under a bilinear operation (apart from a special case of general linear quantum group). As a consequence, there are no general root vectors whereas a quantum analog of the Cartan subalgebra, $U_q(\mathfrak{h})$, is naturally defined. Although simple root vectors are incorporated in the very definition of $U_q(\mathfrak{g})$ as Chevalley generators, vectors of composite roots are not given for granted unlike in the classical case.

There is a recipe of constructing non-simple root vectors that is due to Lusztig (and Khoroshkin–Tolstoy in the case of quantum supergroups) [2, 3]. The resulting vectors depend on a chosen normal order on the set of positive roots and possess a number of nice properties. Of them the principal two are a) the presence of quantum $\mathfrak{sl}(2)$-pairs that comprise the elements of opposite roots, and b) an analogue of Poincare–Birkhoff–Witt basis in $U_q(\mathfrak{g})$ they generate.

At the same time, there are situations when the standard LKT root vectors are not suitable for the task. Consider a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ (a reductive Lie subalgebra of maximal rank whose simple roots are simple

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for $\mathfrak{g}$) and regard $\mathfrak{g}$ as the adjoint $\mathfrak{l}$-module. Decompose $\mathfrak{g}$ as a direct sum $\mathfrak{g} = \mathfrak{m}_- \oplus \mathfrak{m}_+ \oplus \mathfrak{l}$, where $\mathfrak{m}_\pm$ are nilradicals of the parabolic extensions $\mathfrak{p}_\pm \supset \mathfrak{l}$ relative to the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ (compatible with the quantum group deformation of $U(\mathfrak{g})$. Contrary to the classical case, general LKT root vectors do not span (nor even generate) finite dimensional $U_q(\mathfrak{l})$-submodules under the adjoint action on $U_q(\mathfrak{g})$.

Let us illustrate this on the example of $\mathfrak{g} = \mathfrak{sp}(4)$, where $\mathfrak{l} = \mathfrak{sp}(2)$ is built upon the long simple root β. Denote by α the short simple root. The subspace $\mathfrak{m}_+$ contains a 1-dimensional trivial $\mathfrak{l}$-submodule spanned by the vector

$$e_\delta = [f_\alpha, [f_\alpha, f_\beta]].$$

of a composite root $\delta = 2\alpha + \beta$. The LKT quantum root vectors (there are exactly two, by the number of normal orderings on positive roots) are

$$[e_\alpha, [e_\alpha, e_\beta]_{\pm q^2}], \quad \text{where} \quad [a, b]_c = ab - cba.$$

At the same time, the "root vector" generating trivial $U_q(\mathfrak{l})$-submodules in $U_q(\mathfrak{g}_\pm)$, are

$$e_\delta = [e_\alpha, [e_\alpha, e_\beta]_{q^2}]_{q^{-2}}, \quad f_\delta = [f_\alpha, [f_\alpha, f_\beta]_{q^2}]_{q^{-2}}.$$

Note that e_δ and f_δ do not generate quantum $\mathfrak{sl}(2)$-subalgebras. However, they are still central elements in $U_q(\mathfrak{g}_+)$ and, respectively, $U_q(\mathfrak{g}_-)$. One can easily see that, along with LKT vectors of roots $\pm(\alpha + \beta)$ and the simple root vectors, they generate a PBW basis in $U_q(\mathfrak{g})$, over $U_q(\mathfrak{h})$.

A need in such alternative root vectors arises in various problems of representation theory. To name a few, we would mention the problem of basis in pseudo-parabolic Verma modules [4] and construction of Mickelsson algebras via inverse Shapovalov form. In a recent paper [5] of the second author, such a construction has been is suggested. In the case of classical universal enveloping algebras an explicit expression of the PBW basis is given in terms of the root vectors. For a pair of quantum groups, $U_q(\mathfrak{g}) \supset U_q(\mathfrak{l})$, root vectors are replaced with special entries of (reduced) quantum Lax operators with the quantized adjoint module $\mathfrak{g}$ as an auxiliary vector space. We calculate them using a technique of Hasse diagrams associated with representations of quantum groups.

2. Quantum group and its fundamental adjoint module

Let $\mathfrak{g}$ be a simple Lie algebra of rank n. Denote by R its root system, by R^+ the subset of simple positive roots relative to the fixed polarization

$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ with the Cartan subalgebra $\mathfrak{h}$, and by $\Pi \subset \mathrm{R}^+$ the basis of simple roots.

Let $U_q(\mathfrak{g})$-be the quantum group generated by

$$e_\alpha, \quad f_\alpha, \quad q^{\pm h_\alpha}, \quad \alpha \in \Pi.$$

For the definition of $U_q(\mathfrak{g})$, see [2]. In particular, these elements satisfy commutation relations

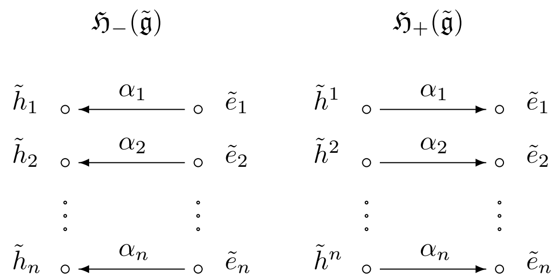
$$q^{h_\alpha} e_\beta = q^{(\alpha,\beta)} e_\beta q^{h_\alpha}, \quad q^{h_\alpha} f_\beta = q^{-(\alpha,\beta)} f_\beta q^{h_\alpha}, \quad [e_\alpha, f_\beta] = \delta_{\alpha,\beta} [h_\alpha]_q,$$

where $[z]_q = \frac{q^{h_\alpha} - q^{-h_\alpha}}{q - q^{-1}}$. Note with care that we are working with a different normalization of f_α as in [2]. That results in the normalization of the commutator $[e_\alpha, f_\alpha]$ and the absence of factor $1/(q_\alpha - q_\alpha^{-1})$ with $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$ in $[h_\alpha]_q$.

Elements $q^{\pm h_\alpha}$ are inverse to each other and generate a quantum version of the Cartan subalgebra $U_q(\mathfrak{h})$. The generators e_α and f_α also satisfy quantum Serre relations whose exact form will not be needed.

Let $\tilde{\mathfrak{g}}$ denote the finite dimensional $U_q(\mathfrak{g})$-module whose highest weight is the maximal root $\xi \in \mathrm{R}^+$. This is a quantum analog of the adjoint $\mathfrak{g}$-module $\mathfrak{g}$. Denote by π the representation homomorphism. The set of weights of $\tilde{\mathfrak{g}}$ is the union $\mathrm{R} \cup \{0\}$. The subspaces $\tilde{\mathfrak{g}}[\alpha]$ of non-zero weight are 1-dimensional, while $\dim \tilde{\mathfrak{g}}[0] = n$. Given a weight basis in $\tilde{\mathfrak{g}}$, we associate with it two Hasse diagram $\mathfrak{H}_\pm(\tilde{\mathfrak{g}})$, where the nodes are elements of the basis. The arrows in $\mathfrak{H}_+(\tilde{\mathfrak{g}})$ are labelled with $\alpha \in \Pi$ and depict the action of e_α. Such an arrow is directed from node j to node i if the matrix entry $\pi(e_\alpha)_{ij}$ is not zero. The arrows in $\mathfrak{H}_-(\tilde{\mathfrak{g}})$ are defined similarly upon replacement of e_α-s with f_α-s. By default, we will mean by $\mathfrak{H}(\tilde{\mathfrak{g}})$ the diagram $\mathfrak{H}_-(\tilde{\mathfrak{g}})$

Fix a non-zero vector $\tilde{e}_\alpha$ in each $\tilde{\mathfrak{g}}[\alpha]$, $\alpha \in \Pi$. We will call them simple root vectors, while the generators of the quantum group will be called Chevalley root vectors. Suppose there are two bases $\{\tilde{h}_i\}_{i=1}^n$ and $\{\tilde{h}^i\}_{i=1}^n$ such that the corresponding parts of Hasse diagrams (of nodes with weights 0 and Π) look as



In other words, the basis vectors of zero weight satisfy

$$f_i\tilde{e}_j = \delta_{ij}\tilde{h}_i, \quad e_i\tilde{h}^j \propto \delta_{ij}\tilde{e}_i.$$

We will call $\{\tilde{h}_i\}_{i=1}^n$ a basis of type I and $\{\tilde{h}^i\}_{i=1}^n$ of type II. This classification will be also extended to bases of the entire $\tilde{\mathfrak{g}}$. The basis elements are defined up to a scalar multiple. In the classical limit, they are scalar multiples of vectors that are dual to, respectively, simple roots and fundamental weights.

Our goal is to calculate certain entries of truncated quantum L-operator in the basis of type II in $\tilde{\mathfrak{g}}$. However some entries of concern are easier to compute in the basis of type I, so we need to know the transition matrix between the two types of bases. It is also worthy to note that elements $\{\tilde{h}^i\}_{i=1}^n$ make sense for us only up to non-zero scalar multipliers.

3. Quantum Cartan matrix

Recall that a classical Cartan matrix A is an invertible $n \times n$-matrix whose entries are integers numbers that are expressed through the inner product on $\mathfrak{h}^*$ by

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad i, j = 1, \ldots, n.$$

A q-version of the Cartan matrix was introduced in [6] in connection with quantum W-algebras. It appears in this presentation as a (part of) transition matrix between bases of types I and II in the $U_q(\mathfrak{g})$-module $\tilde{\mathfrak{g}}$.

We define the q-Cartan matrix by setting

$$A_{ij} = \left[\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right]_{q_i} = \frac{q^{(\alpha_i, \alpha_j)} - q^{-(\alpha_i, \alpha_j)}}{q^{\frac{(\alpha_i, \alpha_i)}{2}} - q^{-\frac{(\alpha_i, \alpha_i)}{2}}}, \quad i, j = 1, \ldots, n.$$

For the four infinite series, the matrices A along with their inverses are given in Appendix.

Given bases $\{\tilde{h}_i\}_{i=1}^n$ and $\{\tilde{h}^i\}_{i=1}^n$ of type I and II, respectively, define the transition matrix $B \in \mathrm{End}(\tilde{\mathfrak{g}}[0])$ via

$$\tilde{h}^i = \sum_{j=1}^n B_{ji}\tilde{h}_j, \quad i = 1, \dots, n.$$

Let $\bar{B}$ be the inverse matrix with entries $\bar{B}_{ij}$.

Introduce an involutive anti-automorphism ω of the algebra $U_q(\mathfrak{g})$ by setting it on the generators as the assignment

$$\omega\colon f_\alpha \to e_\alpha, \quad \omega\colon e_\alpha \to f_\alpha, \quad \omega\colon q^{h_\alpha} \to q^{h_\alpha},$$

Recall that a contravariant (relative to ω) form on a $U_q(\mathfrak{g})$-module V is a symmetric complex valued inner product $\langle -,-\rangle$ satisfying $\langle xv, w\rangle = \langle v, \omega(x)w\rangle$ for all $v, w \in V$ and all $x \in U_q(\mathfrak{g})$. There is a unique contravariant form on the module $\tilde{\mathfrak{g}}$ normalized to $\langle \tilde{e}_\xi, \tilde{e}_\xi\rangle = 1$. We call it canonical.

Lemma 3.1. *Suppose that $\mathfrak{g}$ has rank 2 and is of type $\mathfrak{G}_2$ or $\mathfrak{B}_2$. Then we can pick up $\{\tilde{h}_i\}_{i=1}^2$ and $\{\tilde{h}^i\}_{i=1}^2$ such that B is the inverse q-Cartan matrix. Furthermore, in the case of B_2, the basis $\{\tilde{h}_i\}_{i=1}^2$ descends from simple root vectors $\tilde{e}_i$ of the same squared length relative to the canonical contravariant form on $\tilde{\mathfrak{g}}$.*

Proof. We confine ourselves with the case of B_2 only, as the our main focus is $\mathfrak{g}$ of classical series. Denote by $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = 2\varepsilon_2$ the short and, respectively, the long simple roots expressed through an orthonormal basis $\{\varepsilon_1, \varepsilon_2\} \subset \mathfrak{h}^*$. Set

$$\tilde{e}_1 = f_2 f_1 \tilde{e}_\xi, \quad \tilde{e}_2 = -f_1^2 \tilde{e}_\xi, \tag{1}$$

where and $\xi = 2\varepsilon_1$ is the maximal positive root. It is easy to check that $\langle e_i, e_i\rangle = [2]_q^2$ for both $i = 1, 2$. We will search for $\tilde{h}^i$, $i = 1, 2$, in the form

$$\tilde{h}^1 = B_{11}\tilde{h}_1 + B_{21}\tilde{h}_2,$$
$$\tilde{h}^2 = -\frac{1}{[2]_q}(B_{12}\tilde{h}_1 + B_{22}\tilde{h}_2),$$

where

$$\tilde{h}_1 = f_1\tilde{e}_1, \quad \tilde{h}_2 = f_2\tilde{e}_2.$$

We require that $e_i\tilde h^j=\delta_i^j\tilde e_i$, which gives rise to the system of equations

$$\begin{aligned}
e_1\tilde h^1&=([2]_qB_{11}-[2]_qB_{21})\tilde e_1=\tilde e_1,\\
e_2\tilde h^1&=[2]_q(-B_{11}+B_{21}[2]_{q^2})\tilde e_2=0,\\
e_1\tilde h^2&=([2]_qB_{12}-[2]_qB_{22})\tilde e_1=0,\\
e_2\tilde h^2&=(-B_{12}+B_{22}[2]_{q^2})\tilde e_2=\tilde e_2,
\end{aligned}$$

or, in the matrix form,

$$\begin{pmatrix}[2]_q & -[2]_q\\ -1 & [2]_{q^2}\end{pmatrix}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

This proves the first statement for $\mathfrak{g}$ of type B_2 if we replace $\tilde h^2$ with $-[2]_q^{-1}\tilde h^2$. The second statement holds because $\tilde e_i$ have equal squared norm. □

Remark that we need to know $\tilde h^i$ up to a scalar multiple, in contrast to $\tilde h_i$. Observe that $\{\tilde h_i\}$ are related with $\{\tilde e_i\}$ by

$$e_i\tilde h_k=A_{jk}\tilde e_j, \tag{2}$$

which is the key identity to secure.

Our next goal is to extend Lemma 3.1 to general $\mathfrak{g}$.

Proposition 3.2. *There exists a pair of type I and II bases related by the q-Cartan matrix. The basis of type I descends from simple root vectors $\{\tilde e_i\}$ of equal squared norm $\langle\tilde e_i,\tilde e_i\rangle$. Such pair is unique up to a common scalar multiplier.*

Proof. Two bases of same type may differ only by a diagonal transition matrix. Then uniqueness follows from the following fact: the only diagonal matrix C that is A-conjugated to a diagonal matrix is scalar. Let us proceed to the proof of existence.

We will do induction on rank n. The statement is obvious for $n=1$, and we have already checked it for the case $n=2$ (with connected Dynkin diagrams). Notice that every Dynkin diagram D_n of rank $n>2$ can be obtained from a diagram D_{n-1} by appending a root α_1 that is orthogonal to all roots from D_{n-1} except for, say, α_2 of the same squared length. In other words, α_1 and α_2 generate a root system of type $\mathfrak{A}_2$. We may assume that $(\alpha_1,\alpha_1)=2=(\alpha_2,\alpha_2)$ and $(\alpha_1,\alpha_2)=-1$; in this normalization $q=q_1=q_2$.

Suppose that we have already constructed the bases $\{\tilde{h}^i\}_{i=2}^n$, and $\{\tilde{h}_i\}_{i=2}^n$. Let A_{n-1} be the inverse to the transition matrix satisfying $e_i\tilde{h}_k = A_{ik}\tilde{e}_i$ for $i,k = 2,\dots,n$, see (2). By induction assumption, it is the q-Cartan matrix of the diagram D_{n-1}. We are going to extend it to a q-Cartan matrix A_n.

Construct the element $\tilde{h}_1$ by setting

$$\tilde{h}_1 = f_1\tilde{e}_1, \quad \text{where} \quad \tilde{e}_1 = -f_2e_1\tilde{e}_2. \tag{3}$$

Notice that $e_2e_1\tilde{e}_2 = 0$ because there is no root $\alpha_2 + 2\alpha_1$ in R^+. Also, $f_1\tilde{e}_2 = 0$ as $\alpha_2 - \alpha_1$ is not a root. Then

$$\begin{aligned} e_2\tilde{h}_1 &= -e_2f_1f_2e_1\tilde{e}_2 = -f_1e_2f_2e_1\tilde{e}_2 = -f_1[h_2]_qe_1\tilde{e}_2 = -f_1e_1[h_2-1]_q\tilde{e}_2 \\ &= -f_1e_1\tilde{e}_2 = [h_1]_q\tilde{e}_2 = -\tilde{e}_2. \end{aligned}$$

Furthermore, since $e_1\tilde{e}_1 = 0$, we have

$$e_1\tilde{h}_1 = e_1f_1\tilde{e}_1 = [h_1]_q\tilde{e}_1 = [(\alpha_1,\alpha_1)]_q\tilde{e}_1 = [2]_q\tilde{e}_1.$$

Finally, $e_1\tilde{h}_2 = e_1f_2\tilde{e}_2 = f_2e_1\tilde{e}_2 = -\tilde{e}_1$. This proves of the first statement because (2) is fulfilled. Now let us calculate the squared norm of $\tilde{e}_2$:

$$\langle\tilde{e}_1,\tilde{e}_1\rangle = \langle f_2e_1\tilde{e}_2, f_2e_1\tilde{e}_2\rangle = \langle e_1\tilde{e}_2, e_2f_2e_1\tilde{e}_2\rangle = \langle e_1\tilde{e}_2, [h_2]e_1\tilde{e}_2\rangle$$

because $e_2e_1\tilde{e}_2 = 0$. We continue as

$$= \langle e_1\tilde{e}_2, e_1\tilde{e}_2\rangle = \langle\tilde{e}_2, f_1e_1\tilde{e}_2\rangle = -\langle\tilde{e}_2, [h_1]_q\tilde{e}_2\rangle = \langle\tilde{e}_2,\tilde{e}_2\rangle,$$

as required. Here we again have used $f_1\tilde{e}_2 = 0$. The second statement has been proved. □

For $\mathfrak{g} \neq G_2$, the unit squared norm of simple root vectors fixes the pair of bases up to a common sign. We call them canonical, as well as the basis of simple root vectors $\tilde{e}_i$ that are parent to $\tilde{h}_i$. Observe that fixing the norm of $\tilde{e}_i$ determines it up to a sign, so it remains to choose the signs coherently.

It is clear from the proof that Proposition 3.2 is equivalent to the following.

Proposition 3.3. *There exists a unique, up to a common factor, basis of simple roots $\{\tilde{e}_\alpha\}_{\alpha\in\Pi}$ in $\tilde{\mathfrak{g}}$ such that*

$$e_\beta f_\alpha\tilde{e}_\alpha = A_{\beta\alpha}\tilde{e}_\beta, \quad \forall\alpha,\beta\in\Pi.$$

Unless $\mathfrak{g}$ is $\mathfrak{G}_2$, it can be chosen orthonormal relative to the canonical contravariant form.

4. Canonical root vectors

In this section we construct basis elements in the non-negative part of $\tilde{\mathfrak{g}}$. In particular, we need to ensure that vectors $\tilde{e}_\alpha \in \tilde{\mathfrak{g}}$ of weight $\alpha \in \Pi_\mathfrak{g}$ are normalized by the condition $\langle \tilde{e}_\alpha, \tilde{e}_\alpha \rangle = 1$. This is a necessary condition for transition between bases $\{\tilde{h}_\alpha\}$ and $\{\tilde{h}^\alpha\}$ via the inverse q-Cartan matrix.

We will be using Hasse diagrams associated with $U_q(\mathfrak{g}_-)$-modules. As was agreed, by $\mathcal{H}(V)$ we mean the diagram $\mathcal{H}_-(V)$ whose arrows designate the action of negative simple root vectors.

Pick up the highest vector $\tilde{e}_\xi \in \tilde{\mathfrak{g}}$. Other root vectors are constructed via the following process. The Hasse sub-diagram in $\mathfrak{H}(\tilde{\mathfrak{g}}_+)$ that comprises all paths from e_ξ to e_1 determines every vector in between up to a scalar multiplier, as a monic monomial in f_i applied to $\tilde{e}_\xi$. We set all those multipliers to 1. Then we pass to the subalgebra in $\tilde{\mathfrak{g}}$ of co-rank 1 without root α_1. Its highest vector is obtained from $\tilde{e}_\xi$ as

$$f_1\tilde{e}_\xi \quad \text{for} \quad \mathfrak{g} = \mathfrak{gl}(N), \qquad f_1^2\tilde{e}_\xi \quad \text{for} \quad \mathfrak{g} = \mathfrak{sp}(N), \qquad f_1f_2\tilde{e}_\xi \quad \text{for} \quad \mathfrak{g} = \mathfrak{so}(N).$$

These two processes yield a weight basis in $\tilde{\mathfrak{g}}_-$. Next we will calculate normalization coefficients of $\tilde{e}_\alpha$ in each type of $\mathfrak{g}$ separately.

Proposition 4.1. *Let the highest vector $\tilde{e}_\xi$ be normalized to $\langle \tilde{e}_\xi, \tilde{e}_\xi \rangle = 1$. Then*

$$\tilde{e}_k = (-1)^{k+1}(f_{k+1}\dots f_n)(f_{k-1}\dots f_1)\tilde{e}_\xi, \quad \mathfrak{g} = \mathfrak{gl}(n+1),$$

$$\tilde{e}_k = \frac{(-1)^{k+1}}{[2]_q^k}(f_{k+1}\dots f_n \dots f_k)(f_{k-1}^2\dots f_1^2)\tilde{e}_\xi, \quad \mathfrak{g} = \mathfrak{sp}(2n),$$

$$\tilde{e}_k = \frac{(-1)^{k}}{[2]_q^{k-1}}(f_{k+1}\dots f_n f_n \dots f_{k+1})(f_{k-1}f_k)\dots(f_1f_2)\tilde{e}_\xi, \quad \mathfrak{g} = \mathfrak{so}(2n+1),$$

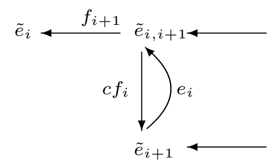
$$\tilde{e}_k = \frac{(-1)^{k+(n-1)\delta_{kn}}}{[2]_q^{k-1}}(f_{k+1}\dots f_{n-2}f_n f_{n-1} \dots f_{k+1})(f_{k-1}f_k)\dots(f_1f_2)\tilde{e}_\xi, \quad \mathfrak{g} = \mathfrak{so}(2n),$$

for $k = 1, \dots, n$, form a canonical basis of positive simple root vectors in $\tilde{\mathfrak{g}}$.

Proof. It is straightforward to check that $\langle \tilde{e}_k, \tilde{e}_k \rangle = 1$ for all $k = 1, \dots, n$.

Let us explain the alternating signs. We define non-normalized $\tilde{e}_\alpha$ via certain paths in the Hasse diagram $\mathfrak{H}_-(\tilde{\mathfrak{g}}_+)$ starting from the highest vector $\tilde{e}_\xi$ and assuming that each arrow $\stackrel{\alpha}{\longleftarrow}$ adds a factor of f_α. One can check that arrows that fall beyond the union of these paths add the factor of cf_α

with some $c \in \frac{1}{[\mathbb{N}]_q}$. In particular, the procedure of appending a root to a root system of smaller rank can be described by the following piece of the Hasse diagram.



The (non-normalized) simple root vectors $\tilde{e}_{i+1}$ and $\tilde{e}_i$ here are defined via horizontal arrows. The numerical coefficient c is calculated to be positive. The arched arrow e_i is the lift from (3), where $i = 1$. Thus the non-normalized vector $\tilde{e}_i$ is to change the sign as in (3). Examining (1) we conclude that the rule of signs applies to all n, including $n = 2$. □

5. Root vectors in quantum groups

A description [5] of Mickelsson algebras associated with pairs of quantum groups $U_q(\mathfrak{g}) \supset U_q(\mathfrak{l})$ [7, 8]is using two objects: an $U_q(\mathfrak{l})$-invariant element $\Psi_X \in X \otimes U_q(\mathfrak{g})$, where X is an irreducible submodule in $\tilde{\mathfrak{g}}/\tilde{\mathfrak{l}}$, and the Shapovalov matrix $S \in \mathrm{End}(X) \otimes U_q(\mathfrak{b}_+)$. Constructing the element Ψ is equivalent to constructing a $U_q(\mathfrak{l})$-submodule $X^* \subset U_q(\mathfrak{g})$ that is the left dual to X. The module X lies either in $\tilde{\mathfrak{g}}_-$ or in $\tilde{\mathfrak{g}}_+$, then X^* is in $U_q(\mathfrak{g}_+)$ or, respectively, in $U_q(\mathfrak{g}_-)$. In the classical situation such modules lie within the Lie algebra $\mathfrak{g}$ and are generated by certain root vectors of lowest and highest weights. With $q \neq 1$, we need to define the appropriate root vectors as their deformations. Unfortunately the standard root vectors of Lusztig-Khoroshkin–Tolstoy are not up suitable in general.

It turns out that the role of such vectors can be given to certain entries of (reduced) Lax operators $L^{\pm}$ of the of the quantum group $U_q(\mathfrak{g})$ associated with the auxiliary module $\tilde{\mathfrak{g}}$.

Introduce a Hopf algebra structure on $U_q(\mathfrak{g})$ with a comultiplication

$$\Delta(f_\alpha) = f_\alpha \otimes 1 + q^{-h_\alpha} \otimes f_\alpha, \quad \Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha},$$
$$\Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha$$

defined on the generators. We will also consider a comultiplication

$$\tilde{\Delta}(f_\alpha) = f_\alpha \otimes 1 + q^{h_\alpha} \otimes f_\alpha, \quad \tilde{\Delta} q^{\pm h_\alpha} = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha},$$
$$\tilde{\Delta}(e_\alpha) = e_\alpha \otimes q^{-h_\alpha} + 1 \otimes e_\alpha,$$

for all $\alpha \in \Pi$.

Let $\mathcal{R}$ denote a quasitriangular structure or $U_q(\mathfrak{g})$ relative to Δ and set $\check{\mathcal{R}} = q^{-\sum_i h_i \otimes h_i}\mathcal{R}$, where $\{h_i\}_{i=1}^n$ is an orthonormal basis in $\mathfrak{h}$. We choose $\mathcal{R}$ such that $\check{\mathcal{R}} \in U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{g}_-)$ (a completed tensor product). The element $\check{\mathcal{R}}$ is a twist that relates the two comultiplications:

$$\check{\mathcal{R}}\Delta(u) = \tilde{\Delta}(u)\check{\mathcal{R}}, \quad \forall u \in U_q(\mathfrak{g}). \tag{4}$$

The matrix $\check{\mathcal{R}}$ plays a central role in this exposition.

Fix a $U_q(\mathfrak{g})$-module V with representation homomorphism $\pi\colon U_q(\mathfrak{g}) \to \mathrm{End}(V)$ and consider the matrices

$$L_V^- = (\pi \otimes \mathrm{id})(\check{\mathcal{R}}^{-1}) \in \mathrm{End}(V) \otimes U_q(\mathfrak{g}_-),$$
$$L_V^+ = (\mathrm{id} \otimes \pi)(\check{\mathcal{R}}) \in U_q(\mathfrak{g}_+) \otimes \mathrm{End}(V).$$

We call them (reduced or truncated) quantum Lax operators. In a chosen weight basis $\{v_i\} \subset V$ the matrix entries $L_{ij}^{\pm}$ will be also denoted by $L^{\pm}(v_i, v_j)$.

Suppose that $V^{\mathfrak{l}} \neq \{0\}$ and pick up a $U_q(\mathfrak{l})$-invariant non-zero weight vector $v_0 \in V^{\mathfrak{l}}$. Define

$$\psi_i^- = L_{i0}^- = L^-(v_i, v_0), \quad \psi_i^+ = L^+(v_i, v_0)q^{h_{\nu_i}} = L_{i0}^+ q^{h_{\nu_i}},$$

where ν_i are the weights of v_i. These elements of $U_q(\mathfrak{g})$ carry weights $\mathrm{wt}(\psi_i^-) = \nu_0 - \nu_i < 0$ and $\mathrm{wt}(\psi_i^+) = \nu_0 - \nu_i > 0$. The weight ν_0 is orthogonal to $\Pi_{\mathfrak{l}}$. It is shown in [5] that non-zero elements $\psi_{i0}^{\pm}$ span finite dimensional $U_q(\mathfrak{l})$-modules with respect to the adjoint action on $U_q(\mathfrak{g})$. In order to describe them, we need to evaluate their lowest and highest vectors.

Define a map $\mathrm{R}_{\mathfrak{g}}^+ \to \Pi_{\mathfrak{g}}$, $\alpha \mapsto \alpha'$, as follows:

$$\begin{array}{ccccc}
2\varepsilon_i & \to & \alpha_i' & & \mathfrak{g} = \mathfrak{sp}(2n), \\
\varepsilon_i + \varepsilon_{i+1} & \to & \alpha_{i+1}', & & \mathfrak{g} = \mathfrak{so}(N), \\
\varepsilon_i + \varepsilon_j & \to & \alpha_i' & i < j,\ \mathfrak{g} = \mathfrak{sp}(N), & \text{and} \quad i < j-1,\ \mathfrak{g} = \mathfrak{so}(N), \\
\varepsilon_i - \varepsilon_j & \to & \alpha_i', & & i < j,\ \mathfrak{g} = \mathfrak{gl}(N), \mathfrak{so}(N), \mathfrak{sp}(N), \\
\varepsilon_i & \to & \alpha_i' & & \mathfrak{g} = \mathfrak{so}(2n+1).
\end{array}$$

Here ε_i, $n = 1, \ldots, n$, form the standard orthonormal basis of weights. By construction, this assignment is identical on $\Pi \in \mathrm{R}^+$.

Let $\{h^\alpha\}_{\alpha \in \Pi}$ be the basis in the Cartan subalgebra of $\mathfrak{g}$ that is dual to $\{h_\alpha\}_{\alpha \in \Pi}$.

Lemma 5.1. *Suppose that $\mathfrak{g}$ is of non-exceptional type. Let $\mathfrak{l} \subset \mathfrak{g}$ be a semi-simple Lie algebra with $\Pi_{\mathfrak{l}} \subset \Pi$. Suppose that $\beta \subset \mathrm{R}^+$ is the lowest weight of an $\mathfrak{l}$-submodule in $\mathfrak{m}^+ = \mathfrak{g}_+/\mathfrak{l}_+$. Then $\beta' \notin \Pi_{\mathfrak{l}}$ and $(\omega_{\beta'}, \beta) \neq 0$. As a consequence, $h^{\beta'} \subset \mathfrak{g}[0]$ is $\mathfrak{l}$-invariant.*

Proof. A direct examination. □

The condition $(\omega_{\beta'}, \beta) \neq 0$ implies that the lowest vector e_β of a $\mathfrak{l}$-submodule in $\mathfrak{g}$ is obtained as a matrix entry of a "classical Lax operator" $L^+ = \sum_{\alpha \in \mathrm{R}^+} \pi(f_\alpha) \otimes e_\alpha$.

Now we proceed to the quantum version $\tilde{\mathfrak{g}}$ of the adjoint module $\mathfrak{g}$. Consider an irreducible decomposition

$$\tilde{\mathfrak{m}}_\pm = \sum_i X_i^\pm$$

and let $\mu_i \in \mathrm{R}^+$ be the lowest weight in X_i^+. Denote by $\Lambda(X_i)$ the set of weights in X_i (they all are of multiplicity 1).

Proposition 5.2. *The vector subspace in $U_q(\mathfrak{g})$ spanned by matrix entries*

$$X_i^- = \mathrm{Span}\{L^-(\tilde{e}_\mu, \tilde{h}^{\mu_i'})\}_{\mu \in \Lambda(X_i^+)} \subset U_q(\mathfrak{g}_-),$$
$$X_i^+ = \mathrm{Span}\{L^+(\tilde{f}_\mu, \tilde{h}^{\mu_i'})q^{-h_\mu}\}_{\mu \in \Lambda(X_i^+)} \subset U_q(\mathfrak{g}_+)U_q(\mathfrak{h}_{\mathfrak{g}}),$$

is a $U_q(\mathfrak{l})$-submodule with respect to adjoint action on $U_q(\mathfrak{g})$.

It is clear that $X_i^\pm \simeq (X_i^\mp)^*$. In order to find the highest/lowest vectors of these submodules, we need to calculate the matrix entries

$$F_\alpha = L^-(\tilde{e}_\alpha, \tilde{h}^{\alpha'}), \quad E_\alpha = L^+(\tilde{f}_\alpha, \tilde{h}^{\alpha'}), \quad \forall \alpha \in \mathrm{R}^+_{\mathfrak{g}}.$$

That will be done with the use of a technique based on Hasse diagrams associated with representations of quantum groups.

6. Hasse diagrams associated with $U_q(\mathfrak{g}_-)$-modules

Hasse diagrams are associated with any partially ordered set: the nodes are elements of the set and arrows connect two ordered nodes if there is no other nodes in between. We will use Hasse diagrams associated with modules over $U_q(\mathfrak{g})$ in order to calculate quantum Lax matrix. There are two possibilities to assign a Hasse diagram to a module depending on wether we consider negative or positive operators as arrows. In this paper, we will focus on diagrams associated with $U_q(\mathfrak{g}_-)$-modules only, which we use for calculation of L^-. The matrix L^+ can be obtained from L^- by a transformation which we describe below. Of course, it can be calculated directly via a similar algorithm as for L^- but with the use of Hasse diagrams relative to the $U_q(\mathfrak{g}_+)$-action.

Let V be an irreducible finite dimensional $U_q(\mathfrak{g})$-module with representation homomorphism π. Denote by V' the $U_q(\mathfrak{g})$-module V twisted by σ, with the representation homomorphism $\pi' = \pi \circ \sigma$. It is isomorphic to the dual module V^*.

Put

$$\bar{L}^-_{V'} = (\pi' \otimes \mathrm{id})(\check{\mathcal{R}}) = (L^-_{V'})^{-1} \in \mathrm{End}(V') \otimes U_q(\mathfrak{g}_-).$$

Lemma 6.1. *The matrices $L^\pm$ are related by the equality $(\sigma \otimes \mathrm{id})(L^+_V) = (\bar{L}^-_{V'})_{21}$.*

Proof. The Chevalley involution flips the R-matrix: $(\sigma \otimes \sigma)(\mathcal{R}) = \mathcal{R}_{21}$ as it is a coalgebra anti-automorphism. Then

$$(\sigma \otimes \mathrm{id})(L^+_V) = (\sigma \otimes \pi)(\check{\mathcal{R}}) = (\mathrm{id} \otimes \pi')(\check{\mathcal{R}}_{21}) = (\bar{L}^-_{V'})_{21},$$

as required. □

Next we remind the basics of Hasse diagrams that are important for this presentations. The reader is referred to [9]. It will be convenient to orient the diagrams so that arrows are directed leftwards. We construct the Hasse diagram $\mathfrak{H}(V)$ as follows. Pick up a basis of elements $v_i \in V$ carrying weights ν_i, $i \in I$. We will also identify the nodes with elements of the index set I. Denote by π the representation homomorphism $U_q(\mathfrak{g}_+) \to \mathrm{End}(V)$. Arrows in $\mathfrak{H}(V)$ are simple roots $\alpha \in \Pi$; we set $j \stackrel{\alpha}{\longleftarrow} i$ if $\pi(f_\alpha)_{ji} \neq 0$. It follows that the weight ν_i and ν_j of the nodes satisfy the equality $\nu_i - \nu_j = \alpha$. We write $j \prec i$ if the nodes j and i can be connected by a sequence of arrows from i to j and call such sequence a path. Any ordered sequence of nodes $j = j_1 \prec j_1 \prec \ldots \prec j_k = i$ is called a route from i to j.

The diagram $\mathfrak{H}(V)$ depends on the choice of basis. Let us demonstrate that on the example of $V = \tilde{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sl}(4)$. A part of the diagram corresponding to for the positive Borel subalgebra in $\tilde{\mathfrak{g}}$ is displayed on the Figure 1.

Given a diagram $\mathfrak{H}(V)$ one can define a "derived" diagram corresponding to the module $V \otimes V^*$. As the transition from V to $V \otimes V^*$ complicates the diagram, it is convenient to stay within $\mathfrak{H}(V)$ and consider pairs of nodes (j, i). We will restrict our consideration to only pairs with $j \preccurlyeq i$ and

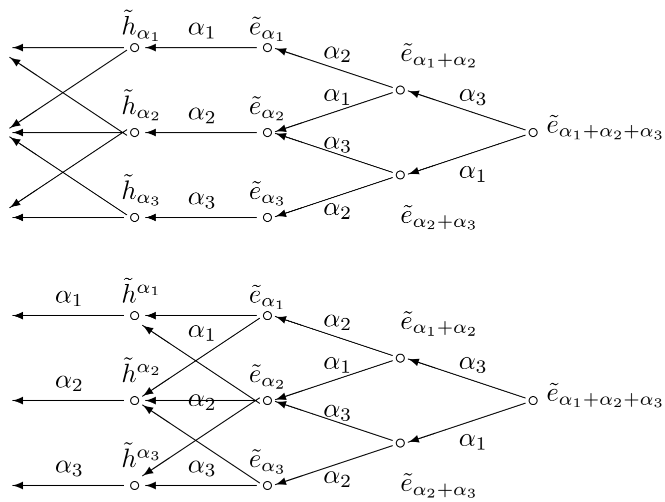


Figure 1. Hasse diagrams for $\mathfrak{sl}(4)$.

denote such a diagram by $\mathfrak{H}(V,V)$. An α-arrow in $\mathfrak{H}(V,V)$ that starts at $(j,i) \in \mathfrak{H}(V,V)$ is either $\stackrel{\alpha}{\longleftarrow} j$ or $i \stackrel{\alpha}{\longleftarrow}$ viewed in $\mathfrak{H}(V)$.

Definition 6.2. A node $i \in \mathfrak{H}(V)$ is said to have α-branching if there are more the one α-arrows outgoing from i.

An example of branching can be seen at $\tilde{\mathfrak{h}}_{\alpha_i}$ in the upper diagram and at $\tilde{e}_{\alpha_i}$ in the lower diagram in Figure 1.

The concept of branching is interesting for us when applied to derived diagrams $\mathfrak{H}(V,V)$. In terms of $\mathfrak{H}(V)$, an α-branching at a node $(j,i) \in \mathfrak{H}(V,V)$ means the presence of a subdiagram

$$\stackrel{\alpha}{\longleftarrow} j \longleftarrow \ldots \longleftarrow i \stackrel{\alpha}{\longleftarrow}$$

With every pair $(j,i) \in \mathfrak{H}(V,V)$ we associate the matrix entry $L^-_{ij} \in U_q(\mathfrak{g}_-)$. Write down the intertwining relation (4) as an equation for L^- in the coordinate form:

$$\sum_{k\in I} \pi(f_\alpha)_{ik} L^-_{kj} - \sum_{k\in I} L^-_{ik}\pi(f_\alpha)_{kj} = q^{(\alpha,\nu_j)} L^-_{ij} f_\alpha - q^{-(\alpha,\nu_i)} f_\alpha L^-_{ij}, \quad \forall i,j \in I, \quad \alpha \in \Pi_{\mathfrak{g}}. \tag{5}$$

It relates the matrix entry L^-_{ij} of weight $\nu_j - \nu_i$ with entries of weight $\nu_j - \nu_i - \alpha$. If the sums in the left-hand-side contract to just one term, $\pi(f_\alpha)_{ik} L^-_{kj}$ or $-L^-_{ik}\pi(f_\alpha)_{kj}$, the corresponding matrix entry is proportional to a q-commutator of L^-_{ij} with f_α. This is exactly the case when the node (j,i) has no α-branching in $\mathfrak{H}(V,V)$. Graphically the matrix entry L^-_{ij} is "inflated" in one of the two directions:

$$\text{either}\quad L^-_{ik}\colon k \stackrel{\alpha}{\longleftarrow} j \stackrel{L^-_{ij}}{\dashleftarrow} i, \quad \text{or} \quad L^-_{kj}\colon j \stackrel{L^-_{ij}}{\dashleftarrow} i \stackrel{\alpha}{\longleftarrow} k,$$

by taking the q-commutator with f_α and multiplying the result by a scalar.

In particular, if there is no α-branching at a pair (i,i), we get

$$\pi(f_\alpha)_{ik} L^-_{ki} \quad \text{or} - L^-_{ik}\pi(f_\alpha)_{ki} = (q^{(\alpha,\nu_j)} - q^{-(\alpha,\nu_i)}) f_\alpha, \quad \alpha \in \Pi_{\mathfrak{g}}, \tag{6}$$

which gives the entries of L^- for the simple pairs of nodes in V connected with an α-arrow. Remark that for a simple pair of nodes (j,i), that is, $\nu_i - \nu_j = \alpha \in \Pi$, the entry L_{ij} is always proportional to f_α. This follows from an explicit formula for the universal R-matrix, see e.g. [3].

Given that the matrix elements of L^- are known for simple pairs of nodes, one can recursively construct other L^-_{ij} provided there is a path from a simple pair to (i,j) without branching.

Definition 6.3. We call the diagram $\mathfrak{H}(V,V)$ solvable if for every pair (j,i) of nodes in $\mathfrak{H}(V)$ such that $j \prec i$ there is $k \in \mathfrak{H}(V)$, $j \preccurlyeq k \preccurlyeq i$, and a path from (k,k) to (j,i) without branching.

Now set $V = \tilde{\mathfrak{g}}$ and choose a basis $\tilde{h}_\alpha \subset \tilde{\mathfrak{g}}[0]$ of type I. Let ξ denote the maximal root vector $\xi \in \mathrm{R}^+$. Note with care that specification of a path $j \dashleftarrow i$ in $\mathfrak{H}(V)$ does not fix a path in $\mathfrak{H}(V,V)$ with the terminating node (j,i). Practically we will use either (i,i) or (j,j) as the start node and inflate accordingly either along or, respectively against the orientation of $j \dashleftarrow i$.

Lemma 6.4. *For each $\alpha \in \Pi$ there is a path*

$$(\tilde{h}_\alpha, \tilde{e}_\xi) \longleftarrow \ldots \longleftarrow (\tilde{e}_\xi, \tilde{e}_\xi)$$

without branching.

Proof. Any path from $\tilde{e}_\xi$ to $\tilde{h}_\alpha$ in the diagram $\mathfrak{H}(\tilde{\mathfrak{g}})$ has a form

$$\tilde{h}_\alpha \stackrel{\alpha}{\longleftarrow} \tilde{e}_\alpha \longleftarrow \ldots \longleftarrow \tilde{e}_{\xi-\nu} \stackrel{\nu}{\longleftarrow} \tilde{e}_\xi$$

for some $\nu \in \Pi$. Furthermore, there are no incoming arrows at $\tilde{e}_\xi$ because it is the highest vector in $\tilde{\mathfrak{g}}$. On the other hand, for each $\beta \in \Pi$, there is at most one outgoing β-arrow at all $\tilde{e}_\mu$, $\mu \in \mathrm{R}^+$. For non-semisimple μ that is because $\dim \tilde{\alpha}[\mu] = 1$. For $\mu = \alpha$, the displayed arrow is the only one outgoing from $\tilde{e}_\alpha$, by definition of $\tilde{h}_\alpha$. □

Now let us change the basis in $\tilde{\mathfrak{g}}[0]$ to $\{h^\alpha\}_{\alpha\in\Pi}$. The previous lemma is no longer valid in general if we replace $\tilde{h}_\alpha$ with $\tilde{h}^\alpha$ (it remains true for $\mathfrak{g} = \mathfrak{gl}(n)$ though). The problematic are simple root vectors $\tilde{e}_\mu$ because the are multiple arrows $\tilde{h}^\alpha \overset{\mu}{\longleftarrow} \tilde{e}_\mu$ for each $\alpha \in \Pi$, see Figure 1. However we can fix it for some pairs of nodes if we change the direction of inflation.

Lemma 6.5. *Suppose that a simple root α enters a positive root β with multiplicity* 1*. Then any path*

$$(\tilde{h}^\alpha, \tilde{e}_\beta) \longleftarrow \ldots \longleftarrow (\tilde{h}^\alpha, \tilde{e}_\alpha)$$

in $\mathfrak{H}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$ has no branching.

Proof. Indeed, the only outgoing arrow at the node $\tilde{h}^\alpha$ in the diagram $\mathfrak{H}(\mathfrak{g})$ is $\tilde{f}_\alpha \overset{\alpha}{\longleftarrow} \tilde{h}^\alpha$. If we start from the node $(\tilde{h}^\alpha, \tilde{e}_\alpha)$, there will be no arrow $\overset{\alpha}{\longleftarrow}$ in the path $\tilde{e}_\alpha \longleftarrow \ldots \longleftarrow \tilde{e}_\beta$ anymore because α enters β only once. □

Note that, unlike in Lemma 6.4, in Lemma 6.5 we start with the opposite end of the path

$$\tilde{h}_\alpha \overset{\alpha}{\longleftarrow} \tilde{e}_\alpha \longleftarrow \ldots \longleftarrow \tilde{e}_\beta$$

in $\mathfrak{H}(\tilde{\mathfrak{g}})$ and inflate against the direction of the path.

In view of Proposition 5.2, we need to calculate matrix entries of L^- in the basis of type II. Lemma 6.5 provides such an algorithm for all roots if $\mathfrak{g}$ is general linear and "almost all" otherwise. Lemma 6.4 allows to calculate the remaining matrix entries using the relation between the two types of bases via the q-Cartan matrix. Matrix entries covered by Lemma 6.5 turn out that to be LKT root vectors, as shown in the next section.

Once we have calculated L^-, we readily find L^+ from L^- thanks to Lemma 6.1. The involution σ gives rise to a linear isomorphism that takes the irreducible module of highest weight λ to its dual module of lowest weight $-\lambda$. Then $(\sigma \otimes \sigma)(L^+) = \bar{L}^-$ implies

$$L^+_{\sigma(i),\sigma(j)}(q) = \sigma(L^-_{i,j}(q^{-1})). \tag{7}$$

Indeed, the intertwining relations for $\bar{L}^-$ and L^- go over to each other under the flip $q \to q^{-1}$. Furthermore, the basis of type II in $\tilde{\mathfrak{g}}$ is taken by σ

to a basis of the same type. Then the matrix entries of L^+ we are concerned with are obtained from the entries of L^- by the transformation (7).

7. Lax operators and standard LKT root vectors

In this section, we use the weight basis in $\tilde{\mathfrak{g}}_-$ constructed in Section 4 via the Hasse diagrams. We do not care of its normalization unless otherwise is stated. That is redundant when we are able to compute the entries of the L-matrix directly in the basis of type II. We only need it when proceeding from the basis $\{\tilde{h}_i\}$ to $\{\tilde{h}^i\}$ as in the next Section.

In what follows, we use the notation $i' = N+1-i$, where N is the dimension of the natural representation of the classical Lie algebra $\mathfrak{g}$. Our calculations are based on Lemma 6.5 and the intertwining relation (5). Suppose that $i<j<i'$ and, in the case of $\mathfrak{g}=\mathfrak{so}(N)$, $j\neq i'-1$. Let us mark the basis element $\tilde{h}^i$ with (i,i), $i=1,\dots,n$. We denote by $F_{i,j}$ the matrix entry $L^-_{ij,ii}$, relative to the basis of type II.

Proposition 7.1. *Up to a non-zero scalar multiplier the element F_{ij} equals*

$$F_{i,j} = [\dots[f_i, f_{i+1}]_q \dots, f_{j-1}]_q$$

if $j\leqslant n$ for $\mathfrak{g}=\mathfrak{sp}(2n),\mathfrak{so}(2n)$, and $j\leqslant n+1$ for $\mathfrak{g}=\mathfrak{so}(2n+1)$,

$$F_{i,j'} = [\dots[F_{i,n'}, f_{n-1}]_q \dots, f_j]_q$$

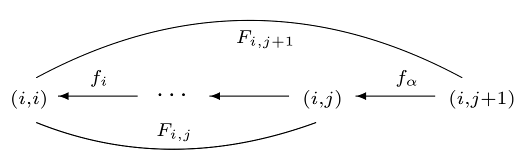
if $j<n$ for $\mathfrak{g}=\mathfrak{sp}(2n),\mathfrak{so}(2n)$, and $j<n+1$ for $\mathfrak{g}=\mathfrak{so}(2n+1)$, where

$$F_{i,n'} = [F_{i,n}, f_n]_{q^2}, \quad F_{i,n'} = [F_{i,n-1}, f_n]_q, \quad F_{i,n'} = [F_{i,n+1}, f_n],$$

for, respectively, $\mathfrak{g}=\mathfrak{sp}(2n)$, $\mathfrak{g}=\mathfrak{so}(2n)$, $\mathfrak{g}=\mathfrak{so}(2n+1)$,

Proof. The intertwining relations (5) gives rises to the recursion relation

$$F_{i,j+1} = [F_{i,j}, f_\alpha]_{q^{-(\alpha,\nu_{ij})}}$$



where the weights ν_{ij} for orthogonal and symplectic case are

$$0, \quad \varepsilon_i - \varepsilon_{i+1}, \quad \dots, \quad \begin{cases} \varepsilon_{i+1} + \varepsilon_i, & \mathfrak{g} = \mathfrak{sp}(N), \\ \varepsilon_{i+2} + \varepsilon_i, & \mathfrak{g} = \mathfrak{so}(N). \end{cases}$$

Developing this recursion proves the statement. □

Define $E_{j,i}$ to be the matrix element $L^+_{ji,ii}$ for all pairs (i, j) as above.

Proposition 7.2. *Up to a non-zero scalar multiplier the element $E_{j,i}$ equals*

$$E_{j,i} = [\dots[e_i, e_{i+1}]_{\bar q} \dots, e_{j-1}]_{\bar q}$$

if $j \leqslant n$ for $\mathfrak{g} = \mathfrak{sp}(2n), \mathfrak{so}(2n)$, and $j \leqslant n+1$ for $\mathfrak{g} = \mathfrak{so}(2n+1)$,

$$E_{j',i} = [\dots[E_{n',i}, e_{n-1}]_{\bar q} \dots, e_j]_{\bar q}$$

if $j < n$ for $\mathfrak{g} = \mathfrak{sp}(2n), \mathfrak{so}(2n)$, and $j < n+1$ for $\mathfrak{g} = \mathfrak{so}(2n+1)$, where

$$E_{n',i} = [E_{n,i}, e_n]_{\bar q^2}, \quad E_{n',i} = [E_{n-1,i}, e_n]_{\bar q}, \quad E_{n',i} = [E_{n+1,i}, e_n],$$

for, respectively, $\mathfrak{g} = \mathfrak{sp}(2n)$, $\mathfrak{g} = \mathfrak{so}(2n)$, $\mathfrak{g} = \mathfrak{so}(2n+1)$.

Proof. Follows from the formulas for F_{ij} and transformation (7). □

The elements $F_{i,j}$ and $E_{i,j}$ calculated above turn out to be LKT root vectors. They are deformations of classical root vectors being iterations of deformed commutators. For a given Levi $\mathfrak{l} \subset \mathfrak{g}$, vectors of this type generate tensor products of minimal fundamental modules for simple blocks in $U_q(\mathfrak{l})$. The remaining vectors generate symmetric or skew symmetric tensor squared minimal modules for general linear blocks of of $U_q(\mathfrak{l})$ (for symplectic $\mathfrak{g}$ they are exactly long root vectors). Their weights can be characterized as the maximal/minimal roots of certain Lie subalgebras from a natural increasing sequence of corank one in each term. The matrix entries of this type cannot be computed directly in the basis of type II because of branching. We do it in the basis of type I, then proceed to the required basis by the quantum (inverse) Cartan matrix. They are also deformations of classical root vectors being linear combinations or iterated deformed commutators.

8. Maximal root vectors

In this subsection we define maximal root vectors for each Lie subalgebra from the chain

$$\mathfrak{g}_\ell \subset \mathfrak{g}_{\ell+1} \subset \dots \subset \mathfrak{g}_n,$$

where $\mathfrak{g}_k$ is either symplectic or orthogonal Lie algebra of rank k whose root system is generated by $\alpha_{n-k+1}, \ldots, \alpha_n$. The integer ℓ equals 2 for symplectic, 3 for odd and 4 for even orthogonal Lie algebras.

Like the root vectors defined in the preceding sections, they are special entries of matrices $L^{\pm}$ in the basis of second type. However our calculation algorithm for them will be different. We find matrix entries in the basis of type I, and then proceed to the basis of type II with the help of the inverse q-Cartan matrix. This complicates the expression for the root vectors, which is failed to be a composition of deformed commutators. We nevertheless conjecture that such a presentation is possible. That is supported by the simplest case of the $\mathfrak{g} = \mathfrak{sp}(4)$ discussed in the introduction.

8.1. Long root vectors for $\mathfrak{g} = \mathfrak{sp}(2n)$. We extend the canonical basis $\{\tilde{h}_\alpha\}_{\alpha\in\Pi} \subset \tilde{\mathfrak{g}}[0]$ to a basis in $\sum_{\alpha\geqslant 0} \tilde{\mathfrak{g}}[\alpha]$ as follows. Pick up a non-zero vector $\tilde{e}_{1,1'}$, which is the highest in $\tilde{\mathfrak{g}}$. Put $\tilde{e}_{i,i'} = f_{i-1}^2 \ldots f_1^2 \tilde{e}_{1,1'}$, for $1 < i \leqslant n$. The vector $\tilde{e}_{i,m}$ for $m < i < m'$ is then found from the path

$$\tilde{e}_{m,i} \longleftarrow \tilde{e}_{m,i+1} \longleftarrow \ldots \longleftarrow \tilde{e}_{m,m'-1} \longleftarrow \tilde{e}_{m,m'}$$

where the arrows are determined uniquely, and the corresponding non-zero matrix elements of $\pi(f_\alpha)$ are set to be 1. These paths are included in the Hasse diagram $\mathfrak{H}(\tilde{\mathfrak{g}})$.

We label the nodes of non-zero weight in $\tilde{\mathfrak{g}}$ with pairs (i, j) where $1 \leqslant i \neq j \leqslant N = 1'$. The node (i, i) will stand for $\tilde{h}_i$.

Lemma 8.1. *For each m, $1 \leqslant m < n$, the element $L^-_{(m,m'),(m,m)}$ equals*

$$(q^{-2} - q^2)q^{-2}[f_m, [f_{m+1}, \ldots, [f_n, \ldots [f_{m+1}, f_m]_q \ldots]_{q^2} \ldots]_q]_{q^4}, \quad m < n,$$

and $L^-_{(n,n');(n,n)} = (q^{-4} - q^{4)})f_n$.

Lemma 8.2. *Suppose that $1 \leqslant l < m < n$. Then*

$$\begin{aligned} L^-_{(l,l')(l+1,l'-1)} &= (q^{-2} - q^2)(q^{-2} - 1)f_l^2, \\ L^-_{(l,l')(m+1,m'-1)} &= (q^{-2} - q^2)(q^{-2} - 1)[f_m, [f_m, L_{(l,l')(m,m')}]_{q^2}]. \end{aligned}$$

The recurrent relation stated by the lemma gives rise to an explicit expression for $L^-_{(l,l'),(m,m')}$ for all pairs $l < m$.

Proposition 8.3. *Suppose that $1 \leqslant l < m < n$. Then the entry $L^-_{(l,l')(m,m)}$ can be calculated by the formula*

$$L^-_{(l,l')(m,m)} = [f_m, \dots [f_{n-1}, [f_n, [f_{n-1}, \dots, [f_{m+1}, [f_m, L^-_{(l,l')(m-1,m'+1)}]_{q^2}]_q \dots]_q]_{q^2}]_q \dots]_q,$$
$$L^-_{(l,l')(n,n)} = [f_n, L^-_{(l,l')(n-1,n'+1)}]_{q^4}.$$

Along with Lemma 8.1*, this gives all matrix entries $L_{(l,l')(m,m)}$ for $1 \leqslant l \leqslant m \leqslant n$.*

8.2. The case of $\mathfrak{so}(N)$.

Lemma 8.4. *The matrix entry $L_{(m,m'-1)(m,m)}$ equals*

$$(q^{-1} - q)q^{-1}[f_{n-1}, [f_n, f_n]_q]_{q^2}, \quad m = n-1,$$
$$(q^{-1} - q)q^{-1}[f_1, [f_2, [f_3, \dots [f_{n-1}, [f_n, [f_n, \dots, [f_3, f_2]_q \dots]_q]]_q \dots]_q]_{q^2}]_{q^2}, \quad 1 \leqslant m < n-1,$$

for $N = 2n+1$ and

$$(q^{-1} - q)q^{-1}[f_{n-2}[f_n, f_{n-1}]_{q^2}]_{q^2}, \quad m = n-2,$$
$$(q^{-1} - q)q^{-1}[f_1, [f_2, [f_3, \dots [f_{n-2}, [f_n, [f_{n-1}, \dots, [f_3, f_2]_q \dots]_q]_q]_q \dots]_q]_{q^2}]_{q^2}, \quad 1 \leqslant m < n-2.$$

Proof. It is sufficient to consider the case of $m = 1$. We do it by inflating along the path

$$(1,1) \xleftarrow{f_1} (1,2) \longleftarrow \dots \longleftarrow (1,3') \xleftarrow{f_3} (1,3') \xleftarrow{f_2} (1,2')$$

in $\mathfrak{H}(\tilde{\mathfrak{g}})$. It is isomorphic to a subdiagram in the Hasse diagram of the natural module of $\mathfrak{so}(N)$. It gives rise to a path in $\mathfrak{H}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$ without branching and generates a recurrent relation

$$q^{-(\alpha,\varepsilon_1+\varepsilon_2)} f_\alpha L^-_{(1,2');(1,l)} \quad = \quad L^-_{(1,2');(1,l)} q^{(\alpha,\varepsilon_1-\varepsilon_l)} f_\alpha + L^-_{(1,2');(1,l-1)},$$

where l is ranging from $2'$ to 2, and the root α equals $\varepsilon_l - \varepsilon_{l-1}$ (in the case of $N = 2n$, $l-1$ should be replaced with $l-2$ when $l = n+1$). □

Consider the following path in $\mathfrak{H}(\tilde{\mathfrak{g}})$, assuming $m < n$:

$$\xleftarrow{f_{m+1}} (m, m'-1) \xleftarrow{f_{m-1}} (m-1, m'-1) \xleftarrow{f_m} (m-1, m') \longleftarrow \dots$$
$$\longleftarrow (2,3') \xleftarrow{f_1} (1,3') \xleftarrow{f_2} (1,2').$$

We will calculate the matrix entry L^-_{ij} for $i=(1,2')$ and $j=(m-1,m')$ for all $1<m\leqslant \frac{N-1}{2}$ by inflating in the direction determined by this path. Thus we arrive at the following result.

Lemma 8.5. *Suppose that $\mathfrak{g}=\mathfrak{so}(2n+1)$ with $n\geqslant 3$, or $\mathfrak{g}=\mathfrak{so}(2n)$ with $n\geqslant 4$. Put $c=q^{-1}-q$. Then the matrix entries $L^-_{(1,2');(m-1,m')}$ are equal to*

$$L^-_{(1,2');(2,3')}=c[f_1,f_2]_q,\quad L^-_{(1,2');(3,4')}=cq^{-1}[f_2,[f_3,[f_1,f_2]_q]_q]_{q^2},$$
$$L^-_{(1,2');(m-1,m')}=[f_{m-1},[f_m,\dots[f_3,[f_4,L^-_{(1,2');(3,4')}]_q]_q\dots]_{q^{1-\delta_{m,n-1}}}]_{q^{1-\delta_{m,n}}},\quad 3\leqslant m\leqslant n,$$

for $\mathfrak{g}=\mathfrak{so}(2n+1)$ and

$$L^-_{(1,2');(m-1,m')}=[f_{m-1},[f_m,\dots[f_3,[f_4,L^-_{(1,2');(3,4')}]_q]_q\dots]_q]_q,\quad 3\leqslant m\leqslant n.$$

Now we compute the matrix entries $L^-_{(1,2');(m,m)}$, where (m,m) designates $\tilde h_m$, a basis element in $\tilde{\mathfrak{g}}[0]$ of type I. It will be done via the path

$$(m,m)\xleftarrow{f_m}\dots\xleftarrow{f_{m+1}}(m,m'-1)\xleftarrow{f_{m-1}}\dots\xleftarrow{f_2}(1,2')$$

in $\mathfrak{H}(\tilde{\mathfrak{g}})$, where the rightmost part has been considered above, and the path to (m,m) from $(m,m'-1)$ is the piece of the Hasse diagram of the natural representation of orthogonal $\mathfrak{g}$ of rank m.

Proposition 8.6. *Let s vary from 1 to $\left[\frac{N-1}{2}\right]-1$ and m from $s+1$ to $\left[\frac{N-1}{2}\right]$. Then*

$$L^-_{(s,s'-1);(m,m)}=q^{\delta_{m,2}}[f_m,\dots[f_{n-1},[f_n,[f_n,\dots[f_{m+1},L^-_{(s,s'-1);(m,m'-1)}]_q,\dots]_q]]_q\dots]_{q^{1+\delta_{m,2}}}$$

for $N=2n+1$, and

$$L^-_{(s,s'-1);(m,m)}=q^{\delta_{m,2}}[f_m,\dots[f_{n-2},[f_n,[f_{n-1},\dots[f_{m+1},L^-_{(s,s'-1);(m,m'-1)}]_q,\dots]_q]_q]_q\dots]_{q^{1+\delta_{m,2}}}$$

$$L^-_{(s,s'-1);(n-1,n-1)}=[f_{n-1},[f_{n-2},[f_n,L^-_{(s,s'-1);(n-2,n+2)}]_q]_q]_q$$
$$L^-_{(s,s'-1);(n,n)}=[f_n,L^-_{(s,s'-1);(n-1,n+1)}]_q$$

for $N=2n$.

Proof. It is sufficient to prove it for $s = 1$. We set , $i = (1, 2')$, $j = (m, l)$ and $k = (m, l-1)$ in the intertwining relation (5), for $m < l < m'$ and $2 \leqslant m \leqslant \frac{N-1}{2}$. This translates (5) to

$$q^{-(\alpha,\varepsilon_1+\varepsilon_2)} f_\alpha L^-_{(1,2');(m,l)} \quad = \quad L^-_{(1,2');(m,l)} q^{(\alpha,\varepsilon_m-\varepsilon_l)} f_\alpha + L^-_{(1,2');(m,l-1)},$$

where $\alpha = \varepsilon_{l-1} - \varepsilon_l$. Then induction on l completes the proof. □

Now we are in possession to define maximal root vectors for subalgebras $(\mathfrak{g}_k)_{k=\ell}^n$.

$$F_{k,k'} = \sum_{m=1}^n L^-_{(k,k')(mm)} (-1)^{m+1} [2]_q^m \bar{A}_{mk}, \quad k = 1, \ldots, n,$$

$$F_{k,k'-1} = \sum_{m=1}^n L^-_{(k,k'-1)(mm)} (-1)^{m+1} [2]_q^{m-1} \bar{A}_{m,k+1}, \quad k = 1, \ldots, n-1,$$

$$F_{k,k'-1} = \sum_{m=1}^n L^-_{(k,k'-1)(mm)} (-1)^{k+(n-1)\delta_{mn}} [2]_q^{m-1} \bar{A}_{m,k+1}, \ \ k = 1, \ldots, n-2.$$

Matrix entries of L^- whose weight is a simple root are not presented because they are always proportional to the simple root vectors.

The positive counterparts of these root vectors are defined via transformation (7): by applying the involutive automorphism σ and replacing q with its reciprocal.

In conclusion, let us comment on two problems concerning the root vectors introduced in this presentation.

- It would be desirable to find a simple presentation for the non-LKT root vectors, e.g. in a form of iterated commutators. We believe that such a presentation is possible.
- The non-LKT root vectors are deformations of their classical counterparts. Therefore they generate a PBW basis in $U_q(\mathfrak{g})$ over $U_q(\mathfrak{h})$ upon extension over the ring of formal power series in $\hbar = \ln(q)$. A question is if that is true over $\mathbb{C}$ for q not a root of unity.

As discussed in the introduction, the answer to these questions is positive for the simplest $\mathfrak{g}$ of type $\mathfrak{B}_2$.

Appendix A. The Q-Cartan matrices

This addendum contains quantum Cartan matrices and their inverses for four infinite series of quantum groups. Note that our version of $A_{\mathfrak{so}(2n+1)}$

differs from ([6]) by replacement of q with q^2.

$$A_{\mathfrak{sl}(n)} = \begin{pmatrix} [2]_q & -1 & 0 & \dots & 0 \\ -1 & [2]_q & -1 & \dots & 0 \\ 0 & -1 & [2]_q & \dots & 0 \\ \dots & \dots & \dots & \dots & -1 \\ 0 & 0 & \dots & -1 & [2]_q \end{pmatrix},$$

$$A_{\mathfrak{so}(2n)} = \begin{pmatrix} [2]_q & -1 & \dots & \dots & \dots & \dots \\ -1 & [2]_q & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & [2]_q & -1 & -1 \\ 0 & \dots & \dots & -1 & [2]_q & 0 \\ 0 & \dots & \dots & -1 & 0 & [2]_q \end{pmatrix},$$

$$A_{\mathfrak{sp}(n)} = \begin{pmatrix} [2]_q & -1 & \dots & \dots & \dots & \dots \\ -1 & [2]_q & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & [2]_q & -1 & 0 \\ 0 & \dots & \dots & -1 & [2]_q & -[2]_q \\ 0 & \dots & \dots & 0 & -1 & [2]_{q^2} \end{pmatrix},$$

$$A^{-1}_{\mathfrak{so}(2n+1)} = \begin{pmatrix} [2]_q & -1 & \dots & \dots & \dots & \dots \\ -1 & [2]_q & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & [2]_q & -1 & 0 \\ 0 & \dots & \dots & -1 & [2]_q & -1 \\ 0 & \dots & \dots & 0 & -[2]_{q^{\frac{1}{2}}} & [2]_{q^{\frac{1}{2}}} \end{pmatrix},$$

$$A^{-1}_{\mathfrak{sl}(n+1)} = \frac{1}{[n+1]_q} \begin{pmatrix} [n]_q & [n-1]_q & [n-2]_q & \dots & 1 \\ [n-1]_q & [2]_q[n-1]_q & [2]_q[n-2]_q & \dots & [2]_q \\ [n-2]_q & [n-2]_q[2]_q & [n-2]_q[3]_q & \dots & [3]_q \\ \dots & \dots & \dots & \dots & \dots \\ 1 & [2]_q & \dots & [n-1]_q & [n]_q \end{pmatrix},$$

$$A^{-1}_{\mathfrak{so}(2n)} = \frac{1}{[2]_{q^n}}$$

$$\times \begin{pmatrix} [2]_{q^{n-1}} & [2]_{q^{n-2}} & [2]_{q^{n-3}} & \dots & [2]_q & 1 & 1 \\ [2]_{q^{n-2}} & [2]_q[2]_{q^{n-2}} & [2]_q[2]_{q^{n-3}} & \dots & [2]_q^2 & [2]_q & [2]_q \\ [2]_{q^{n-3}} & [2]_q[2]_{q^{n-3}} & [3]_q[2]_{q^{n-3}} & \dots & [3]_q[2]_q & [3]_q & [3]_q \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ [2]_q & [2]_q^2 & [2]_q[3]_q & \dots & [2]_q[n-1]_q & [n-1]_q & [n-1]_q \\ 1 & [2]_q & [3]_q & \dots & [n-1]_q & \frac{[n+1]_q}{2_q} & \frac{[n-1]_q}{[2]_q} \\ 1 & [2]_q & [3]_q & \dots & [n-1]_q & \frac{[n-1]_q}{[2]_q} & \frac{[n+1]_q}{2_q} \end{pmatrix},$$

$$A^{-1}_{\mathfrak{sp}(n)} = \frac{1}{[2]_{q^{n+1}}} \begin{pmatrix} [2]_{q^n} & [2]_{q^{n-1}} & [2]_{q^{n-2}} & \dots & [2]_q \\ [2]_{q^{n-1}} & [2]_q[2]_{q^{n-1}} & [2]_q[2]_{q^{n-2}} & \dots & [2]_q^2 \\ [2]_{q^{n-2}} & [2]_q[2]_{q^{n-2}} & [3]_q[2]_{q^{n-2}} & \dots & [3]_q[2]_q \\ \dots & \dots & \dots & \dots & \dots \\ 1 & [2]_q & \dots & [n-1]_q & [n]_q \end{pmatrix},$$

$$A^{-1}_{\mathfrak{so}(2n+1)} = \frac{1}{[2]_{q^{n-\frac{1}{2}}}} \begin{pmatrix} [2]_{q^{n-\frac{3}{2}}} & [2]_{q^{n-\frac{5}{2}}} & [2]_{q^{n-\frac{7}{2}}} & \dots & [2]_{q^{\frac{1}{2}}} \\ [2]_{q^{n-\frac{5}{2}}} & [2]_q[2]_{q^{n-\frac{5}{2}}} & [2]_q[2]_{q^{n-\frac{7}{2}}} & \dots & [2]_q[2]_{q^{\frac{1}{2}}} \\ [2]_{q^{n-\frac{7}{2}}} & [2]_q[2]_{q^{n-\frac{7}{2}}} & [3]_q[2]_{q^{n-\frac{7}{2}}} & \dots & [3]_q[2]_{q^{\frac{1}{2}}} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & [2]_q & \dots & [n-1]_q & [n]_q \end{pmatrix}^T.$$

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