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ASYMPTOTICS OF SOLUTIONS OF THE DEGENERATE THIRD PAINLEVÉ EQUATION IN THE NEIGHBOURHOOD OF THE REGULAR SINGULAR POINT: THE ISOMONODROMY DEFORMATION APPROACH

Abstract. This paper contains several technical refinements of our previously obtained results on the monodromy parametrisation of small-τ asymptotics of solutions $u(\tau)$ of the degenerate third Painlevé equation,

$$u''(\tau)=\frac{(u'(\tau))^2}{u(\tau)}-\frac{u'(\tau)}{\tau}+\frac{1}{\tau}\big(-8\varepsilon(u(\tau))^2+2ab\big)+\frac{b^2}{u(\tau)},$$

where $\varepsilon=\pm1$, $\varepsilon b>0$, $a\in\mathbb{C}$, and of its associated *mole function*, $\varphi(\tau)$, which satisfies $\varphi'(\tau)=\frac{2a}{\tau}+\frac{b}{u(\tau)}$. We also describe three families of three-real-parameter solutions $u(\tau)$ which have infinite sequences of zeros converging to the origin of the complex τ-plane. Furthemore, for $a=0$, a numerical visualisation of the formulae connecting the asymptotics as $\tau\to0$ and $\tau\to+\infty$ of solutions $u(\tau)$ and $\varphi(\tau)$ having logarithmic behaviour as $\tau\to0$ is given.

§1. Introduction

The Degenerate Third Painlevé equation (DP3E),

$$u''(\tau)=\frac{(u'(\tau))^2}{u(\tau)}-\frac{u'(\tau)}{\tau}+\frac{1}{\tau}\big(-8\varepsilon(u(\tau))^2+2ab\big)+\frac{b^2}{u(\tau)},\qquad \varepsilon=\pm1,\quad \varepsilon b>0,\quad a\in\mathbb{C}, \tag{1.1}$$

has garnered recent interest not only with respect to the description of the asymptotic properties of its solutions in the algebroid [21], algebraic [3,4], and elliptic [23] function classes, but also its manifestations in differential geometry and theoretical and applied physics [2,5–9,13,24].

The immediate goal of our current research on the DP3E is to obtain a complete description of the small-τ asymptotic behaviour of all its solutions $u(\tau)$. This description is based on the Method of Isomonodromy

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Deformations, which provides not only the asymptotics, but also their parametrisation(s) in terms of the monodromy data of an associated first-order 2×2 matrix linear ODE whose isomonodromy deformations are described in terms of solutions of the DP3E (see [16] for details). This work is, in fact, an elaboration of Section 5 of our paper [16]; in particular, notation, formulae, and statements from [16] are used with little, or no further, explanation, except where absolutely necessary. The reader who wants to fully comprehend the contents of this paper should refer to [16]; otherwise, only the main scheme of the derivations and formulations of the results will be clear. Here, we present the "core" asymptotic results as $\tau \to 0$, which, in the notation of [16], means that $\varepsilon_1 = \varepsilon_2 = 0$, that is, $\varepsilon_1 = 0 \Rightarrow \arg(\tau) = 0$ and $\varepsilon_2 = 0 \Rightarrow \varepsilon b > 0$.

For the reader's convenience, we recall the definition of the *manifold of the monodromy data*, $\mathcal{M}$, which is important for understanding the results presented in this paper. Consider $\mathbb{C}^8$ with co-ordinates $(a,\ s_0^0,\ s_0^\infty,\ s_1^\infty,\ g_{11},\ g_{12},\ g_{21},\ g_{22})$, where a, the parameter of formal monodromy, s_0^0, s_0^∞, and s_1^∞, the Stokes multipliers, and $g_{ij} := (G)_{ij}$, $i, j = 1, 2$, the elements of the connection matrix, G, constitute the *monodromy data* [16]. The monodromy data are related by the set of algebraic equations [1]

$$s_0^\infty s_1^\infty = -1 - \mathrm{e}^{-2\pi a} - \mathrm{i} s_0^0 \mathrm{e}^{-\pi a}, \tag{1.2}$$

$$g_{21} g_{22} - g_{11} g_{12} + s_0^0 g_{11} g_{22} = \mathrm{i}\mathrm{e}^{-\pi a}, \tag{1.3}$$

$$g_{11}^2 - g_{21}^2 - s_0^0 g_{11} g_{21} = \mathrm{i} s_0^\infty \mathrm{e}^{-\pi a}, \tag{1.4}$$

$$g_{22}^2 - g_{12}^2 + s_0^0 g_{12} g_{22} = \mathrm{i} s_1^\infty \mathrm{e}^{\pi a}, \tag{1.5}$$

$$g_{11} g_{22} - g_{12} g_{21} = 1. \tag{1.6}$$

For the unique parametrisation of solutions of the DP3E in terms of the monodromy data, one has to identify (glue) points of $\mathcal{M}$ that correspond to matrices G and $-G$, that is, one has to consider $G \in \mathrm{PSL}(2, \mathbb{C})$. This parametrisation can be used to connect the asymptotics as $\tau \to 0$ to the asymptotics as $\tau \to \infty$, that is, in obtaining *connection formulae* for asymptotics of solutions of the DP3E [16, 17, 21]. It is also of paramount importance for the complete classification of the asymptotic behaviour(s) of solutions, because if asymptotics of solutions of the DP3E are prescribed in a one-to-one manner to each point of $\mathcal{M}$, then all conceivable asymptotic behaviours are exhausted.

[1] In these equations, $\mathrm{e}^{\pi a}$ is considered to be a parameter.

This paper is the first in a series of works devoted to the goal of obtaining the complete asymptotic behaviour as $\tau \to 0$ of solutions $u(\tau)$ of the DP3E and of its associated *mole function*, $\varphi(\tau)$, which satisfies the ODE

$$\varphi'(\tau) = \frac{2a}{\tau} + \frac{b}{u(\tau)}. \tag{1.7}$$

This paper contains several technical results that are implied by, and/or can be derived from, the isomonodromy deformation framework developed in [16], in which we obtained monodromy-data-dependent parametrisations for the $\tau \to 0$ and $\tau \to \infty$ asymptotic formulae for the general solution of the DP3E. Asymptotic results for the function $\varphi(\tau)$ were actually obtained in [16] but were not explicitly stated there since the importance of this function in the theory of Painlevé equations was not understood at that time and the principal object of our concern was the Painlevé function $u(\tau)$. In our recent work on algebroid solutions of the DP3E [21], the $\tau \to 0^+$ (resp., $\tau \to +\infty$) asymptotics for $\varphi(\tau)$ is given in Appendix B (resp., Appendix C). Note that the notation $\tau \to 0^+$ is understood in the extended sense as $|\tau| \to 0$ and $|\arg \tau| \leqslant \phi_0 < \pi$.

The asymptotic results of [16] were stated in terms of a plethora of notations that appeared during the course of their derivation, which, inadvertently, had the effect of adding a degree of unreadability to them, and thereby making the results not wholly transparent to those readers who are not fully immersed in the technical aspects of the Isomonodromy Deformation Method; furthermore, scant attention was paid to various special cases that are inconspicuous to extract from the general asymptotic formulae in [16]: one of the purposes of this paper is to rectify these shortcomings.

The aforementioned asymptotics contain several restrictions on the monodromy data. Some of the restrictions are important: when these restrictions are violated, the qualitative behaviour of the asymptotics changes, and other restrictions turn out to be related to our methodology [16] for obtaining the parametrisation(s) of asymptotics rather than to the properties of the solutions themselves. One purpose of the present paper is to remove those restrictions that do not reflect the essential properties of the solutions; in [16], for example, we derived monodromy-data-dependent parametrisations for the $\tau \to 0$ and $\tau \to \infty$ asymptotics of $u(\tau)$ under the "generic" condition $g_{11}g_{22} \neq 0$. This restriction is an essential constraint for asymptotics at the point at infinity, whilst for asymptotics in the neighbourhood of $\tau=0$, it is related with our method for obtaining this result, and can, therefore, be eliminated.

This paper is organised as follows. In Section 2, the generic condition $g_{11}g_{22}\neq 0$ is removed, and power-like asymptotics for $u(\tau)$ and $\varphi(\tau)$ corresponding to special sets of monodromy data are given. In Section 3, logarithmic asymptotics for the functions $u(\tau)$ and $\varphi(\tau)$ are considered; in particular, the notation for the general logarithmic asymptotic formula stated in Theorem 3.5 of [16] is simplified, and non-trivial, special cases of this formula for $a=0$ are specified. In Section 4, three families of solutions $u(\tau)$, depending on three real parameters, having zeros accumulating at the origin are studied; the corresponding mole function, $\varphi(\tau)$, which depends on one additional complex parameter, has movable logarithmic branch points at these zeros. In Appendix A, an inconsistency in Propositions 5.1 and 5.2 of [16] is amended. Finally, in Appendix B, a numerical verification of a connection result for asymptotics of a solution of the DP3E for $a=0$ having logarithmic behaviour as $\tau\to 0$ is given.

§2. The condition $g_{11}g_{22}\neq 0$ and power-like asymptotics

The DP3E has a regular singular point at $\tau=0$; therefore, the general solution has a branching point at $\tau=0$. In order to characterise this non-single-valued behaviour, we introduced a branching parameter, ρ, such that $|\mathrm{Re}(\rho)|<1/2$. In Theorem 3.4 of [16], the leading term of asymptotics of $u(\tau)$ is stated under the assumption $g_{11}g_{22}\neq 0$. This assumption was essential for the derivation of the formula relating ρ and the monodromy data, namely,

$$\cos(2\pi\rho)=-\frac{\mathrm{i}s_0^0}{2}=\cosh(\pi a)+\frac{1}{2}s_0^{\infty}s_1^{\infty}\mathrm{e}^{\pi a}, \tag{2.1}$$

which was derived in Proposition 5.6 of [16]. In the course of the proof of Proposition 5.6, we used the following relation, which is valid for $\rho\neq 0$ (the case $\rho=0$ is considered in Section 3):[2]

$$\begin{aligned}\big(\rho-\tfrac{\mathrm{i}a}{2}\big)\mathfrak{p}(a,\rho)\mathfrak{p}(-a,-\rho)\mathrm{e}^{\pi\mathrm{i}\rho}\chi_1(\rho)\chi_2(-\rho)&\\ +\big(\rho+\tfrac{\mathrm{i}a}{2}\big)\mathfrak{p}(a,-\rho)\mathfrak{p}(-a,\rho)\mathrm{e}^{-\pi\mathrm{i}\rho}\chi_1(-\rho)\chi_2(\rho)&=0,\end{aligned} \tag{2.2}$$

[2] For $\varepsilon_1=\varepsilon_2=0$, the monodromy functions in Theorem 3.4 of [16] simplify as follows: $s_0^0(0,0):=s_0^0$, $s_j^{\infty}(0,0):=s_j^{\infty}$, $j=0,1$, $g_{kl}(0,0):=g_{kl}$, $k,l=1,2$, $\chi_m(\vec{g}(0,0);*):=\chi_m(*)$, $m=1,2$, and $\varpi_n^{\natural}(0,0;*):=\varpi_n(*)$, $n=1,2$.

where $\mathfrak{p}(z_1,z_2)$, $\chi_1(z_3)$, and $\chi_2(z_4)$ are given in equations (47) and (48) of [16].[3] Via equations (47) of [16] and the gamma function identities [10] $\Gamma(\frac{1}{2}-z)\Gamma(\frac{1}{2}+z)=\frac{\pi}{\cos(\pi z)}$, $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$, and $\Gamma(1+z)=z\Gamma(z)$, it follows from equations (1.3) and (2.2) that

$$\frac{\pi(\rho-\mathrm{i}a/2)(\rho+\mathrm{i}a/2)}{\sin(\pi(\rho+\mathrm{i}a/2))}\Big(\mathrm{e}^{-\pi a}+\mathrm{i}s_0^0g_{11}g_{22}+g_{11}g_{22}\mathrm{e}^{2\pi\mathrm{i}\rho}+g_{12}g_{21}\mathrm{e}^{-2\pi\mathrm{i}\rho}\Big)\mathrm{e}^{\pi\mathrm{i}\rho}$$
$$+\frac{\pi(\rho-\mathrm{i}a/2)(\rho+\mathrm{i}a/2)}{\sin(\pi(\rho-\mathrm{i}a/2))}\Big(\mathrm{e}^{-\pi a}+\mathrm{i}s_0^0g_{11}g_{22}+g_{11}g_{22}\mathrm{e}^{-2\pi\mathrm{i}\rho}+g_{12}g_{21}\mathrm{e}^{2\pi\mathrm{i}\rho}\Big)\mathrm{e}^{-\pi\mathrm{i}\rho}=0. \tag{2.3}$$

(Equation (2.3) is also valid for $\rho=\pm\mathrm{i}a/2$.) For $\rho\neq\pm\mathrm{i}a/2$, equation (2.3) simplifies to

$$g_{11}g_{22}\big(2\cos(2\pi\rho)+\mathrm{i}s_0^0\big)=0, \tag{2.4}$$

which, for $g_{11}g_{22}\neq 0$, implies equation (2.1).[4] In Proposition 2.1 below, we present a direct calculation for the Stokes multiplier s_0^0 that is independent of any assumptions on the connection matrix.

Proposition 2.1. *For all connection matrices $G\in\mathrm{PSL}(2,\mathbb{C})$,*

$$s_0^0=2\mathrm{i}\cos(2\pi\rho). \tag{2.5}$$

Proof. The Stokes multiplier s_0^0 is defined in terms of the canonical solutions $X_k^0(\mu)$, $k=0,1$, of the linear auxiliary system (12) on p. 1169 of [16], where $\mu\in\Omega_k^0:=\{\mu\in\mathbb{C};\ |\mu|<\delta,\ -\pi+\pi k<\arg(\mu)-\frac{1}{2}\arg(\tau)-\frac{1}{2}\arg(\varepsilon b)<\pi+\pi k\}$, with $\delta>0$. Using equations (20)–(25) on p. 1171 of [16], one shows that the defining relation for the determination of the Stokes multiplier s_0^0 can be expressed in terms of one canonical solution corresponding to $k=0$:

$$\big(X_0^0(\mu)\big)^{-1}\sigma_3X_0^0(\mathrm{e}^{-\pi\mathrm{i}}\mu)=\begin{pmatrix}1 & s_0^0\\ 0 & 1\end{pmatrix}\sigma_1, \tag{2.6}$$

where $\sigma_3=\mathrm{diag}(1,-1)$ and $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. For $\arg(\tau)=\arg(\varepsilon b)=0$, an asymptotic formula for $X_0^0(\mu)\in\mathrm{SL}(2,\mathbb{C})$ in terms of the Hankel functions of the first and second kinds, $H_*^{(1)}(\cdot)$ and $H_*^{(2)}(\cdot)$, respectively, is presented

[3] Actually, in Proposition 5.6 of [16], we used the branching function $\widehat{\rho}=\widehat{\rho}(\tau)$, which is related to the branching parameter ρ via the asymptotic relation $\widehat{\rho}\underset{\tau\to0^+}{=}\rho(1+o(\tau^{\delta}))$, $\delta>0$; thus, equation (2.2) is, in fact, the leading term of the relation that was used in the proof of Proposition 5.6.

[4] Considering the limits $\rho\to\pm\mathrm{i}a/2$ in equation (2.3), one arrives, again, at equation (2.4), but with $\rho=\pm\mathrm{i}a/2$.

in Section 5 of [16]. From Propositions 2.1, 5.3, and 5.4 of [16], it follows that, with $z:=\sqrt{\tau\varepsilon b}/\mu$,

$$X_0^0(\mathrm{e}^{-\pi\mathrm{i}k}\mu) \underset{\tau\to0^+}{=} \mathcal{P}(\mathrm{e}^{-\pi\mathrm{i}k}\mu)\mathfrak{B}(\mathrm{e}^{\pi\mathrm{i}k}z)\big(\mathrm{I}+o\big((\tau\mu^2)^{\delta}\big)\big), \quad k=0,1, \quad \delta>0, \tag{2.7}$$

where $\mathrm{I}=\mathrm{diag}(1,1)$,

$$\mathcal{P}(\mu)=\begin{pmatrix} \frac{1}{\sqrt{\mu}} & \frac{1}{\sqrt{\mu}}\frac{\widehat{A}(\tau)}{\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}} \\ 0 & \sqrt{\mu} \end{pmatrix}, \tag{2.8}$$

$$\begin{aligned}
\mathfrak{B}_{11}(z)&=\mathfrak{z}_{11}\Big(\mathfrak{r}^{\uparrow}(\tau)H^{(2)}_{\nu_0}(z)+zH^{(2)}_{\nu_0-1}(z)\Big),\\
\mathfrak{B}_{12}(z)&=\mathfrak{z}_{12}\Big(\mathfrak{r}^{\uparrow}(\tau)H^{(1)}_{\nu_0}(z)+zH^{(1)}_{\nu_0-1}(z)\Big),\\
\mathfrak{B}_{21}(z)&=\mathfrak{z}_{21}\Big(\mathfrak{r}^{\downarrow}(\tau)H^{(2)}_{\nu_0}(z)+zH^{(2)}_{\nu_0-1}(z)\Big),\\
\mathfrak{B}_{22}(z)&=\mathfrak{z}_{22}\Big(\mathfrak{r}^{\downarrow}(\tau)H^{(1)}_{\nu_0}(z)+zH^{(1)}_{\nu_0-1}(z)\Big),
\end{aligned} \tag{2.9}$$

with

$$\begin{aligned}
\mathfrak{z}_{11}&:=-\frac{\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}(\frac{\pi\nu_0}{2}+\frac{\pi}{4})}\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}}{2\sqrt{\varepsilon b}\,\sqrt{\widehat{B}(\tau)}},\\
\mathfrak{z}_{12}&:=-\frac{\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}(\frac{\pi\nu_0}{2}+\frac{\pi}{4})}\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}}{2\sqrt{\varepsilon b}\,\sqrt{\widehat{B}(\tau)}},\\
\mathfrak{z}_{21}&:=-\frac{\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}(\frac{\pi\nu_0}{2}+\frac{\pi}{4})}\sqrt{\widehat{B}(\tau)}}{2\sqrt{\varepsilon b}},\\
\mathfrak{z}_{22}&:=-\frac{\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}(\frac{\pi\nu_0}{2}+\frac{\pi}{4})}\sqrt{\widehat{B}(\tau)}}{2\sqrt{\varepsilon b}},\\
\mathfrak{r}^{\uparrow}(\tau)&:=-\nu_0+\mathrm{i}a+\frac{2\tau\widehat{B}(\tau)\widehat{C}(\tau)}{\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}},\\
\mathfrak{r}^{\downarrow}(\tau)&:=-\nu_0-\mathrm{i}a+\frac{2\tau\widehat{D}(\tau)\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}}{\widehat{B}(\tau)},\\
\nu_0^2&=4\widehat{\rho}^2,
\end{aligned} \tag{2.10}$$

and $\widehat{A}(\tau)=A(\tau)\tau^{-\mathrm{i}a}$, $\widehat{B}(\tau)=B(\tau)\tau^{\mathrm{i}a}$, $\widehat{C}(\tau)=C(\tau)\tau^{-\mathrm{i}a}$, and $\widehat{D}(\tau)=D(\tau)\tau^{\mathrm{i}a}$ are the elements of the coefficient matrices of the linear auxiliary system (12), where $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ solve the system of isomonodromy deformations (5) on p. 1167 of [16], which are equivalent to the system (1.1), (1.7) for the functions $u(\tau)$ and $\varphi(\tau)$, that is, $u(\tau)=\varepsilon\tau\sqrt{-A(\tau)B(\tau)}$, $\varepsilon=\pm1$, and $\varphi(\tau)=-\mathrm{i}\ln\big(\sqrt{-A(\tau)B(\tau)}/B(\tau)\big)$.[5] The index of the Hankel functions is given by $\nu_0=2\widehat{\rho}$, where $\widehat{\rho}$ is the branching function.[3] Using the fact that (cf. equation (2.8)) $(\mathcal{P}(\mu))^{-1}\sigma_3\mathcal{P}(\mathrm{e}^{-\pi\mathrm{i}}\mu)=\mathrm{i}\,\mathrm{I}$, it follows from equations (2.6) and (2.7) that

$$(\mathfrak{B}(z))^{-1}\,\mathfrak{B}(\mathrm{e}^{\pi\mathrm{i}}z)\underset{\tau\to0^+}{=}\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}\begin{pmatrix}s_0^0 & 1\\ 1 & 0\end{pmatrix}+o_{2\times2}\big((\tau\mu^2)^{\delta}\big),$$

where $o_{2\times2}\big((\tau\mu^2)^{\delta}\big)$ denotes a 2×2 matrix each of whose entries are $o\big((\tau\mu^2)^{\delta}\big)$, thus

$$\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}s_0^0+o\big((\tau\mu^2)^{\delta}\big)\underset{\tau\to0^+}{=}\mathfrak{B}_{22}(z)\mathfrak{B}_{11}(\mathrm{e}^{\pi\mathrm{i}}z)-\mathfrak{B}_{12}(z)\mathfrak{B}_{21}(\mathrm{e}^{\pi\mathrm{i}}z). \tag{2.11}$$

Since, from equations (2.10),

$$\mathfrak{z}_{11}\mathfrak{z}_{22}=\mathfrak{z}_{12}\mathfrak{z}_{21}=\frac{\pi\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}}{4(\varepsilon b)}, \tag{2.12}$$

it follows from equations (2.9), (2.11), and (2.12) that

$$\begin{aligned}\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}s_0^0+o\big((\tau\mu^2)^{\delta}\big)&\underset{\tau\to0^+}{=}\frac{\pi\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}}{4(\varepsilon b)}\big(\mathfrak{r}^{\uparrow}(\tau)-\mathfrak{r}^{\downarrow}(\tau)\big)\\ &\times\Big(H_{\nu_0}^{(1)}(z)H_{\nu_0-1}^{(2)}(\mathrm{e}^{\pi\mathrm{i}}z)+H_{\nu_0-1}^{(1)}(z)H_{\nu_0}^{(2)}(\mathrm{e}^{\pi\mathrm{i}}z)\Big)z.\end{aligned} \tag{2.13}$$

Using the Hankel function identities [11]

$$\begin{aligned}H_{\nu_0}^{(1)}(\mathrm{e}^{\pi\mathrm{i}}z)&=-\mathrm{e}^{-\pi\mathrm{i}\nu_0}H_{\nu_0}^{(2)}(z),\\ H_{\nu_0}^{(2)}(\mathrm{e}^{\pi\mathrm{i}}z)&=\frac{\sin(2\pi\nu_0)}{\sin(\pi\nu_0)}H_{\nu_0}^{(2)}(z)+\mathrm{e}^{\pi\mathrm{i}\nu_0}H_{\nu_0}^{(1)}(z),\\ H_{\nu_0}^{(1)}(z)H_{\nu_0-1}^{(2)}(z)-H_{\nu_0-1}^{(1)}(z)H_{\nu_0}^{(2)}(z)&=-\frac{4\mathrm{i}}{\pi z},\end{aligned}$$

[5] As discussed in Appendix A, due to the recalibration of the gauge of the canonical solutions at the point at infinity (see [18], Section 7), the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, $\widehat{\gamma}$, and $\widehat{\delta}$ appearing on the left-hand sides of equations (139)–(145) (resp., equations (147)–(153)) in Proposition 5.5 (resp. Proposition 5.7) of [16] must be changed to $A(\tau)\tau^{-\mathrm{i}a}$, $B(\tau)\tau^{\mathrm{i}a}$, $C(\tau)\tau^{-\mathrm{i}a}$, $D(\tau)\tau^{\mathrm{i}a}$, $\widehat{\gamma}\tau^{-\mathrm{i}a}$, and $\widehat{\delta}\tau^{\mathrm{i}a}$, respectively.

it follows from equation (2.13) that

$$\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}s_0^0+o\big(\tau^{\delta}\big)\underset{\tau\to0^+}{=}\frac{2\mathrm{e}^{\frac{\pi\mathrm{i}}{2}}\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}}{\varepsilon b}\big(\mathfrak{r}^{\uparrow}(\tau)-\mathfrak{r}^{\downarrow}(\tau)\big)\cos(\pi\nu_0). \tag{2.14}$$

One shows from equations (13) and (14) of [16], in conjunction with equations (2.10), that

$$\begin{aligned}\mathfrak{r}^{\uparrow}(\tau)-\mathfrak{r}^{\downarrow}(\tau)&=\frac{2}{\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}}\left(\mathrm{i}a\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}+\tau\left(\widehat{A}(\tau)\widehat{D}(\tau)+\widehat{B}(\tau)\widehat{C}(\tau)\right)\right)\\&=-\frac{\mathrm{i}\varepsilon b}{\sqrt{-\widehat{A}(\tau)\widehat{B}(\tau)}};\end{aligned} \tag{2.15}$$

hence, taking the $\tau\to0^+$ limit in equation (2.14) and recalling that $\nu_0=2\widehat{\rho}\to2\rho$,[3] one arrives at equation (2.5). □

Remark 2.1. Proposition 2.1 implies that one can use the $\tau\to0^+$ asymptotic results in Theorem B.1 of [21] for the cases $g_{11}=0$ (and $g_{12}g_{21}g_{22}\neq0$) or $g_{22}=0$ (and $g_{11}g_{12}g_{21}\neq0$); the corresponding monodromy data for these cases read: (i) for $g_{11}=0$,

$$\begin{gathered}a,g_{21}\in\mathbb{C}\setminus\{0\},\quad|\mathrm{Im}(a)|<1,\quad \mathrm{i}s_0^0\in\mathbb{C}\setminus\{\mp2\},\\ s_0^{\infty}=\mathrm{i}\mathrm{e}^{\pi a}g_{21}^2,\quad s_1^{\infty}=\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{21}^2}\big(1+\mathrm{e}^{-2\pi a}+\mathrm{i}\mathrm{e}^{-\pi a}s_0^0\big),\\ g_{11}=0,\quad g_{12}=-\frac{1}{g_{21}},\quad g_{22}=\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{21}};\end{gathered} \tag{2.16}$$

and (ii) for $g_{22}=0$,

$$\begin{gathered}a,g_{12}\in\mathbb{C}\setminus\{0\},\quad|\mathrm{Im}(a)|<1,\quad \mathrm{i}s_0^0\in\mathbb{C}\setminus\{\mp2\},\\ s_0^{\infty}=\frac{\mathrm{i}\mathrm{e}^{\pi a}}{g_{12}^2}\big(1+\mathrm{e}^{-2\pi a}+\mathrm{i}\mathrm{e}^{-\pi a}s_0^0\big),\quad s_1^{\infty}=\mathrm{i}\mathrm{e}^{-\pi a}g_{12}^2,\\ g_{11}=-\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{12}},\quad g_{21}=-\frac{1}{g_{12}},\quad g_{22}=0.\end{gathered} \tag{2.17}$$

Substituting the particular values of the monodromy data (2.16) and (2.17) into the generic asymptotic formulae given in Theorem B.1 of [21], one arrives at the corresponding asymptotics for the functions $u(\tau)$ and $\varphi(\tau)$. Despite the fact that the monodromy data (2.16) and (2.17) look rather special, the corresponding $\tau\to0^+$ asymptotic expansions for the functions

$u(\tau)$ and $\varphi(\tau)$ resemble, in form, the generic $\tau \to 0^+$ asymptotic formulae stated in Theorem B.1 of [21], and will, therefore, not be presented here.

The generic $\tau \to 0^+$ asymptotic formulae for the functions $u(\tau)$ and $\varphi(\tau)$ stated in Theorem B.1 of [21] (and also for $u(\tau)$ in Theorem 3.4 of [16]) are valid for all $\rho \neq 0$ such that $|\mathrm{Re}(\rho)| < 1/2$. In Corollary 2.1 below, we consider the special cases $\rho = \pm \mathrm{i}a/2$, because, in these cases, the aforementioned asymptotic formulae require reparametrisation and one of their coefficients vanishes; this fact can be gleaned from equations (2.2) and (2.3).

Corollary 2.1. *Let* $(u(\tau), \varphi(\tau))$ *be a solution of the system* (1.1), (1.7) *corresponding to the monodromy data*

$$\begin{gathered} a, g_{21} \in \mathbb{C} \setminus \{0\}, \ \ \mathrm{Im}(a) \in [0,1), \ \ s_0^0 = 2\mathrm{i}\cosh(\pi a), \ \ s_0^\infty = 0, \ \ s_1^\infty \in \mathbb{C}, \\ g_{11} = \mathrm{i} g_{21} \mathrm{e}^{-\pi a}, \ \ g_{22} = \frac{\mathrm{i} - s_1^\infty g_{21}^2 \mathrm{e}^{\pi a}}{2 g_{21} \sinh(\pi a)}, \ \ g_{12} = -\frac{\mathrm{e}^{\pi a} + \mathrm{i} s_1^\infty g_{21}^2}{2 g_{21} \sinh(\pi a)}; \end{gathered} \tag{2.18}$$

then,

$$\begin{aligned} u(\tau) \underset{\tau \to 0^+}{=} & -\frac{b\tau}{2a} \left(\mathrm{e}^{\frac{\pi}{2}(a+\mathrm{i})} \left(\frac{\sinh(\pi a)}{\pi a} \right)^2 (\Gamma(1-\mathrm{i}a))^3 s_1^\infty g_{21}^2 \left(\frac{\varepsilon b \tau^2}{2} \right)^{\mathrm{i}a} + 1 \right) \\ & \times \left(1 + \mathcal{O}(\tau^\delta)\right), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\varphi(\tau)} \underset{\tau \to 0^+}{=} & \frac{\mathrm{e}^{\pi a}}{2\pi a g_{21}^2} \left(\frac{4}{\varepsilon b} \right)^{\mathrm{i}a} \left(\mathrm{e}^{\frac{\pi a}{2}} \Gamma(1-\mathrm{i}a) s_1^\infty g_{21}^2 \left(\frac{\varepsilon b \tau^2}{2} \right)^{\mathrm{i}a} - \mathrm{i}(\Gamma(1+\mathrm{i}a))^2 \right) \\ & \times \left(1 + \mathcal{O}(\tau^\delta)\right), \end{aligned} \tag{2.20}$$

where $\Gamma(*)$ *is the gamma function* [10], *and* $\delta > 0$.

Let $(u(\tau), \varphi(\tau))$ *be a solution of the system* (1.1), (1.7) *corresponding to the monodromy data*

$$\begin{gathered} a, g_{12} \in \mathbb{C} \setminus \{0\}, \ \ \mathrm{Im}(a) \in (-1,0], \ \ s_0^0 = 2\mathrm{i}\cosh(\pi a), \ \ s_0^\infty \in \mathbb{C}, \ \ s_1^\infty = 0, \\ g_{11} = \frac{s_0^\infty g_{12}^2 \mathrm{e}^{-\pi a} - \mathrm{i}}{2 g_{12} \sinh(\pi a)}, \ \ g_{22} = -\mathrm{i} g_{12} \mathrm{e}^{-\pi a}, \ \ g_{21} = -\frac{\mathrm{e}^{\pi a} + \mathrm{i} s_0^\infty g_{12}^2 \mathrm{e}^{-2\pi a}}{2 g_{12} \sinh(\pi a)}; \end{gathered} \tag{2.21}$$

then,

$$u(\tau) \underset{\tau\to0^+}{=} \frac{b\tau}{2a}\left(\mathrm{e}^{-\frac{3\pi}{2}(a-\mathrm{i})}\left(\frac{\sinh(\pi a)}{\pi a}\right)^2(\Gamma(1+\mathrm{i}a))^3 s_0^{\infty} g_{12}^2\left(\frac{\varepsilon b\tau^2}{2}\right)^{-\mathrm{i}a}-1\right) \times\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{2.22}$$

$$\mathrm{e}^{-\mathrm{i}\varphi(\tau)} \underset{\tau\to0^+}{=} -\frac{\mathrm{e}^{\pi a}}{2\pi a g_{12}^2}\left(\frac{\varepsilon b}{4}\right)^{\mathrm{i}a}\left(\mathrm{e}^{-\frac{3\pi a}{2}}\Gamma(1+\mathrm{i}a)s_0^{\infty}g_{12}^2\left(\frac{\varepsilon b\tau^2}{2}\right)^{-\mathrm{i}a}-\mathrm{i}(\Gamma(1-\mathrm{i}a))^2\right) \times\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right). \tag{2.23}$$

Let $(u(\tau),\varphi(\tau))$ *be a solution of the system* (1.1), (1.7) *corresponding to the monodromy data*

$$\begin{gathered} a, g_{12}\in\mathbb{C}\setminus\{0\}, \quad |\mathrm{Im}(a)|<1, \quad s_0^0=2\mathrm{i}\cosh(\pi a), \quad s_0^{\infty}=s_1^{\infty}=0, \\ g_{11}=-\frac{\mathrm{i}}{2g_{12}\sinh(\pi a)}, \quad g_{21}=-\frac{\mathrm{e}^{\pi a}}{2g_{12}\sinh(\pi a)}, \quad g_{22}=-\mathrm{i}g_{12}\mathrm{e}^{-\pi a}; \end{gathered} \tag{2.24}$$

then, the functions $u(\tau)$ *and* $\varphi(\tau)$ *are holomorphic at the origin* [18, 20], *and*

$$u(\tau) \underset{\tau\to0^+}{=} -\frac{\tau b}{2a}\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{2.25}$$

$$\mathrm{e}^{\mathrm{i}\varphi(\tau)} \underset{\tau\to0^+}{=} 2\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}\mathrm{e}^{-\pi a}g_{12}^2\sinh(\pi a)\left(\frac{4}{\varepsilon b}\right)^{\mathrm{i}a}\frac{\Gamma(1+\mathrm{i}a)}{\Gamma(1-\mathrm{i}a)}\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{2.26}$$

where $\delta=1$ *if* $a=\pm\mathrm{i}/2$ *and* $\delta=2$ *if* $a\neq\pm\mathrm{i}/2$. [6]

Proof. If $\rho=\pm\mathrm{i}a/2$, then it follows from Proposition 2.1 that

$$s_0^0=2\mathrm{i}\cosh(\pi a). \tag{2.27}$$

Since the asymptotic results for $u(\tau)$ presented in Theorem 3.4 of [16] are symmetric with respect to the change $\rho\to-\rho$, it suffices to consider the

[6] The results formulated in this corollary (and Theorems 3.1 and 4.1 below) are extended in [22] for all $a\neq0$; since the removal of the restriction $|\mathrm{Im}(a)|<1$ used in [16] requires a somewhat more elaborate technique, we continue to employ it in the current work. We did not discuss a precise estimate for $\delta>0$ in [16], so in equations (2.18) and (2.21) a more accurate restriction for $\mathrm{Im}(a)$, rather than $|\mathrm{Im}(a)|<1$, is stated. As a matter of fact, the corresponding asymptotics are valid for $|\mathrm{Im}(a)|<1$.

case $\rho = \mathrm{i}a/2$, which, from equation (2.2), implies

$$\chi_1(-\mathrm{i}a/2)\chi_2(\mathrm{i}a/2) = 0, \tag{2.28}$$

since $\mathfrak{p}(a, -\mathrm{i}a/2)\mathfrak{p}(-a, \mathrm{i}a/2) = (2/a)^2 \neq 0$. In conjunction with equations (1.2)–(1.6) and (2.27), equation (2.28) gives rise to the following three—distinct—cases for the determination of the corresponding monodromy data: (i) if $\chi_1(-\mathrm{i}a/2) = 0$ and $\chi_2(\mathrm{i}a/2) \neq 0$, then the monodromy data are given by equations (2.18); (ii) if $\chi_1(-\mathrm{i}a/2) \neq 0$ and $\chi_2(\mathrm{i}a/2) = 0$, then the monodromy data are provided by equations (2.21); and (iii) if $\chi_1(-\mathrm{i}a/2) = \chi_2(\mathrm{i}a/2) = 0$, then one arrives at the monodromy data (2.24). Furthermore, via equations (46) of [16],[2] it follows that: (i) $\chi_1(-\mathrm{i}a/2) = 0 \Rightarrow \varpi_1(-\mathrm{i}a/2) = 0$; (ii) $\chi_2(\mathrm{i}a/2) = 0 \Rightarrow \varpi_2(\mathrm{i}a/2) = 0$; and (iii) $\chi_1(-\mathrm{i}a/2) = \chi_2(\mathrm{i}a/2) = 0 \Rightarrow \varpi_1(-\mathrm{i}a/2) = \varpi_2(\mathrm{i}a/2) = 0$. Finally, substituting the monodromy data (2.18), (2.21), and (2.24) into the corresponding asymptotic formula in Theorem 3.4 of [16], one arrives at the $\tau \to 0^+$ asymptotics for $u(\tau)$ stated in equations (2.19), (2.22), and (2.25), respectively, whilst from the results of Theorem B.1 of [21], one arrives at the $\tau \to 0^+$ asymptotics for $\varphi(\tau)$ stated in equations (2.20), (2.23), and (2.26), respectively. □

Remark 2.2. Proposition 2.1 also implies that one can use the $\tau \to 0^+$ asymptotics of Corollary 2.1 for the cases $g_{11} = 0$ (and $g_{12}g_{21}g_{22} \neq 0$) or $g_{22} = 0$ (and $g_{11}g_{12}g_{21} \neq 0$); the corresponding monodromy data for these cases read: (i) for $g_{11} = 0$,

$$\begin{gathered} a, g_{12} \in \mathbb{C} \setminus \{0\}, \quad \operatorname{Im}(a) \in (-1, 0], \quad s_0^0 = 2\mathrm{i}\cosh(\pi a), \\ s_0^\infty = \frac{\mathrm{i}e^{\pi a}}{g_{12}^2}, \quad s_1^\infty = 0, \\ g_{11} = 0, \quad g_{21} = -\frac{1}{g_{12}}, \quad g_{22} = -\mathrm{i}e^{-\pi a} g_{12}; \end{gathered} \tag{2.29}$$

and (ii) for $g_{22} = 0$,

$$\begin{gathered} a, g_{21} \in \mathbb{C} \setminus \{0\}, \quad \operatorname{Im}(a) \in [0, 1), \\ s_0^0 = 2\mathrm{i}\cosh(\pi a), \quad s_0^\infty = 0, \quad s_1^\infty = \frac{\mathrm{i}e^{-\pi a}}{g_{21}^2}, \\ g_{11} = \mathrm{i}e^{-\pi a} g_{21}, \quad g_{12} = -\frac{1}{g_{21}}, \quad g_{22} = 0. \end{gathered} \tag{2.30}$$

Substituting the particular values of the monodromy data (2.29) (resp., (2.30)) into the asymptotic formulae (2.22) and (2.23) (resp., (2.19) and

(2.20)), one arrives at the corresponding asymptotics for the functions $u(\tau)$ and $\varphi(\tau)$.

§3. Logarithmic asymptotics

In [16], we obtained small-τ asymptotics of the function $u(\tau)$ for the case $\rho=0$: the corresponding results were presented in Theorem 3.5 of [16], where it was shown that these asymptotics possess logarithmic behaviour; the logarithmic asymptotics for the function $\varphi(\tau)$, however, was not considered in [16, 21]. As a consequence of Proposition 2.1, the restriction $g_{11}g_{22}\neq 0$ can be removed from the formulation of Theorem 3.5 in [16]. We also analyse, more carefully, the case $a=0$, and formulate the corresponding results in Corollary 3.1.

The following theorem is an extension of Theorem 3.5 in [16]:[7] it includes a new formula for the $\tau\to 0^+$ asymptotics of the function $\varphi(\tau)$, and also a simplified expression for the $\tau\to 0^+$ asymptotics of the function $u(\tau)$.

Theorem 3.1. *Let* $(u(\tau),\varphi(\tau))$ *be a solution of the system* (1.1), (1.7) *corresponding to the monodromy data* $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. *Suppose that*

$$a\in\mathbb{C}\setminus\{0\},\qquad |\mathrm{Im}(a)|<1,\qquad s_0^0=2\mathrm{i};\tag{3.1}$$

then,[8]

$$\begin{aligned}u(\tau)\underset{\tau\to0^+}{=}&-\frac{ab\tau}{4}\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(-\tfrac{\mathrm{i}a}{2}\right)-\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}(g_{12}+\mathrm{i}g_{22})}{(g_{12}-\mathrm{i}g_{22})}\right)\\&\times\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(1-\tfrac{\mathrm{i}a}{2}\right)-\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}(g_{12}+\mathrm{i}g_{22})}{(g_{12}-\mathrm{i}g_{22})}\right)\left(1+\mathcal{O}\left(\tau^\delta\right)\right),\end{aligned}\tag{3.2}$$

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\varphi(\tau)}\underset{\tau\to0^+}{=}&\frac{\mathrm{e}^{\frac{\pi}{2}(a+\mathrm{i})}}{\pi a}(g_{12}-\mathrm{i}g_{22})^2\left(\Gamma\left(1-\tfrac{\mathrm{i}a}{2}\right)\right)^2(2\tau^2)^{\mathrm{i}a}\\&\times\left(\frac{\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(1-\frac{\mathrm{i}a}{2}\right)-\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}(g_{12}+\mathrm{i}g_{22})}{(g_{12}-\mathrm{i}g_{22})}}{\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(-\frac{\mathrm{i}a}{2}\right)-\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}(g_{12}+\mathrm{i}g_{22})}{(g_{12}-\mathrm{i}g_{22})}}\right)\left(1+\mathcal{O}\left(\tau^\delta\right)\right),\end{aligned}\tag{3.3}$$

[7] For $\varepsilon_1=\varepsilon_2=0$, the monodromy functions in Theorem 3.5 of [16] simplify as follows: $s_0^0(0,0):=s_0^0$, $s_j^\infty(0,0):=s_j^\infty$, $j=0,1$, $g_{kl}(0,0):=g_{kl}$, $k,l=1,2$, $\chi_m(\vec{g}(0,0);0):=\chi_m$, $m=1,2$, and $\varpi_n^\flat(0,0):=\varpi_n^\flat$, $n=1,2,3,4$.

[8] There exists another logarithmic expansion as $\tau\to0^+$ for the functions $u(\tau)$ and $\varphi(\tau)$ corresponding to $s_0^0=-2\mathrm{i}$ [22].

where $\psi(z) := \frac{\mathrm{d} \ln \Gamma(z)}{\mathrm{d}z}$ *is the digamma function,*

$$\gamma = -\psi(1) = 0.5772156649015328606065l2\ldots$$

is the Euler–Mascheroni constant, and $\delta > 0$.

Proof. The corresponding $\tau \to 0^+$ asymptotic expansion of $u(\tau)$ given in Theorem 3.5, equation (51) of [16] reads, after multiplying out the various expressions in parentheses,

$$u(\tau) \underset{\tau \to 0^+}{=} \frac{\tau b \mathrm{e}^{\frac{\pi a}{2}}}{2a \sinh(\pi a/2)} \Big(\varpi_1^\flat \varpi_3^\flat + \big(\varpi_2^\flat \varpi_3^\flat - \varpi_1^\flat \varpi_4^\flat\big) \ln \tau - \varpi_2^\flat \varpi_4^\flat (\ln \tau)^2 \Big) \times \big(1 + \mathcal{O}\big(\tau^\delta\big)\big), \quad (3.4)$$

where the coefficients $\varpi_k^\flat$, $k = 1, 2, 3, 4$, are given in equations (52) and (53) of [16]. Using the digamma function identities [10] $\psi(1+z) = \psi(z) + 1/z$, $\psi(1-z) = \psi(z) + \pi \cot(\pi z)$, $\psi(\frac{1}{2}+z) - \psi(\frac{1}{2}-z) = \pi \tan(\pi z)$, and $\psi(1/2) = \psi(1) - 2 \ln 2$, the reflection formula [10] $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, the algebraic relations (1.2)–(1.6) (with $s_0^0 = 2\mathrm{i}$) for the monodromy data, and the corresponding equations (52) and (53) in [16], one shows that

$$\begin{aligned} -\varpi_2^\flat \varpi_4^\flat &= a^2 \omega_1(\mathbf{g};1)\omega_2(\mathbf{g};1), \\ \varpi_2^\flat \varpi_3^\flat - \varpi_1^\flat \varpi_4^\flat &= \frac{a^2}{2} \big(\omega_1(\mathbf{g};1)\big(\pi \mathrm{i}\, \omega_2(\mathbf{g};-1) + \omega_2(\mathbf{g};1)\Phi(\mathbf{g};-1)\big) \\ &\quad + \omega_2(\mathbf{g};1)\big(\pi \mathrm{i}\, \omega_1(\mathbf{g};-1) + \omega_1(\mathbf{g};1)\Phi(\mathbf{g};1)\big)\big), \\ \varpi_1^\flat \varpi_3^\flat &= \frac{a^2}{4} \big(\pi \mathrm{i}\, \omega_1(\mathbf{g};-1) + \omega_1(\mathbf{g};1)\Phi(\mathbf{g};1)\big) \\ &\quad \times \big(\pi \mathrm{i}\, \omega_2(\mathbf{g};-1) + \omega_2(\mathbf{g};1)\Phi(\mathbf{g};-1)\big), \end{aligned} \quad (3.5)$$

where, for $k = 1, 2$ and $l = \pm 1$,

$$\omega_k(\mathbf{g};l) := g_{1k} \mathrm{e}^{\frac{\pi \mathrm{i}}{4}} + l g_{2k} \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}}, \quad (3.6)$$

$$\Phi(\mathbf{g};l) := 4\gamma + \psi\big(1 + l \tfrac{\mathrm{i}a}{2}\big) + \ln\big(\tfrac{\varepsilon b}{2}\big) + l \tfrac{\pi \mathrm{i}}{2}. \quad (3.7)$$

Via the identities (3.5), one shows that the asymptotics (3.4) for $u(\tau)$ can be presented in the factorised form

$$u(\tau) \underset{\tau \to 0^+}{=} \frac{ab \mathrm{e}^{\frac{\pi a}{2}} \tau}{8 \sinh(\pi a/2)} \big(\omega_1(\mathbf{g};1) \ln(\tau^2) + \pi \mathrm{i}\, \omega_1(\mathbf{g};-1) + \omega_1(\mathbf{g};1)\Phi(\mathbf{g};1)\big) \times \big(\omega_2(\mathbf{g};1) \ln(\tau^2) + \pi \mathrm{i}\, \omega_2(\mathbf{g};-1) + \omega_2(\mathbf{g};1)\Phi(\mathbf{g};-1)\big)\big(1 + \mathcal{O}\big(\tau^\delta\big)\big). \quad (3.8)$$

Recall the following expression for the function $\varphi(\tau)$ given in [19]:

$$\varphi(\tau) = -\mathrm{i}\ln\left(\frac{\sqrt{-A(\tau)B(\tau)}}{B(\tau)}\right);$$

substituting into the latter expression the $\tau \to 0^+$ asymptotics for the functions $\sqrt{-A(\tau)B(\tau)}$ and $\sqrt{B(\tau)}$ given in Proposition 5.7 of [16],[5] using the identities (3.5), and simplifying, one shows that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\varphi(\tau)} \underset{\tau\to 0^+}{=} & \frac{2\pi(2\tau^2)^{\mathrm{i}a}}{a\sinh(\pi a/2)(\Gamma(\mathrm{i}a/2))^2} \\ & \times \left(\frac{\omega_2(\mathbf{g};1)\ln(\tau^2)+\pi\mathrm{i}\,\omega_2(\mathbf{g};-1)+\omega_2(\mathbf{g};1)\Phi(\mathbf{g};-1)}{\omega_1(\mathbf{g};1)\ln(\tau^2)+\pi\mathrm{i}\,\omega_1(\mathbf{g};-1)+\omega_1(\mathbf{g};1)\Phi(\mathbf{g};1)}\right)\left(1+\mathcal{O}(\tau^{\delta})\right). \end{aligned} \tag{3.9}$$

Recalling that $s_0^0 = 2\mathrm{i}$, one arrives at, after using the algebraic relations (1.2)–(1.6) and the definition(s) (3.6),

$$\begin{gathered} \omega_1(\mathbf{g};1)\omega_2(\mathbf{g};1) = -2\mathrm{e}^{-\frac{\pi a}{2}}\sinh(\pi a/2), \qquad \frac{\omega_2(\mathbf{g};1)}{\omega_1(\mathbf{g};1)} = \frac{g_{12}-\mathrm{i}g_{22}}{g_{11}-\mathrm{i}g_{21}}, \\ \frac{\omega_1(\mathbf{g};-1)}{\omega_1(\mathbf{g};1)} = \frac{g_{11}+\mathrm{i}g_{21}}{g_{11}-\mathrm{i}g_{21}}, \qquad \frac{\omega_2(\mathbf{g};-1)}{\omega_2(\mathbf{g};1)} = \frac{g_{12}+\mathrm{i}g_{22}}{g_{12}-\mathrm{i}g_{22}}: \end{gathered} \tag{3.10}$$

using equations (3.10), one simplifies the asymptotics (3.8) and (3.9), respectively, as follows:

$$\begin{aligned} u(\tau) \underset{\tau\to 0^+}{=} & -\frac{ab\tau}{4}\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(1+\tfrac{\mathrm{i}a}{2}\right)+\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}\,(g_{11}+\mathrm{i}g_{21})}{(g_{11}-\mathrm{i}g_{21})}\right) \\ & \times\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(1-\tfrac{\mathrm{i}a}{2}\right)-\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}(g_{12}+\mathrm{i}g_{22})}{(g_{12}-\mathrm{i}g_{22})}\right)\left(1+\mathcal{O}(\tau^{\delta})\right), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\varphi(\tau)} \underset{\tau\to 0^+}{=} & \frac{2\pi(g_{12}-\mathrm{i}g_{22})(2\tau^2)^{\mathrm{i}a}}{a\sinh(\pi a/2)(\Gamma(\mathrm{i}a/2))^2(g_{11}-\mathrm{i}g_{21})} \\ & \times\left(\frac{\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(1-\frac{\mathrm{i}a}{2}\right)-\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}\,(g_{12}+\mathrm{i}g_{22})}{(g_{12}-\mathrm{i}g_{22})}}{\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+4\gamma+\psi\left(1+\frac{\mathrm{i}a}{2}\right)+\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}(g_{11}+\mathrm{i}g_{21})}{(g_{11}-\mathrm{i}g_{21})}}\right)\left(1+\mathcal{O}(\tau^{\delta})\right). \end{aligned} \tag{3.12}$$

In order to reduce the number of parameters appearing in the asymptotics (3.11) and (3.12), one shows from the definition(s) (3.6) and equations

(3.10) that

$$\frac{g_{11}+\mathrm{i}g_{21}}{g_{11}-\mathrm{i}g_{21}}=\frac{\mathrm{e}^{\frac{\pi}{2}(a-\mathrm{i})}(g_{11}g_{12}+g_{21}g_{22})}{2\sinh(\pi a/2)}-\frac{\mathrm{e}^{\frac{\pi a}{2}}}{2\sinh(\pi a/2)}, \tag{3.13}$$

$$\frac{g_{12}+\mathrm{i}g_{22}}{g_{12}-\mathrm{i}g_{22}}=\frac{\mathrm{e}^{\frac{\pi}{2}(a-\mathrm{i})}(g_{11}g_{12}+g_{21}g_{22})}{2\sinh(\pi a/2)}+\frac{\mathrm{e}^{\frac{\pi a}{2}}}{2\sinh(\pi a/2)}, \tag{3.14}$$

whence

$$\frac{g_{11}+\mathrm{i}g_{21}}{g_{11}-\mathrm{i}g_{21}}=\frac{g_{12}+\mathrm{i}g_{22}}{g_{12}-\mathrm{i}g_{22}}-\frac{\mathrm{e}^{\frac{\pi a}{2}}}{\sinh(\pi a/2)}. \tag{3.15}$$

Finally, via equation (3.15) and the digamma and gamma function identities given at the beginning of the proof, one simplifies the asymptotics (3.11) and (3.12), respectively, in order to arrive at the $\tau \to 0^+$ asymptotics (3.2) and (3.3) for $u(\tau)$ and $\varphi(\tau)$. □

Remark 3.1. If one defines

$$c:=4\gamma+\psi(-\mathrm{i}a/2)-\frac{\pi\mathrm{i}}{2}+\frac{\pi\mathrm{i}\,(g_{12}+\mathrm{i}g_{22})}{(g_{12}-\mathrm{i}g_{22})}+\frac{\mathrm{i}}{a}+\ln(\varepsilon b/2), \tag{3.16}$$

then the asymptotics (3.2) and (3.3), respectively, can be presented in the simplified form

$$u(\tau) \underset{\tau\to 0^+}{=} -ab\tau\left((\ln\tau)^2+c\ln\tau+\frac{1}{4}\left(c^2+\frac{1}{a^2}\right)\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{3.17}$$

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\varphi(\tau)} \underset{\tau\to 0^+}{=} &\frac{\mathrm{e}^{\frac{\pi}{2}(a+\mathrm{i})}}{\pi a}(g_{12}-\mathrm{i}g_{22})^2\left(\Gamma\left(1-\tfrac{\mathrm{i}a}{2}\right)\right)^2(2\tau^2)^{\mathrm{i}a}\\ &\times\left(\frac{\ln\tau+\frac{1}{2}(c+\mathrm{i}/a)}{\ln\tau+\frac{1}{2}(c-\mathrm{i}/a)}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \end{aligned} \tag{3.18}$$

where $\delta>0$.

For $a=0$, the asymptotic results stated in Theorem 3.1 require reparametrisation: this is given in the following corollary.

Corollary 3.1. *Let $(u(\tau),\varphi(\tau))$ be a solution of the system* (1.1), (1.7) *corresponding to the monodromy data*

$$\begin{gathered} a=s_0^{\infty}=0, \quad s_0^0=2\mathrm{i}, \quad g_{21}\in\mathbb{C}\setminus\{0\}, \quad s_1^{\infty}=\frac{\mathrm{i}}{g_{21}^2}, \quad g_{11}=\mathrm{i}g_{21},\\ g_{12}=-\frac{1+\mathrm{i}c_1}{2g_{21}}, \quad g_{22}=-\frac{c_1+\mathrm{i}}{2g_{21}}, \quad c_1\in\mathbb{C}; \end{gathered} \tag{3.19}$$

then,

$$u(\tau) \underset{\tau\to 0^+}{=} \frac{\mathrm{i}b\tau}{2}\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+3\gamma-\pi c_1-\frac{\pi\mathrm{i}}{2}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{3.20}$$

$$\mathrm{e}^{\mathrm{i}\varphi(\tau)} \underset{\tau\to 0^+}{=} -\frac{1}{2\pi g_{21}^2}\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+3\gamma-\pi c_1-\frac{\pi\mathrm{i}}{2}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{3.21}$$

where $\gamma=-\psi(1)=0.577215664901532860606512\ldots$ *is the Euler–Mascheroni constant, and* $\delta>0$.

Let $(u(\tau),\varphi(\tau))$ *be a solution of the system* (1.1), (1.7) *corresponding to the monodromy data*

$$\begin{gathered} a=s_1^{\infty}=0, \quad s_0^0=2\mathrm{i}, \quad g_{12}\in\mathbb{C}\setminus\{0\}, \quad s_0^{\infty}=\frac{\mathrm{i}}{g_{12}^2}, \quad g_{22}=-\mathrm{i}g_{12}, \\ g_{11}=\frac{c_2-\mathrm{i}}{2g_{12}}, \quad g_{21}=-\frac{3+\mathrm{i}c_2}{2g_{12}}, \quad c_2\in\mathbb{C}; \end{gathered} \tag{3.22}$$

then,

$$u(\tau) \underset{\tau\to 0^+}{=} -\frac{\mathrm{i}b\tau}{2}\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+3\gamma+\pi c_2-\frac{3\pi\mathrm{i}}{2}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{3.23}$$

$$\mathrm{e}^{-\mathrm{i}\varphi(\tau)} \underset{\tau\to 0^+}{=} -\frac{1}{2\pi g_{12}^2}\left(\ln\left(\frac{\varepsilon b\tau^2}{2}\right)+3\gamma+\pi c_2-\frac{3\pi\mathrm{i}}{2}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right). \tag{3.24}$$

Proof. For $a=0$ and $s_0^0=2\mathrm{i}$ ($\rho=0$), equation (1.2) implies that $s_0^{\infty}s_1^{\infty}=0$. The case $s_0^{\infty}=s_1^{\infty}=0$ contradicts equation (1.6); therefore, the following two cases are left: (i) $s_0^{\infty}=0$ and $s_1^{\infty}\neq 0$; and (ii) $s_0^{\infty}\neq 0$ and $s_1^{\infty}=0$. Consider case (i). An analysis of equations (1.3)–(1.6) shows that the monodromy data can be presented as in equations (3.19). To derive the asymptotics of the functions $u(\tau)$ and $\varphi(\tau)$, one notes that the parameter c (cf. equation (3.16)) has the following asymptotic behaviour as $a\to 0$:

$$c=-\frac{\mathrm{i}}{a}+\widehat{c}+\mathcal{O}(a), \tag{3.25}$$

where

$$\widehat{c}:=3\gamma-\frac{\pi\mathrm{i}}{2}-\pi c_1+\ln(\varepsilon b/2). \tag{3.26}$$

Write, now, the asymptotics (3.17) and (3.18), respectively, in the form

$$u(\tau) \underset{\tau\to 0^+}{=} -b\tau\left(a(\ln\tau)^2+ac\ln\tau+\frac{a}{4}\left(c+\frac{\mathrm{i}}{a}\right)\left(c-\frac{\mathrm{i}}{a}\right)\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{3.27}$$

and

$$e^{i\varphi(\tau)} \underset{\tau\to 0^+}{=} \frac{e^{\frac{\pi}{2}(a+i)}}{\pi g_{21}^2}\left(\Gamma\left(1-\tfrac{ia}{2}\right)\right)^2 e^{ia\ln(2\tau^2)}\left(\frac{\ln\tau+\frac{1}{2}(c+i/a)}{a\ln\tau+\frac{a}{2}(c-i/a)}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right); \tag{3.28}$$

substituting into equations (3.27) and (3.28) the relation for c given in equation (3.25) and taking the limit as $a\to 0$, one gets

$$u(\tau) \underset{\tau\to 0^+}{=} \frac{ib\tau}{2}\left(\ln\tau^2+\widehat{c}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right), \tag{3.29}$$

and

$$e^{i\varphi(\tau)} \underset{\tau\to 0^+}{=} -\frac{1}{2\pi g_{21}^2}\left(\ln\tau^2+\widehat{c}\right)\left(1+\mathcal{O}\left(\tau^{\delta}\right)\right); \tag{3.30}$$

thus, one arrives at the asymptotics (3.20) and (3.21), respectively. The analysis for case (ii), with corresponding monodromy data given by equations (3.22), is similar; however, there is a difference, because the denominator of the fractional term $i\pi(g_{12}+ig_{22})/(g_{12}-ig_{22})$ appearing in the asymptotics (3.2) and (3.3) vanishes. In order to resolve this problem, one has to exploit equation (3.15); by doing so, the following representation for the asymptotics of the parameter c as $a\to 0$ is obtained:

$$c=\frac{i}{a}+\check{c}+\mathcal{O}(a), \tag{3.31}$$

where

$$\check{c}:=3\gamma-\frac{3\pi i}{2}+\pi c_2+\ln(\varepsilon b/2). \tag{3.32}$$

Proceeding, now, as delineated above, one arrives at the asymptotics (3.23) and (3.24). □

Remark 3.2. The validation of the limiting procedure as $a\to 0$ studied in Corollary 3.1 is based on the justification scheme for the Isomonodromy Deformation Method [15]. The calculation of the monodromy data in [16] for the case $\rho=0$ was undertaken in accordance with the condition $|\mathrm{Im}(a)|<1$ and certain assumptions on the 4-tuple of coefficient functions $(A(\tau),B(\tau),C(\tau),D(\tau))$ of a first-order 2×2 matrix linear ODE whose isomonodromy deformations are described in terms of solutions of the DP3E: these assumptions are valid for all a satisfying the restriction $|\mathrm{Im}(a)|<1$, including, in particular, $a=0$; therefore, the associated asymptotic formulae for the functions $u(\tau)$ and $\varphi(\tau)$ should be valid for all a such that $\mathrm{Im}(a)\in(-1,1)$. The manifestation of this problem is principally due to the fact that one can not assign a unique parametrisation for $\mathcal{M}$

that subsumes both sets of the monodromy data (3.19) and (3.22); more precisely, for a close to 0, one can not introduce variables that parametrise the asymptotics smoothly in the neighbourhoods of two disjoint curves on $\mathcal{M}$, thus the neighbourhood of each curve requires its own parametrisation (cf. equations (3.26) and (3.32)). After the implementation of such a parametrisation, one simply sets $a=0$ in the corresponding asymptotic formulae; in fact, the *raison d'être* of the proof is to delineate a paradigm for how one finds proper, smooth parametrisations for asymptotics in the neighbourhoods of each of the two disjoint curves on $\mathcal{M}$.

Remark 3.3. The results presented in Corollary 3.1 allow one to check the two logarithmic asymptotics for $\tau^{-1/2}\mathrm{e}^{u}$ that appear directly below equations (19) on p. 2082 in [14]; the result of this comparison shows that the right-hand sides of both formulae for $\tau^{-1/2}\mathrm{e}^{u}$ must be multiplied by 2 (see, also, Appendix A of [13]).

§4. Distribution of zeros

In many, but not all, cases, at a point where a solution of a Painlevé equation has a zero or a pole, the corresponding Fuchs-Garnier pair for this Painlevé equation is not defined. Even when the Fuchs-Garnier pair is well defined, at a zero, say, the definition of the canonical solutions may require a modification depending on the type of the zero: this is precisely the case we encounter in the study of the DP3E.

On the other hand, in case an asymptotic formula for the corresponding solution of a Painlevé equation has zeros or poles accumulating at a singular point where we construct the asymptotics, the standard asymptotic technique based on the isomonodromy deformations also fails in some neighbourhood of these points; therefore, the poles and zeros of the asymptotic formulae also require a more accurate treatment.

Taking the DP3E as our principal example, we recall the standard interpretation of the asymptotic results in case the leading term of asymptotics has zeros accumulating at the origin. Of course, for the description of the asymptotic behaviour of the solution in a neighbourhood of zeros or poles of the leading term of its asymptotics accumulating at the corresponding singular point, one can invoke the correction terms of the asymptotic expansion; this, however, is the next step of the asymptotic analysis, because the initial step is to use the leading term of asymptotics in order to decide whether or not, in a neighbourhood of its zeros or poles, there are zeros and/or poles of the corresponding solution: this is the second question that we address in this section. To conclude the introductory part of this section, we note that, as follows from a local analysis of equation (1.1), all zeros of its solutions are of order one and all poles are of order two.

We recall the asymptotic formulae as $\tau \to 0$ for the functions $u(\tau)$ and $\varphi(\tau)$ given in Theorem B.1 of [21]. These formulae contain a branching parameter, ρ; for $\rho = \mathrm{i}\varkappa$, $\varkappa \in \mathbb{R} \setminus \{0\}$, these asymptotics read:

$$u(\tau) \underset{\tau\to 0}{=} \frac{\tau b \mathrm{e}^{\frac{\pi a}{2}}}{16\pi} \left(\mathcal{A}(\varkappa)\tau^{2\mathrm{i}\varkappa} + \mathcal{A}(-\varkappa)\tau^{-2\mathrm{i}\varkappa}\right) \times \left(\mathcal{B}(\varkappa)\tau^{2\mathrm{i}\varkappa} + \mathcal{B}(-\varkappa)\tau^{-2\mathrm{i}\varkappa}\right)\left(1 + \mathcal{O}\left(\tau^{\delta}\right)\right), \tag{4.1}$$

$$\mathrm{e}^{\mathrm{i}\varphi(\tau)} \underset{\tau\to 0}{=} \mathrm{e}^{\pi\mathrm{i}}(2\tau^2)^{\mathrm{i}a} \left(\frac{\mathcal{B}(\varkappa)\tau^{2\mathrm{i}\varkappa} + \mathcal{B}(-\varkappa)\tau^{-2\mathrm{i}\varkappa}}{\mathcal{A}(\varkappa)\tau^{2\mathrm{i}\varkappa} + \mathcal{A}(-\varkappa)\tau^{-2\mathrm{i}\varkappa}}\right)\left(1 + \mathcal{O}\left(\tau^{\delta}\right)\right), \tag{4.2}$$

where

$$\mathcal{A}(\varkappa) = \left(\tfrac{1}{2}(\varepsilon b)\mathrm{e}^{\frac{\pi \mathrm{i}}{2}}\right)^{\mathrm{i}\varkappa} \frac{\Gamma(1-2\mathrm{i}\varkappa)}{\Gamma(1+2\mathrm{i}\varkappa)} \frac{\Gamma(1+\frac{\mathrm{i}a}{2}+\mathrm{i}\varkappa)}{\mathrm{i}\varkappa} \times \left(g_{11}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\mathrm{e}^{-\pi\varkappa} + g_{21}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\mathrm{e}^{\pi\varkappa}\right), \tag{4.3}$$

$$\mathcal{B}(\varkappa) = \left(\tfrac{1}{2}(\varepsilon b)\mathrm{e}^{-\frac{\pi \mathrm{i}}{2}}\right)^{\mathrm{i}\varkappa} \frac{\Gamma(1-2\mathrm{i}\varkappa)}{\Gamma(1+2\mathrm{i}\varkappa)} \frac{\Gamma(1-\frac{\mathrm{i}a}{2}+\mathrm{i}\varkappa)}{\mathrm{i}\varkappa} \times \left(g_{12}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\mathrm{e}^{-\pi\varkappa} + g_{22}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\mathrm{e}^{\pi\varkappa}\right). \tag{4.4}$$

If, in addition, we assume that the Stokes multipliers corresponding to $u(\tau)$ and $\varphi(\tau)$ satisfy the condition $s_0^{\infty} s_1^{\infty} \neq 0$, then $\mathcal{A}(\varkappa)\mathcal{A}(-\varkappa)\mathcal{B}(\varkappa)\mathcal{B}(-\varkappa) \neq 0$; under this assumption, equation (4.1) can be rewritten as

$$u(\tau) = u_{as}(\tau)\left(1 + \mathcal{O}\left(\tau^{\delta}\right)\right) = u_{as}(\tau) + \mathcal{O}\left(\tau^{1+\delta}\right), \tag{4.5}$$

where $u_{as}(\tau)$ can be presented in the following form:

$$u_{as}(\tau) = \frac{\tau b \mathrm{e}^{\frac{\pi a}{2}}}{4\pi} \sqrt{\mathcal{A}(\varkappa)\mathcal{A}(-\varkappa)}\,\sqrt{\mathcal{B}(\varkappa)\mathcal{B}(-\varkappa)}\,\cosh(z_{\mathcal{A}}(\tau))\cosh(z_{\mathcal{B}}(\tau)), \tag{4.6}$$

with

$$z_{\mathcal{A}}(\tau) := 2\mathrm{i}\varkappa \ln\tau + \frac{1}{2}\ln\left(\frac{\mathcal{A}(\varkappa)}{\mathcal{A}(-\varkappa)}\right), \quad z_{\mathcal{B}}(\tau) := 2\mathrm{i}\varkappa \ln\tau + \frac{1}{2}\ln\left(\frac{\mathcal{B}(\varkappa)}{\mathcal{B}(-\varkappa)}\right). \tag{4.7}$$

Equations (4.6) and (4.7) imply that there are two sequences of zeros of $u_{as}(\tau)$ accumulating at the origin:

$$\widehat{\tau}_{m,\mathcal{C}} = \exp\left(-\frac{\pi m}{2|\varkappa|} + \frac{\pi}{4\varkappa} + \frac{\mathrm{i}}{4\varkappa}\ln\left(\frac{\mathcal{C}(\varkappa)}{\mathcal{C}(-\varkappa)}\right)\right), \qquad \mathcal{C} \in \{\mathcal{A}, \mathcal{B}\}, \tag{4.8}$$

where the branches of the ln-functions can be fixed arbitrarily, and $m \in \mathbb{N}$.

Define discs $D_{m,\mathcal{C}}$ of radius $R_{m,\mathcal{C}} = R_0|\widehat{\tau}_{m,\mathcal{C}}|^{1+\delta_d}$, $R_0 > 0$, $0 < \delta_d < \delta$, centred at $\widehat{\tau}_{m,\mathcal{C}}$. If $\delta_d < \delta$, then R_0 can be taken equal to 1, and if $\delta_d = \delta$, then R_0 is the same as the constant[9] in the estimate of the function $\mathcal{O}\big(\tau^{\delta}\big)$ in equation (4.1).

Proposition 4.1. *For large enough $m_1, m_2 \in \mathbb{N}$, the discs $D_{m_1,\mathcal{C}_1}$ and $D_{m_2,\mathcal{C}_2}$ do not intersect.*

Proof. For large enough m_1 and m_2, the distance between the centres of the discs is of the order $\mathcal{O}\big(|\widehat{\tau}_{m,\mathcal{C}_1}|\big)$, where $m := \min\{m_1, m_2\}$, while $R_{m_1,\mathcal{C}_1} + R_{m_2,\mathcal{C}_2} = \mathcal{O}\big(|\widehat{\tau}_{m,\mathcal{C}_1}|^{1+\delta_d}\big)$. □

Consider the sector $\mathcal{S} := \{\tau \in \mathbb{C} : |\arg(\tau)| \leqslant \phi_0 < \pi\}$, and define the complement of the open discs in this sector, $\widehat{\mathcal{S}} := \mathcal{S} \setminus \bigcup_{m\in\mathbb{N},\mathcal{C}\in\{\mathcal{A},\mathcal{B}\}} \mathrm{Int}\, D_{m,\mathcal{C}}$.

Theorem 4.1. *Let $(u(\tau), \varphi(\tau))$ be a solution of the system* (1.1), (1.7) *corresponding to the monodromy data $(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22})$. Suppose that* [10]

$$|\operatorname{Im}(a)| < 1, \qquad \operatorname{Re}(s_0^0) = 0, \quad \operatorname{Im}(s_0^0) > 2, \qquad s_0^\infty s_1^\infty \neq 0, \tag{4.9}$$

and $\varkappa \in \mathbb{R} \setminus \{0\}$ is a solution of the equation $s_0^0 = 2\mathrm{i}\cosh(2\pi\varkappa)$.[11] *Then, the asymptotics of the functions $u(\tau)$ and $\varphi(\tau)$ as $\tau \to 0$ and $\tau \in \mathrm{Int}\,\widehat{\mathcal{S}}$ are given by equations* (4.1) *and* (4.2), *respectively.*

Proof. The functions $u_{as}(\tau)$ and $\mathrm{e}^{\mathrm{i}\varphi_{as}(\tau)}$ are non-vanishing holomorphic functions in $\mathrm{Int}\,\widehat{\mathcal{S}}$.[12] They define the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ (cf. Appendix A of this paper and [16]) which satisfy all the conditions used in the derivation of the asymptotics presented in [16] in $\mathrm{Int}\,\widehat{\mathcal{S}}$. The functions $u(\tau)$ and $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$ are meromorphic functions in $\mathrm{Int}\,\widehat{\mathcal{S}}$ as a consequence of the

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[9] The constant depends on the monodromy data.

[10] Note that the restriction $s_0^\infty s_1^\infty \neq 0$ in the conditions (4.9) is equivalent to (cf. equation (1.2)) $s_0^0 \neq 2\mathrm{i}\cosh(\pi a)$.

[11] The sign of $\varkappa$ can be chosen arbitrarily because of the $\varkappa \to -\varkappa$ symmetry of the asymptotics (4.1) and (4.2).

[12] The function $\varphi_{as}(\tau)$ is defined as the leading term of the asymptotics (4.2).

Painlevé property. Thus, one can use the justification scheme presented in [15], which proves the asymptotic expansion (4.5) and the analogous relation for $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$; in particular, the functions $u(\tau)$ and $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$ have neither zeros nor poles in $\operatorname{Int}\widehat{\mathcal{S}}$. □

Lemma 4.1. *If $u(\tau)$ is the solution of equation* (1.1) *that corresponds to the monodromy data satisfying the restrictions* (4.9)*, then, for all large enough $m \in \mathbb{N}$, the discs $D_{m,\mathcal{C}}$ do not contain poles of the function $u(\tau)$.*

Proof. For $\varepsilon = 1$, the Laurent expansion of the function $u(\tau)$ in a neighbourhood of a pole $\tau_p \neq 0$ reads:

$$\begin{gathered} u(\tau)=\frac{u_{-2}}{(\tau-\tau_p)^2}+\sum_{k=0}^{\infty} u_k(\tau-\tau_p)^k, \quad u_{-2}=-\frac{\tau_p}{4}, \quad u_0\in\mathbb{C}, \quad u_1=-\frac{u_0}{\tau_p}, \\ u_2=\frac{2ab\tau_p-24\tau_p u_0^2+9u_0}{10\tau_p^2}, \quad u_3=-\frac{4(2ab\tau_p-54\tau_p u_0^2+9u_0)}{45\tau_p^3}, \quad \ldots . \end{gathered} \tag{4.10}$$

The coefficients u_k, $k \geqslant 4$, can be uniquely determined in terms of u_0 and τ_p. To introduce the parameter $\varepsilon = \pm 1$ into the expansion (4.10), one can make the following substitutions in the formulae for the coefficients u_k: $b \to \varepsilon b$, $u_k \to \varepsilon u_k$ for all $k = -2, 0, 1, 2, \ldots$, and take into account that $\varepsilon^2 = 1$.

The proof is by contradiction. Assume that $u(\tau)$ has $n \geqslant 1$ poles, $\tau_{p_1}, \ldots, \tau_{p_n}$, in $D_{m,\mathcal{C}}$; then, the following integral can be evaluated explicitly via the Residue Theorem:

$$\frac{1}{2\pi\mathrm{i}} \oint\limits_{\partial D_{m,\mathcal{C}}} \tau(u(\tau) - u_{as}(\tau))\, \mathrm{d}\tau = -\frac{1}{4}\sum_{k=1}^{n} \tau_{p_k}. \tag{4.11}$$

The absolute value of the sum of poles on the right-hand side of equation (4.11) can be estimated from below with the help of the triangle inequality:

$$\begin{gathered} \left|-\frac{1}{4}\sum_{k=1}^{n}\tau_{p_k}\right| = \frac{1}{4}\left|\sum_{k=1}^{n}\widehat{\tau}_{m,\mathcal{C}} + \tau_{p_k} - \widehat{\tau}_{m,\mathcal{C}}\right| \geqslant \frac{1}{4}\sum_{k=1}^{n}\left(|\widehat{\tau}_{m,\mathcal{C}}| - |\tau_{p_k} - \widehat{\tau}_{m,\mathcal{C}}|\right) \\ > \frac{n}{4}\left(|\widehat{\tau}_{m,\mathcal{C}}| - R_{m,\mathcal{C}}\right) = \frac{n}{4}\left|\widehat{\tau}_{m,\mathcal{C}}\right|\left(1 - R_0\left|\widehat{\tau}_{m,\mathcal{C}}\right|^{\delta_d}\right). \end{gathered} \tag{4.12}$$

Now, we evaluate the integral on the left-hand side of equation (4.11) from above:

$$\left|\frac{1}{2\pi\mathrm{i}}\oint\limits_{\partial D_{m,c}}\tau(u(\tau)-u_{as}(\tau))\,\mathrm{d}\tau\right|\leqslant\frac{1}{2\pi}\oint\limits_{\partial D_{m,c}}|\tau||u(\tau)-u_{as}(\tau)|\,|\mathrm{d}\tau|$$
$$<R_{m,c}(|\widehat{\tau}_{m,c}|+R_{m,c})\left|\mathcal{O}\left((|\widehat{\tau}_{m,c}|+R_{m,c})^{1+\delta_d}\right)\right|=\left|\mathcal{O}\left(|\widehat{\tau}_{m,c}|^{3+2\delta_d}\right)\right|. \tag{4.13}$$

Combining inequalities (4.12) and (4.13) and dividing both sides by $|\widehat{\tau}_{m,c}|$, we get

$$\frac{n}{4}\left(1-R_0\,|\widehat{\tau}_{m,c}|^{\delta_d}\right)<\left|\mathcal{O}\left(|\widehat{\tau}_{m,c}|^{2+2\delta_d}\right)\right|. \tag{4.14}$$

Letting $m\to\infty$, i.e., $\widehat{\tau}_{m,c}\to 0$, we arrive at the conclusion that $n=0$. □

Corollary 4.1. *For any solution $u(\tau)$ corresponding to the monodromy data specified in Theorem* 4.1*, there exists a pole-free small enough cut neighbourhood of the origin, i.e., there exists $\epsilon>0$ such that for all $\tau\in\mathbb{C}$ with $|\arg\tau|<\pi$ and $0<|\tau|<\epsilon$ the function $u(\tau)$ has no poles.*

Proof. Follows from Theorem 4.1 and Lemma 4.1. □

Theorem 4.2. *If $u(\tau)$ is the solution of equation* (1.1) *that corresponds to the monodromy data satisfying the restrictions* (4.9)*, then, for all large enough $m\in\mathbb{N}$, the discs $D_{m,c}$ contain one and only one zero of the function $u(\tau)$.*

Proof. This is a consequence of the fact that the function $u_{as}(\tau)$ has, by construction, only one zero located at the centre of $D_{m,c}$ and Rouché's Theorem, which should be applied to the right-hand side of equation (4.5) where the function $\mathcal{O}(\tau^{1+\delta})$ is holomorphic in $D_{m,c}$ (as the difference of two holomorphic functions), and the condition $|u_{as}(\tau)|>|\mathcal{O}(\tau^{1+\delta})|$ is guaranteed for all large enough m by a proper choice of the radius, $R_{m,c}$, of the disc: one either has to take $\delta_d<\delta$, or, for a sharper estimate, take $\delta_d=\delta$ and increase, if necessary, the parameter R_0. □

Theorem 4.2 can also be reformulated as follows.

Corollary 4.2. *Assume that a solution $u(\tau)$ of equation* (1.1) *corresponds to the monodromy data specified in Theorem* 4.1*. Then, there exists a small enough cut neighbourhood of the origin (i.e., there exists $\epsilon>0$ such that for all $\tau\in\mathbb{C}$ with $|\arg\tau|<\pi$ and $0<|\tau|<\epsilon$) where all zeros, $\tau_{m,c}$, of* *the function* $u(\tau)$ *can be approximated by the corresponding zeros,* $\widehat{\tau}_{m,\mathcal{C}}$*, of the leading term of its asymptotics* $u_{as}(\tau)$*:*

$$\tau_{m,\mathcal{C}} \underset{m\to\infty}{=} \widehat{\tau}_{m,\mathcal{C}} \left(1+\mathcal{O}\left(\mathrm{e}^{-\frac{\pi m\delta}{2|\varkappa|}}\right)\right), \tag{4.15}$$

where $\widehat{\tau}_{m,\mathcal{C}}$ *are defined in equation* (4.8)*.*

Proposition 4.2. *Let the function* $\varphi(\tau)$ *be a solution of equation* (1.7) *corresponding to the monodromy data specified in Theorem* 4.1*. Then, in a small enough cut neighbourhood of the origin, all zeros of the function* $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$ *are located at* $\tau_{m,\mathcal{B}}$ *and all poles are located at* $\tau_{m,\mathcal{A}}$*.*

Proof. As follows from equation (1.7), the zeros and poles of $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$ are located at the zeros of $u(\tau)$. The asymptotic formulae (4.1) and (4.2) make explicit which zeros of $u(\tau)$ define the zeros and poles of $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$. □

In case the last condition in (4.9) is not valid, i.e., one of the Stokes multipliers at the point at infinity vanishes, there are also sequences of zeros of the function $u(\tau)$ accumulating at the origin. The asymptotic formulae (4.1) and (4.2) for these cases remain valid, and therefore their analysis does not require any new ideas; nevertheless, these asymptotics can be simplified and presented in a slightly different form: this is done below.

Theorem 4.3. *Assume that a solution* $(u(\tau),\varphi(\tau))$ *of the system* (1.1), (1.7) *corresponds to the monodromy data*

$$\begin{gathered} a=-2\varkappa, \quad \varkappa\in\mathbb{R}\setminus\{0\}, \qquad s_1^{\infty}, g_{21}\in\mathbb{C}\setminus\{0\},\\ s_0^{\infty}=0, \qquad s_0^0=2\mathrm{i}\cosh(2\pi\varkappa),\\ g_{11}=\mathrm{i}g_{21}\mathrm{e}^{2\pi\varkappa}, \;\; g_{22}=\frac{s_1^{\infty}g_{21}^2\mathrm{e}^{-2\pi\varkappa}-\mathrm{i}}{2g_{21}\sinh(2\pi\varkappa)}, \;\; g_{12}=\frac{\mathrm{i}s_1^{\infty}g_{21}^2+\mathrm{e}^{-2\pi\varkappa}}{2g_{21}\sinh(2\pi\varkappa)}; \end{gathered} \tag{4.16}$$

then $\mathcal{A}(\varkappa)=0$*. The functions* $u(\tau)$ *and* $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$ *have only one sequence of zeros,* $\tau_{m,\mathcal{B}}$*,* $m\in\mathbb{N}$*, accumulating at the origin with asymptotic behaviour* (4.15) *for* $\mathcal{C}=\mathcal{B}$*.*

The asymptotics of the functions $u(\tau)$ *and* $\varphi(\tau)$ *are given by equations* (4.1) *and* (4.2)*, respectively, with* $\mathcal{A}(\varkappa)=0$ *and* $\tau\to 0$ *in the sector* $\widehat{\mathcal{S}}_{\mathcal{B}} := \mathcal{S}\setminus \bigcup_{m\in\mathbb{N}} \mathrm{Int}\, D_{m,\mathcal{B}}$*; these asymptotic formulae coincide with those in Corollary* 2.1 *(cf. equations* (2.19) *and* (2.20) *for* $a=-2\varkappa$*).*

Theorem 4.4. *Assume that a solution* $(u(\tau),\varphi(\tau))$ *of the system* (1.1), (1.7) *corresponds to the monodromy data*

$$a=2\varkappa,\quad \varkappa\in\mathbb{R}\setminus\{0\},$$
$$s_0^{\infty},g_{12}\in\mathbb{C}\setminus\{0\},\quad s_1^{\infty}=0,\quad s_0^0=2\mathrm{i}\cosh(2\pi\varkappa),$$
$$g_{22}=-\mathrm{i}g_{12}\mathrm{e}^{-2\pi\varkappa},\ \ g_{11}=\frac{s_0^{\infty}g_{12}^2\mathrm{e}^{-2\pi\varkappa}-\mathrm{i}}{2g_{12}\sinh(2\pi\varkappa)},\ \ g_{21}=-\frac{\mathrm{i}s_0^{\infty}g_{12}^2\mathrm{e}^{-4\pi\varkappa}+\mathrm{e}^{2\pi\varkappa}}{2g_{12}\sinh(2\pi\varkappa)};\tag{4.17}$$

then $\mathcal{B}(\varkappa)=0$*. The function* $u(\tau)$ *(resp.* $\mathrm{e}^{\mathrm{i}\varphi(\tau)}$*) has only one sequence of zeros (resp. poles),* $\tau_{m,\mathcal{A}}$*,* $m\in\mathbb{N}$*, accumulating at the origin with asymptotic behaviour* (4.15) *for* $\mathcal{C}=\mathcal{A}$*.*

The asymptotics of the functions $u(\tau)$ *and* $\varphi(\tau)$ *are given by equations* (4.1) *and* (4.2)*, respectively, with* $\mathcal{B}(\varkappa)=0$ *and* $\tau\to 0$ *in the sector* $\widehat{\mathcal{S}}_{\mathcal{A}}:=\mathcal{S}\setminus\bigcup_{m\in\mathbb{N}}\operatorname{Int}D_{m,\mathcal{A}}$*; these asymptotic formulae coincide with those in Corollary* 2.1 (*cf. equations* (2.22) *and* (2.23) *for* $a=2\varkappa$).

Appendix §A. Revised derivation for the small-τ asymptotics of isomonodromy deformations in [16]

As pointed out in Section 7 of [18], there is an inconsistency in the definition of the canonical asymptotics at the point at infinity: the purpose of this appendix is to address this matter, and to discuss some of its implications for the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ satisfying the system of isomonodromy deformations (5) on p. 1167 of [16] and defining the functions $u(\tau)$ and $\varphi(\tau)$. Although the small-τ asymptotics for $u(\tau)$ stated in Theorems 3.4 and 3.5 of [16] are not impacted by the changes to $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ discussed below, the small-τ asymptotics of $\varphi(\tau)$ presented in this paper, however, are affected by the multiplicative factor $\tau^{\mathrm{i}a}$ (this also justifies the small-τ asymptotics for $\varphi(\tau)$ given in Theorem B.1 of [21]).

Recall that the original matrix linear ODE system for the $\mathrm{SL}(2,\mathbb{C})$-valued function $\Phi(\lambda,\tau):=\Phi(\lambda)$, whose compatibility condition gives rise to the isomonodromy deformations (5) on p. 1167 of [16], was given in Proposition 1.1, equation (4) of [16]. The starting point of the Isomonodromy Deformation Method is the definition of the monodromy data for the first equation of the 2×2 matrix linear ODE system (4) in Proposition 1.1 of [16]. To this end, one studies the fundamental solutions of this matrix differential equation under special normalisation conditions at the singular points 0 and ∞ (the canonical solutions). The normalisation condition of the canonical solutions at the point at infinity requires a minor correction; this correction affects the formulation of Propositions 5.1 and 5.2 in [16]: for the convenience of the reader, amended versions of these propositions, including proofs, are given below.

As shown in Section 5 of [16], for the asymptotic analysis of the fundamental solution(s) in the neighbourhood of the point at infinity, it is convenient to present the first equation of the 2×2 matrix linear ODE system (4) as

$$\frac{\partial}{\partial\lambda}\Phi(\lambda)=(\mathcal{U}_0(\lambda)+\mathcal{V}_0(\lambda))\Phi(\lambda), \tag{A.1}$$

where

$$\begin{aligned}\mathcal{U}_0(\lambda)&=\tau\left(-\mathrm{i}\sigma_3-\frac{\mathrm{i}a}{2\tau\lambda}\sigma_3-\frac{1}{\lambda}\begin{pmatrix}0 & C(\tau)\\ D(\tau) & 0\end{pmatrix}\right),\\ \mathcal{V}_0(\lambda)&=\frac{\mathrm{i}\tau}{2\lambda^2}\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix},\end{aligned} \tag{A.2}$$

whilst for the asymptotic analysis of the fundamental solution(s) in the neighbourhood of the origin, it is suitable to present the first equation of the 2×2 matrix linear ODE system (4) in the form

$$\frac{\partial}{\partial\lambda}\Phi(\lambda)=\big(\widetilde{\mathcal{U}}_0(\lambda)+\widetilde{\mathcal{V}}_0(\lambda)\big)\Phi(\lambda), \tag{A.3}$$

where

$$\begin{gathered}\widetilde{\mathcal{U}}_0(\lambda)=\tau\left(-\frac{\mathrm{i}a}{2\tau\lambda}\sigma_3-\frac{1}{\lambda}\begin{pmatrix}0 & C(\tau)\\ D(\tau) & 0\end{pmatrix}+\frac{\mathrm{i}}{2\lambda^2}\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix}\right),\\ \widetilde{\mathcal{V}}_0(\lambda)=-\mathrm{i}\tau\sigma_3.\end{gathered} \tag{A.4}$$

Consider the system (A.1), wherein the inaccuracy related to the point at infinity occurs.[13] To study the asymptotic representation of equation (A.1) as $\tau\to 0^+$ and $\lambda\to\infty$ ($\arg\lambda=0$), the following "model problem" for the $\mathrm{SL}(2,\mathbb{C})$-valued parametrix $\mathbf{W}(\lambda)$ was considered in Proposition 5.1,

[13]The gauge of system (A.1) was changed, but the corresponding changes in the normalisation of the canonical asymptotics were not done. In this appendix, this inconsistency is corrected.

equation (111) of [16]:

$$\frac{\partial}{\partial\lambda}\mathbf{W}(\lambda)=\mathcal{U}_0(\lambda)\mathbf{W}(\lambda). \tag{A.5}$$

Its fundamental solution in terms of the Whittaker function [12], $W_{z_1,z_2}(z)$, presented in Proposition 5.1, equation (112) of [16], contains an incorrectly typed factor; more precisely, the right-most term $\exp(\mathrm{i}a\ln(\sqrt{2\tau})\sigma_3)$ in equation (112) must be changed to $\exp(\mathrm{i}a\ln(\sqrt{2}\tau)\sigma_3)$.[14] As a result of this correction, one arrives at asymptotics at the point at infinity that are self-consistent with those of the canonical solution discussed in Section 7 of [18]. An amended version of Proposition 5.1 of [16] is now presented.[15]

Proposition A.1 ([16], Proposition 5.1)**.** *Let* $\mathbf{W}(\lambda)\in\mathrm{SL}(2,\mathbb{C})$ *solve equation* (A.5)*. A fundamental solution of equation* (A.5) *is given by*

$$\mathbf{W}(\lambda)=\frac{\mathrm{e}^{-\pi a/4}}{\sqrt{2\mathrm{i}\tau\lambda}}\begin{pmatrix} W_{\varkappa_1,\widehat{\rho}}(2\mathrm{i}\tau\lambda) & \mathrm{i}\widehat{\gamma}W_{-\varkappa_1,\widehat{\rho}}(-2\mathrm{i}\tau\lambda) \\ \widehat{\delta}W_{\varkappa_1-1,\widehat{\rho}}(2\mathrm{i}\tau\lambda) & \mathrm{i}W_{-(\varkappa_1-1),\widehat{\rho}}(-2\mathrm{i}\tau\lambda)\end{pmatrix}\mathrm{e}^{\mathrm{i}a\ln(\sqrt{2}\tau)\sigma_3}, \tag{A.6}$$

where

$$\varkappa_1:=\tfrac{1}{2}(1-\mathrm{i}a), \qquad \widehat{\rho}^2:=\widehat{\gamma}\widehat{\delta}-a^2/4, \qquad \widehat{\gamma}=\tau C(\tau), \qquad \widehat{\delta}=\tau D(\tau); \tag{A.7}$$

moreover,

$$\begin{aligned}\mathbf{W}(\lambda)\underset{\substack{\lambda\to\infty\\ \arg\lambda=0}}{=}&\left(\mathrm{I}+\frac{1}{2\mathrm{i}\tau\lambda}\begin{pmatrix}\widehat{\gamma}\widehat{\delta} & -\widehat{\gamma}\\ \widehat{\delta} & -\widehat{\gamma}\widehat{\delta}\end{pmatrix}+\frac{1}{(2\mathrm{i}\tau\lambda)^2}\begin{pmatrix}\frac{1}{2}\widehat{\gamma}\widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1+\mathrm{i}a)) & \widehat{\gamma}(\widehat{\gamma}\widehat{\delta}-(1-\mathrm{i}a))\\ \widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1+\mathrm{i}a)) & \frac{1}{2}\widehat{\gamma}\widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1-\mathrm{i}a))\end{pmatrix}\right.\\ &\left.+\mathcal{O}\left(\frac{1}{(\tau\lambda)^3}\begin{pmatrix}\widehat{\gamma}\widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1+\mathrm{i}a))(\widehat{\gamma}\widehat{\delta}-2(2+\mathrm{i}a)) & \widehat{\gamma}(\widehat{\gamma}\widehat{\delta}-(1-\mathrm{i}a))(\widehat{\gamma}\widehat{\delta}-2(2-\mathrm{i}a))\\ \widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1+\mathrm{i}a))(\widehat{\gamma}\widehat{\delta}-2(2+\mathrm{i}a)) & \widehat{\gamma}\widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1-\mathrm{i}a))(\widehat{\gamma}\widehat{\delta}-2(2-\mathrm{i}a))\end{pmatrix}\right)\right)\\ &\times\mathrm{e}^{-\mathrm{i}(\tau\lambda+\frac{a}{2}\ln\lambda)\sigma_3}\tau^{\frac{\mathrm{i}a}{2}\sigma_3}.\end{aligned} \tag{A.8}$$

As a consequence of the correction above, the $\tau\to 0^+$ conditions on the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ stated in Proposition 5.2 of [16] require modification:[16] this is the gist of the following proposition.

[14]One consequence of this modification is that the right-most factor in the large-λ asymptotic expansion of $\mathbf{W}(\lambda)$ given in Proposition 5.1, equation (114) of [16] must be changed to $\exp(-\mathrm{i}(\tau\lambda+\frac{a}{2}\ln(\tau/\lambda))\sigma_3)$.

[15]In contrast to the large-λ asymptotic expansion for $\mathbf{W}(\lambda)$ given in Proposition 5.1, equation (114) of [16], higher-order terms are retained in the $\lambda\to\infty$ $(\arg\lambda=0)$ asymptotic expansion of $\mathbf{W}(\lambda)$ presented here.

[16]The conditions (115) in Proposition 5.2 of [16] require modification, whilst the condition (116) remains unchanged, and is therefore not mentioned in Proposition A.2.

Proposition A.2 ([16], Proposition 5.2)**.** *For* $\epsilon_1 > 0$*, the parameters of equation* (A.1) *satisfy the following conditions:*

$$|\mathrm{Im}(a)|<1, \ \widehat{\rho} \underset{\tau\to 0^+}{=} \mathcal{O}(1), \ \tau^{2(1-\mathrm{i}a)}A(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1}), \\ \tau^{2(1+\mathrm{i}a)}B(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1}), \ \tau C(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{2\mathrm{i}a}), \ \tau D(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{-2\mathrm{i}a}). \tag{A.9}$$

Proof. Let $\mathbf{W}(\lambda) \in \mathrm{SL}(2,\mathbb{C})$ be the fundamental solution of equation (A.5) stated in Proposition A.1. Let $\Phi(\lambda) := \mathbf{W}(\lambda)\mathcal{C}(\lambda)$ be a solution of equation (A.1);[17] substituting this representation for $\Phi(\lambda) \in \mathrm{SL}(2,\mathbb{C})$ into equation (A.1), it follows that $\mathcal{C}(\lambda) \in \mathrm{SL}(2,\mathbb{C})$ solves the ODE system

$$\frac{\partial}{\partial\lambda}\mathcal{C}(\lambda) = (\mathbf{W}(\lambda))^{-1}\mathcal{V}_0(\lambda)\mathbf{W}(\lambda)\mathcal{C}(\lambda). \tag{A.10}$$

One shows that the normalised solution of equation (A.10), that is, the one for which $\mathcal{C}(+\infty) = \mathrm{I}$, is given by

$$\mathcal{C}(\lambda) = \mathrm{I} + \sum_{m=1}^{\infty} \mathcal{C}_m(\lambda), \tag{A.11}$$

where

$$\mathcal{C}_m(\lambda) = \int_{+\infty}^{\lambda} (\mathbf{W}(\xi))^{-1}\mathcal{V}_0(\xi)\mathbf{W}(\xi)\mathcal{C}_{m-1}(\xi)\,\mathrm{d}\xi, \quad m \in \mathbb{N}, \tag{A.12}$$

with $\mathcal{C}_0(\ast) = \mathrm{I}$; thus, there exists a fundamental solution $\Phi(\lambda)$ of equation (A.1) with representation

$$\Phi(\lambda) = \mathbf{W}(\lambda)\left(\mathrm{I} + \sum_{m=1}^{\infty} \mathcal{C}_m(\lambda)\right). \tag{A.13}$$

For the purposes of this proof, it suffices to study the $\lambda \to \infty$ ($\arg\lambda = 0$) behaviour of only $\mathcal{C}_1(\lambda)$. Let $z := \tau\lambda$; then, with $\widetilde{\mathcal{C}}_1(z) := \mathcal{C}_1(z/\tau)$, it follows that

$$\widetilde{\mathcal{C}}_1(z) = \int_{+\infty}^{z} \frac{1}{\tau}(\mathbf{W}(\xi/\tau))^{-1}\mathcal{V}_0(\xi/\tau)\mathbf{W}(\xi/\tau)\,\mathrm{d}\xi. \tag{A.14}$$

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[17]For simplicity of notation, the τ-dependence of $\mathcal{C}(\lambda)$ has been suppressed.

Via equations (A.2) and the $\lambda\to\infty$ $(\arg\lambda=0)$ asymptotics for $\mathbf{W}(\lambda)\in\mathrm{SL}(2,\mathbb{C})$ given in equation (A.8), one shows that

$$\begin{aligned}
&\frac{1}{\tau}(\mathbf{W}(\xi/\tau))^{-1}\mathcal{V}_0(\xi/\tau)\mathbf{W}(\xi/\tau)\\
&\underset{\substack{\xi\to\infty\\ \arg\xi=0}}{=} \mathrm{e}^{\mathrm{i}(\xi+\frac{a}{2}\ln\xi)\sigma_3}\tau^{-\mathrm{i}a\sigma_3}\Bigg(\frac{\mathrm{i}\tau^2}{2\xi^2}\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix}\\
&+\frac{\tau^2}{4\xi^3}\begin{pmatrix}A(\tau)\widehat{\delta}+B(\tau)\widehat{\gamma} & -2\widehat{\gamma}(\sqrt{-A(\tau)B(\tau)}+A(\tau)\widehat{\delta})\\ 2\widehat{\delta}(B(\tau)\widehat{\gamma}-\sqrt{-A(\tau)B(\tau)}) & -(A(\tau)\widehat{\delta}+B(\tau)\widehat{\gamma})\end{pmatrix}\\
&+\mathcal{O}\Bigg(\frac{\tau^2}{\xi^4}\Bigg(\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix}\begin{pmatrix}\mathfrak{w}_1 & \mathfrak{w}_2\\ \mathfrak{w}_3 & \mathfrak{w}_4\end{pmatrix}\\
&+\begin{pmatrix}\mathfrak{w}_4 & -\mathfrak{w}_2\\ -\mathfrak{w}_3 & \mathfrak{w}_1\end{pmatrix}\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix}\\
&+\begin{pmatrix}-\widehat{\gamma}\widehat{\delta} & \widehat{\gamma}\\ -\widehat{\delta} & \widehat{\gamma}\widehat{\delta}\end{pmatrix}\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix}\begin{pmatrix}\widehat{\gamma}\widehat{\delta} & -\widehat{\gamma}\\ \widehat{\delta} & -\widehat{\gamma}\widehat{\delta}\end{pmatrix}\Bigg)\Bigg)\Bigg)\\
&\times\mathrm{e}^{-\mathrm{i}(\xi+\frac{a}{2}\ln\xi)\sigma_3}\tau^{\mathrm{i}a\sigma_3}, \qquad (\mathrm{A}.15)
\end{aligned}$$

where

$$\mathfrak{w}_1:=\tfrac{1}{2}\widehat{\gamma}\widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1+\mathrm{i}a)),\qquad \mathfrak{w}_2:=\widehat{\gamma}(\widehat{\gamma}\widehat{\delta}-(1-\mathrm{i}a)),$$
$$\mathfrak{w}_3:=\widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1+\mathrm{i}a)),\qquad \mathfrak{w}_4:=\tfrac{1}{2}\widehat{\gamma}\widehat{\delta}(\widehat{\gamma}\widehat{\delta}-(1-\mathrm{i}a)).$$

Evaluate, now, the leading-order, $\mathcal{O}(\xi^{-2})$, and the next-to-leading-order, $\mathcal{O}(\xi^{-3})$, terms. From equation (A.14) and the asymptotic expansion (A.15), one writes

$$\widetilde{\mathcal{C}}_1(z)\underset{\substack{z\to\infty\\ \arg z=0}}{=}\widetilde{C}_1^{\dagger}(z)+\widetilde{C}_1^{\ddagger}(z)+\cdots, \qquad (\mathrm{A}.16)$$

where

$$\widetilde{C}_1^{\dagger}(z)=\begin{pmatrix}(\widetilde{C}_1^{\dagger}(z))_{11} & (\widetilde{C}_1^{\dagger}(z))_{12}\\ (\widetilde{C}_1^{\dagger}(z))_{21} & -(\widetilde{C}_1^{\dagger}(z))_{11}\end{pmatrix}, \qquad (\mathrm{A}.17)$$

$$\widetilde{C}_1^{\ddagger}(z)=\begin{pmatrix}(\widetilde{C}_1^{\ddagger}(z))_{11} & (\widetilde{C}_1^{\ddagger}(z))_{12}\\ (\widetilde{C}_1^{\ddagger}(z))_{21} & -(\widetilde{C}_1^{\ddagger}(z))_{11}\end{pmatrix}, \qquad (\mathrm{A}.18)$$

with

$$(\widetilde{C}_1^{\dagger}(z))_{11} := \frac{\mathrm{i}\tau^2}{2}\sqrt{-A(\tau)B(\tau)}\lim_{t\to+\infty}\left(\int_t^z \xi^{-2}\right) = -\frac{\mathrm{i}\tau^2\sqrt{-A(\tau)B(\tau)}}{2z}, \tag{A.19}$$

$$\left(\widetilde{C}_1^{\dagger}(z)\right)_{12} := \frac{\mathrm{i}}{2}\tau^{2(1-\mathrm{i}a)}A(\tau)\lim_{t\to+\infty}\left(\int_t^z \mathrm{e}^{2\mathrm{i}\xi}\xi^{\mathrm{i}a-2}\,\mathrm{d}\xi\right), \tag{A.20}$$

$$\left(\widetilde{C}_1^{\dagger}(z)\right)_{21} := \frac{\mathrm{i}}{2}\tau^{2(1+\mathrm{i}a)}B(\tau)\lim_{t\to+\infty}\left(\int_t^z \mathrm{e}^{-2\mathrm{i}\xi}\xi^{-\mathrm{i}a-2}\,\mathrm{d}\xi\right), \tag{A.21}$$

and

$$(\widetilde{C}_1^{\ddagger}(z))_{11} := \frac{\tau^2}{4}\left(A(\tau)\widehat{\delta}+B(\tau)\widehat{\gamma}\right)\lim_{t\to+\infty}\left(\int_t^z \xi^{-3}\right) = -\frac{\tau^2\left(A(\tau)\widehat{\delta}+B(\tau)\widehat{\gamma}\right)}{8z^2}, \tag{A.22}$$

$$(\widetilde{C}_1^{\ddagger}(z))_{12} := -\frac{1}{2}\widehat{\gamma}\tau^{2(1-\mathrm{i}a)}\left(\sqrt{-A(\tau)B(\tau)}+A(\tau)\widehat{\delta}\right)\lim_{t\to+\infty}\left(\int_t^z \mathrm{e}^{2\mathrm{i}\xi}\xi^{\mathrm{i}a-3}\,\mathrm{d}\xi\right), \tag{A.23}$$

$$(\widetilde{C}_1^{\ddagger}(z))_{21} := \frac{1}{2}\widehat{\delta}\tau^{2(1+\mathrm{i}a)}\left(B(\tau)\widehat{\gamma}-\sqrt{-A(\tau)B(\tau)}\right)\lim_{t\to+\infty}\left(\int_t^z \mathrm{e}^{-2\mathrm{i}\xi}\xi^{-\mathrm{i}a-3}\,\mathrm{d}\xi\right). \tag{A.24}$$

Via the well-known integral inequality $|\int f\,\mathrm{d}z| \leqslant \int |f|\,|\mathrm{d}z|$, one shows, after a straightforward integration argument, that the improper integrals appearing in equations (A.20) and (A.21) converge for $\mathrm{Im}(a)+1>0$ and $\mathrm{Im}(a)-1<0$, respectively; therefore, the parameter of formal monodromy, a, must satisfy the inequality $|\mathrm{Im}(a)|<1$ (cf. conditions (A.9)). Without loss of generality, consider, say, the integral appearing in equation (A.20):

$$\mathcal{I}_{12} := \int_t^z \mathrm{e}^{2\mathrm{i}\xi}\xi^{\mathrm{i}a-2}\,\mathrm{d}\xi. \tag{A.25}$$

From Euler's formula and the integrals **2.632** (2. and 4.) and **3.761** (2. and 7.) on pp. 226 and 458, respectively, of [12], one shows that

$$\mathcal{I}_{12}=2^{-\mu_1}\mathrm{e}^{\mathrm{i}\pi\mu_1/2}\Big(\Gamma(\mu_1,2\mathrm{e}^{-\mathrm{i}\pi/2}t)-\Gamma(\mu_1,2\mathrm{e}^{-\mathrm{i}\pi/2}z)\Big), \quad \mu_1:=\mathrm{i}a-1, \tag{A.26}$$

where $\Gamma(\alpha,z)$ is the complementary incomplete gamma function. From equation **6.5.3** on p. 260 of [1] relating $\Gamma(\alpha,z)$ and the incomplete gamma function, $\gamma(\alpha,z)$, that is, $\Gamma(\alpha,z)=\Gamma(\alpha)-\gamma(\alpha,z)$, where $\Gamma(\ast)$ is the gamma function, and equation **6.5.12** on p. 262 of [1] relating $\gamma(\ast,z)$ and the confluent hypergeometric function, $\mathbf{M}(a,b,z)$,[18] that is,

$$\gamma(\alpha,z)=\alpha^{-1}z^{\alpha}\mathbf{M}(\alpha,1+\alpha,-z),$$

one shows that equation (A.26) can be presented in the form

$$\mathcal{I}_{12}=\frac{1}{\mu_1}\Big(z^{\mu_1}\mathbf{M}(\mu_1,1+\mu_1,2\mathrm{e}^{\mathrm{i}\pi/2}z)-t^{\mu_1}\mathbf{M}(\mu_1,1+\mu_1,2\mathrm{e}^{\mathrm{i}\pi/2}t)\Big); \tag{A.27}$$

thus, via equation (A.20), it follows that

$$\big(\widetilde{C}_1^{\dagger}(z)\big)_{12}=\frac{\mathrm{i}}{2\mu_1}\tau^{2(1-\mathrm{i}a)}A(\tau)\Bigg(z^{\mu_1}\mathbf{M}(\mu_1,1+\mu_1,2\mathrm{e}^{\mathrm{i}\pi/2}z) - \lim_{t\to+\infty}t^{\mu_1}\mathbf{M}(\mu_1,1+\mu_1,2\mathrm{e}^{\mathrm{i}\pi/2}t)\Bigg). \tag{A.28}$$

Using, now, the large-z asymptotic expansion for $\mathbf{M}(a,b,z)$ given in equation **13.5.1** on p. 508 of [1], that is,

$$\frac{\mathbf{M}(a,b,z)}{\Gamma(b)} \underset{z\to\infty}{=} \frac{z^{-a}\mathrm{e}^{\pm\pi\mathrm{i}a}}{\Gamma(b-a)}\left(\sum_{n=0}^{\mathrm{R}-1}\frac{(a)_n(1+a-b)_n}{n!(-z)^n}+\mathcal{O}\big(|z|^{-\mathrm{R}}\big)\right) + \frac{z^{a-b}\mathrm{e}^{z}}{\Gamma(a)}\left(\sum_{n=0}^{\mathrm{S}-1}\frac{(b-a)_n(1-a)_n}{n!z^n}+\mathcal{O}\big(|z|^{-\mathrm{S}}\big)\right), \quad (\mathrm{R},\mathrm{S})\in\mathbb{N}\times\mathbb{N},$$

where the upper (resp., lower) sign is taken if $-\pi/2<\arg z<3\pi/2$ (resp., $-3\pi/2<\arg z\leqslant-\pi/2$), and $(\alpha)_n$ is the Pochhammer symbol, namely, $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)=\Gamma(\alpha+n)/\Gamma(\alpha)$, $n\in\mathbb{N}$, one shows that

$$t^{\mu_1}\mathbf{M}(\mu_1,1+\mu_1,2\mathrm{e}^{\pi\mathrm{i}/2}t) \underset{t\to+\infty}{=} \frac{\mu_1\mathrm{e}^{-\pi\mathrm{i}/2}}{2}\frac{\mathrm{e}^{2\mathrm{i}t}t^{\mathrm{i}a}}{t^2}\left(1-\frac{\mathrm{i}(2-\mathrm{i}a)}{2t}+\mathcal{O}(t^{-2})\right); \tag{A.29}$$

[18]Note: $\mathbf{M}(a,b,z)$ is also denoted as ${}_1\mathrm{F}_1(a;b;z)$.

thus, via the inequality $|\mathrm{Im}(a)|<1$, the Squeeze Lemma, and the inequality $t^{-3}<t^{-(2+\mathrm{Im}(a))}<t^{-1}$, it follows that

$$\lim_{t\to+\infty} t^{\mu_1}\mathbf{M}(\mu_1,1+\mu_1,2\mathrm{e}^{\pi\mathrm{i}/2}t)=0,$$

whence

$$\big(\widetilde{C}_1^{\dagger}(z)\big)_{12} \underset{\substack{z\to\infty\\ \arg z=0}}{=} \frac{1}{4}\tau^{2(1-\mathrm{i}a)}A(\tau)\frac{z^{\mathrm{i}a}\mathrm{e}^{2\mathrm{i}z}}{z^2}\left(1-\frac{\mathrm{i}(2-\mathrm{i}a)}{2z}+\mathcal{O}(z^{-2})\right). \tag{A.30}$$

Analogously, one shows that (cf. equation (A.21)), with $\mu_2:=-\mathrm{i}a-1$,

$$\begin{aligned}\big(\widetilde{C}_1^{\dagger}(z)\big)_{21} &= \frac{\mathrm{i}}{2\mu_2}\tau^{2(1+\mathrm{i}a)}B(\tau)z^{\mu_2}\mathbf{M}(\mu_2,1+\mu_2,2\mathrm{e}^{-\pi\mathrm{i}/2}z)\\ &\underset{\substack{z\to\infty\\ \arg z=0}}{=} -\frac{1}{4}\tau^{2(1+\mathrm{i}a)}B(\tau)\frac{z^{-\mathrm{i}a}\mathrm{e}^{-2\mathrm{i}z}}{z^2}\left(1+\frac{\mathrm{i}(2+\mathrm{i}a)}{2z}+\mathcal{O}(z^{-2})\right);\end{aligned} \tag{A.31}$$

thus (cf. equations (A.17), (A.19), (A.30), and (A.31)),

$$\begin{aligned}\widetilde{C}_1^{\dagger}(z) \underset{\substack{z\to\infty\\ \arg z=0}}{=} &-\frac{\mathrm{i}\tau^2\sqrt{-A(\tau)B(\tau)}}{2z}\sigma_3\\ &+\tau^{2(1-\mathrm{i}a)}A(\tau)\frac{z^{\mathrm{i}a}\mathrm{e}^{2\mathrm{i}z}}{4z^2}\left(1-\frac{\mathrm{i}(2-\mathrm{i}a)}{2z}+\mathcal{O}(z^{-2})\right)\sigma_+\\ &-\tau^{2(1+\mathrm{i}a)}B(\tau)\frac{z^{-\mathrm{i}a}\mathrm{e}^{-2\mathrm{i}z}}{4z^2}\left(1+\frac{\mathrm{i}(2+\mathrm{i}a)}{2z}+\mathcal{O}(z^{-2})\right)\sigma_-,\end{aligned} \tag{A.32}$$

where $\sigma_+=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\sigma_-=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$.

Consider, now, the expression for $\widetilde{C}_1^{\ddagger}(z)$ given in equation (A.18) (cf. equations (A.22), (A.23), and (A.24)). Proceeding as delineated above, one shows that, with $\mu_3:=\mathrm{i}a-2$,

$$\begin{aligned}(\widetilde{C}_1^{\ddagger}(z))_{12} &= -\frac{1}{2\mu_3}\widehat{\gamma}\tau^{2(1-\mathrm{i}a)}\Big(\sqrt{-A(\tau)B(\tau)}+A(\tau)\widehat{\delta}\Big)z^{\mu_3}\mathbf{M}(\mu_3,1+\mu_3,2\mathrm{e}^{\pi\mathrm{i}/2}z)\\ &\underset{\substack{z\to\infty\\ \arg z=0}}{=} \mathrm{i}\widehat{\gamma}\tau^{2(1-\mathrm{i}a)}\Big(\sqrt{-A(\tau)B(\tau)}+A(\tau)\widehat{\delta}\Big)\,\frac{z^{\mathrm{i}a}\mathrm{e}^{2\mathrm{i}z}}{4z^3}\left(1-\frac{\mathrm{i}(3-\mathrm{i}a)}{2z}+\mathcal{O}(z^{-2})\right),\end{aligned} \tag{A.33}$$

and, with $\mu_4 := -\mathrm{i}a - 2$,

$$(\widetilde{C}_1^{\ddagger}(z))_{21} = \frac{1}{2\mu_4}\widehat{\delta}\tau^{2(1+\mathrm{i}a)}\Big(B(\tau)\widehat{\gamma} - \sqrt{-A(\tau)B(\tau)}\Big) z^{\mu_4}\mathbf{M}(\mu_4, 1+\mu_4, 2\mathrm{e}^{-\pi\mathrm{i}/2}z)$$
$$\underset{\substack{z\to\infty \\ \arg z=0}}{=} \mathrm{i}\widehat{\delta}\tau^{2(1+\mathrm{i}a)}\Big(B(\tau)\widehat{\gamma} - \sqrt{-A(\tau)B(\tau)}\Big)\frac{z^{-\mathrm{i}a}\mathrm{e}^{-2\mathrm{i}z}}{4z^3}\left(1 + \frac{\mathrm{i}(3+\mathrm{i}a)}{2z} + \mathcal{O}(z^{-2})\right), \tag{A.34}$$

thus,

$$\widetilde{C}_1^{\ddagger}(z) \underset{\substack{z\to\infty \\ \arg z=0}}{=} -\frac{\tau^2(A(\tau)\widehat{\delta} + B(\tau)\widehat{\gamma})}{8z^2}\sigma_3$$
$$+ \mathrm{i}\tau^{2(1-\mathrm{i}a)}\widehat{\gamma}\Big(\sqrt{-A(\tau)B(\tau)} + A(\tau)\widehat{\delta}\Big)\frac{z^{\mathrm{i}a}\mathrm{e}^{2\mathrm{i}z}}{4z^3}\big(1 + \mathcal{O}(z^{-1})\big)\sigma_+$$
$$- \mathrm{i}\tau^{2(1+\mathrm{i}a)}\widehat{\delta}\Big(\sqrt{-A(\tau)B(\tau)} - B(\tau)\widehat{\gamma}\Big)\frac{z^{-\mathrm{i}a}\mathrm{e}^{-2\mathrm{i}z}}{4z^3}\big(1 + \mathcal{O}(z^{-1})\big)\sigma_-. \tag{A.35}$$

Hence, via equations (A.16), (A.32), and (A.35), one arrives at

$$\widetilde{C}_1(z) \underset{\substack{z\to\infty \\ \arg z=0}}{=} -\frac{\tau^2}{2z}\left(\mathrm{i}\sqrt{-A(\tau)B(\tau)} + \frac{1}{4z}\big(A(\tau)\widehat{\delta} + B(\tau)\widehat{\gamma}\big)\right)\sigma_3$$
$$+ \frac{\tau^{2(1-\mathrm{i}a)}}{4}\frac{\mathrm{e}^{2\mathrm{i}z}z^{\mathrm{i}a}}{z^2}\Bigg(A(\tau)\left(1 - \frac{\mathrm{i}(2-\mathrm{i}a)}{2z} + \mathcal{O}(z^{-2})\right)$$
$$+ \mathrm{i}\widehat{\gamma}\big(\sqrt{-A(\tau)B(\tau)} + A(\tau)\widehat{\delta}\big)\frac{1}{z}\big(1 + \mathcal{O}(z^{-1})\big)\Bigg)\sigma_+$$
$$- \frac{\tau^{2(1+\mathrm{i}a)}}{4}\frac{\mathrm{e}^{-2\mathrm{i}z}z^{-\mathrm{i}a}}{z^2}\Bigg(B(\tau)\left(1 + \frac{\mathrm{i}(2+\mathrm{i}a)}{2z} + \mathcal{O}(z^{-2})\right)$$
$$+ \mathrm{i}\widehat{\delta}\big(\sqrt{-A(\tau)B(\tau)} - B(\tau)\widehat{\gamma}\big)\frac{1}{z}\big(1 + \mathcal{O}(z^{-1})\big)\Bigg)\sigma_- + \cdots. \tag{A.36}$$

Since the sought-after class of functions is the one for which the τ-dependent coefficients in the asymptotic expansion (A.36) tend to zero as $\tau \to 0^+$, one must impose certain conditions on $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, $\widehat{\gamma}$, and $\widehat{\delta}$; in particular, from the off-diagonal elements of the expansion (A.36), one demands that, for $|\mathrm{Im}(a)|<1$ and some $\epsilon_1>0$,

$$\tau^{2(1-\mathrm{i}a)}A(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1}), \qquad \tau^{2(1+\mathrm{i}a)}B(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1}),$$
$$\widehat{\gamma}:=\tau C(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}\big(\tau^{2\mathrm{i}a}\big), \qquad \widehat{\delta}:=\tau D(\tau) \underset{\tau\to 0^+}{=} \mathcal{O}\big(\tau^{-2\mathrm{i}a}\big),$$

which, in conjunction with the definitions (A.7), imply that

$$\begin{aligned}
&\widehat{\rho} \underset{\tau\to 0^+}{=} \mathcal{O}(1), \quad \tau^2\sqrt{-A(\tau)B(\tau)} \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1}),\\
&\tau^{2(1-\mathrm{i}a)}\widehat{\gamma}\big(\sqrt{-A(\tau)B(\tau)}+A(\tau)\widehat{\delta}\big) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1}),\\
&\tau^{2(1+\mathrm{i}a)}\widehat{\delta}\big(\sqrt{-A(\tau)B(\tau)}-B(\tau)\widehat{\gamma}\big) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1}),\\
&\tau^2\big(A(\tau)\widehat{\delta}+B(\tau)\widehat{\gamma}\big) \underset{\tau\to 0^+}{=} \mathcal{O}(\tau^{\epsilon_1});
\end{aligned}$$

hence, one arrives at the conditions (A.9). □

The modifications of Propositions A.1 and A.2 given in this appendix suggest the following changes to the connection matrix, G, and the isomonodromy deformations $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, $\widehat{\gamma}:=\tau C(\tau)$, and $\widehat{\delta}:=\tau D(\tau)$:

$$\begin{aligned}
&G\to G\tau^{\frac{\mathrm{i}a}{2}\sigma_3},\\
&\big(A(\tau),B(\tau),C(\tau),D(\tau),\widehat{\gamma},\widehat{\delta}\big)\\
&\quad\to\big(A(\tau)\tau^{-\mathrm{i}a},B(\tau)\tau^{\mathrm{i}a},C(\tau)\tau^{-\mathrm{i}a},D(\tau)\tau^{\mathrm{i}a},\widehat{\gamma}\tau^{-\mathrm{i}a},\widehat{\delta}\tau^{\mathrm{i}a}\big).
\end{aligned} \tag{A.37}$$

The transformations (A.37) imply, in particular, that in Lemmata 5.1 and 5.2 of [16], and on the left-hand sides of the $\tau\to 0^+$ asymptotic expansions given in Propositions 5.5 and 5.7 of [16],[19] the “symbols” A, B, C, D, $\widehat{\gamma}$, and $\widehat{\delta}$ must be replaced by $A\tau^{-\mathrm{i}a}$, $B\tau^{\mathrm{i}a}$, $C\tau^{-\mathrm{i}a}$, $D\tau^{\mathrm{i}a}$, $\widehat{\gamma}\tau^{-\mathrm{i}a}$, and $\widehat{\delta}\tau^{\mathrm{i}a}$, respectively.

[19]Since the product $\widehat{\gamma}\widehat{\delta}$ remains invariant with respect to this transformation, it follows from equations (A.7) that, for either choice of the branch of the square root function, $\widehat{\rho}$, too, is invariant; consequently, Proposition 5.6 of [16] remains unchanged.

Appendix §B. Numerical visualisation of the connection formulae for the small-τ logarithmic asymptotics: $a = 0$

In [21], we studied, both analytically and numerically, a special algebroid solution of the DP3E for $a = 0$. In this appendix, we verify numerically the connection results for asymptotics of solutions of the DP3E for $a = 0$ possessing logarithmic behaviour as $\tau \to 0$ (cf. Corollary 3.1). Since such solutions are specified in terms of the monodromy data, the simplest scheme for the calculations is as follows: (i) choosing monodromy data that satisfy the conditions specified in Corollary 3.1, we compute the corresponding asymptotics of the functions $u(\tau)$ and $\varphi(\tau)$ by means of the formulae given in the said corollary; (ii) using these asymptotics, we then calculate the initial data for the corresponding solutions at some small enough initial point τ_0 and numerically continue the solutions to large enough values of τ; and (iii) these numerical solutions are compared with their large-τ asymptotics constructed with the help of the above-mentioned monodromy data according to the formulae given in Appendix C of [21].

An often occurring problem with this method is that, in order to get a better approximation for the solution via its asymptotics, one has to approach the singular points 0 and ∞ (the closer the approach, the better the approximation); but, on the other hand, the accuracy of the numerical calculations near the singular points becomes progressively worse. Nevertheless, for all the solutions we have studied thus far, there is a fairly large interval in the neighbourhoods of the singular points wherein the numerics continue to function with a good enough accuracy and for which the corresponding asymptotics provide a good enough approximation for the solutions to yield a reliable—for our purposes—result. In this context, what does "good enough" mean? An essential component of the methodology we've employed in our previous studies (see, for example, [21]) is to choose an initial point τ_0 that is close enough to $\tau = 0$, which, in most (but not all!) cases, is $\tau_0 = 10^{-6}$ (a reasonable placement for this point), and construct the corresponding numerical solution; then, we redo the calculation with $\tau_0 = 10^{-12}$, and compare the plots of the numerical solutions obtained: in the event that the plots are visually indistinguishable, we consider this plot as a "good enough" approximation for the solution corresponding to the chosen monodromy data. For the examples considered in this appendix, we varied the initial point τ_0 from 10^{-3} to 10^{-98} and observed that the plots of the solutions remain stable; but, for $\tau_0 = 10^{-99}$, the numerical procedure failed. In some cases, mainly those related to some very special solutions, one needs to investigate further the proper choice(s) for the placement of the initial point τ_0 (some examples are given in [22]).

We can not simultaneously visualise in one figure the neighbourhoods of the origin and the point at infinity, where the asymptotics we study are compared with the numerical solution, because one neighbourhood is centred at the origin, with an approximately $\mathcal{O}(1)$ radius as $\tau \to 0$, and the other, analogous neighbourhood is centred at the point at infinity, extending to $\mathcal{O}(1)$ values. The first neighbourhood is too small for observing the behaviour of the solutions; therefore, theoretically, we have two options: (1) the one described above; or (2) to take initial values in some proper neighbourhood of $\tau = \infty$ and plot the solution and its small-τ asymptotics after making the transformation $\tau \to 1/\tau$, namely, to interchange the roles of the singular points $\tau = 0$ and $\tau = \infty$. This plot would reflect the behaviour of the original solution in the "blown-up neighbourhood" of $\tau = 0$, whilst the behaviour at the point at infinity would be hidden. In this appendix, we decided to keep the original variables, that is, we chose option (1).

We now turn to the description of the calculations presented below. The coefficients of the DP3E used for all the calculations in this appendix are chosen as follows:

$$a = 0, \qquad b = 0.02, \qquad \varepsilon = 1. \tag{B.1}$$

These calculations are done with the help of MAPLE, using its standard programs for solving ODEs and plotting the corresponding results; in particular, both the absolute and relative errors in the `dsolve` procedure were set to 10^{-12}. We checked the stability of our calculations with respect to both the MAPLE parameter `Digits` and the value of τ_0; the parameter `Digits` was successively set to the values 10, 80, and 160, with the variation for τ_0 discussed above. The first 10 digits for all the calculations coincided, so that, visually, the final pictures were virtually identical.[20]

[20] It is a matter of interest to compare the times required for these calculations. Our notebook computer, equipped with a 12th Gen Intel(R) Core(TM) i7-12700H processor, computed the results presented in Figs. 1–4 in roughly 0.3 seconds for the 10-digit calculation, 39 seconds for the 80-digit calculation, and 47 seconds for the 160-digit calculation.

For the calculation of the initial data of the solution presented in Figs. 1–4, we used equations (3.19)–(3.21), where we set

$$c_1 = 2 - \mathrm{i} \qquad \text{and} \qquad g_{21} = 2. \tag{B.2}$$

As a result, we obtained the following initial data for the numerical solution at $\tau_0 = 10^{-12}$:

$$\begin{aligned} u(\tau_0) &= -1.570796327\ldots \times 10^{-14} - \mathrm{i}\,6.441875074\ldots \times 10^{-13}, \\ u'(\tau_0) &= -0.01570796327\ldots - \mathrm{i}\,0.6241875074\ldots, \end{aligned} \tag{B.3}$$

$$\varphi(\tau_0) = 6.258805991\ldots - \mathrm{i}\,0.941530528\ldots. \tag{B.4}$$

The digits that are displayed explicitly in equations (B.3) and (B.4) represent the initial values that are actually used in the 10-digit calculations. We draw the reader's attention to the fact that we plot the mole function $\varphi(\tau)$ which is defined modulo 2π. Our definition of the function $\varphi(\tau)$ suggests that we substitute $e^{\pi \mathrm{i}}$ in lieu of the minus sign on the right-hand side of equation (3.21) and then apply the formal ln-operation to the both sides of the resulting equation.

The large-τ asymptotic formulae for the functions $u(\tau)$ and $\varphi(\tau)$ plotted in Figs. 1–4 are constructed with the help of Theorem C.1 in Appendix C of [21] by using the monodromy data (3.19) with c_1 and g_{21} given in equations (B.2). Taking into consideration that the large-τ asymptotics for the function $\varphi(\tau)$ is defined up to $2\pi k$, for some $k \in \mathbb{Z}$ (cf. Remark C.4 in [21]), the value of k in Fig. 3, which is determined by comparing numerical plots, is equal to $+1$.

The initial data for the solution presented in Figs. 5–8 are obtained with the help of equations (3.22)–(3.24) for the following values of the parameters,

$$c_2 = 1 + 2\mathrm{i} \qquad \text{and} \qquad g_{12} = 2. \tag{B.5}$$

Via the asymptotics (3.23) and (3.24), we calculate the initial data at the point $\tau_0 = 10^{-12}$:

$$\begin{aligned} u(\tau_0) &= 1.570796326\ldots \times 10^{-14} + \mathrm{i}\,5.499397276\ldots \times 10^{-13}, \\ u'(\tau_0) &= 0.01570796326\ldots + \mathrm{i}\,0.5299397276\ldots, \end{aligned} \tag{B.6}$$

$$\varphi(\tau_0) = 0.02855529941\ldots + \mathrm{i}\,0.7834599236\ldots. \tag{B.7}$$

In equations (B.6) and (B.7), the digits that are explicitly shown define the initial data that are used in the 10-digit calculations. In this case, we define the mole function $\varphi(\tau)$ by substituting $e^{-\pi \mathrm{i}}$ in lieu of the minus sign

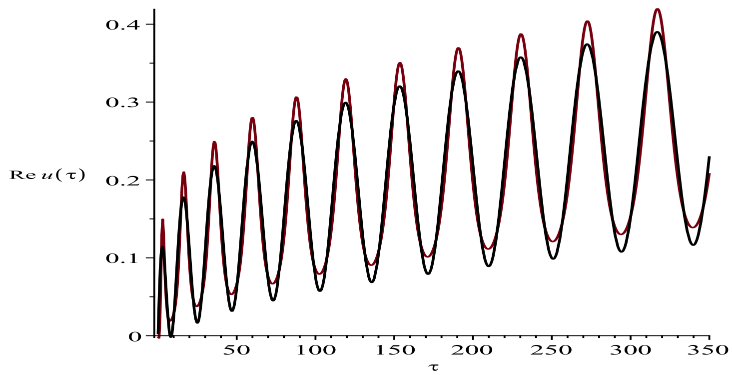

Figure 1. The brown (higher extrema) and black plots are, respectively, the real parts of the numeric and large-τ asymptotic values of the function $u(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.20) for $c_1 = 2 - \mathrm{i}$.

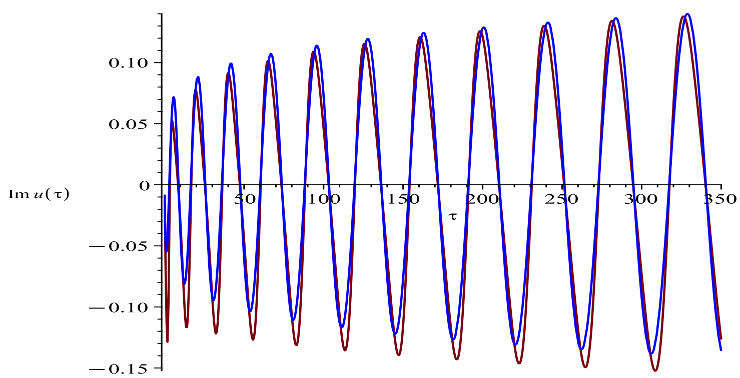

Figure 2. The brown (lower extrema) and blue plots are, respectively, the imaginary parts of the numeric and large-τ asymptotic values of the function $u(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.20) for $c_1 = 2 - \mathrm{i}$.

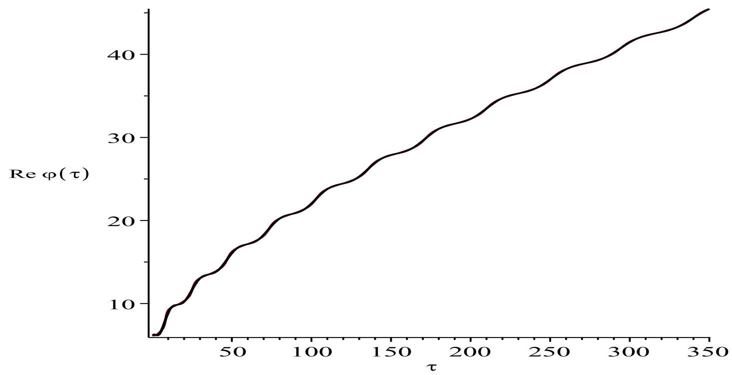

Figure 3. The brown and black plots (virtually coincident) are, respectively, the real parts of the numeric and large-τ asymptotic values of the function $\varphi(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.21) for $c_1 = 2 - \mathrm{i}$.

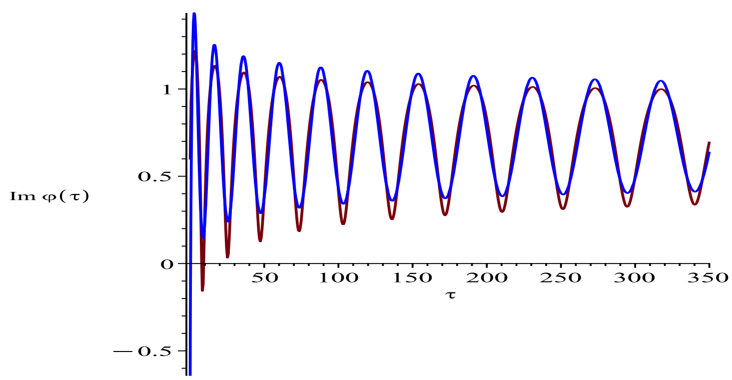

Figure 4. The brown (lower extrema) and blue plots are, respectively, the imaginary parts of the numeric and large-τ asymptotic values of the function $\varphi(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.21) for $c_1 = 2 - \mathrm{i}$.

on the right-hand side of equation (3.24) and then applying the formal ln-operation to both sides of the resulting equation. The large-τ asymptotics in Figs. 5–8 are obtained, once again, using the formulae stated in Theorem C.1 of [21] with winding parameter $k = 0$ for the function $\varphi(\tau)$.

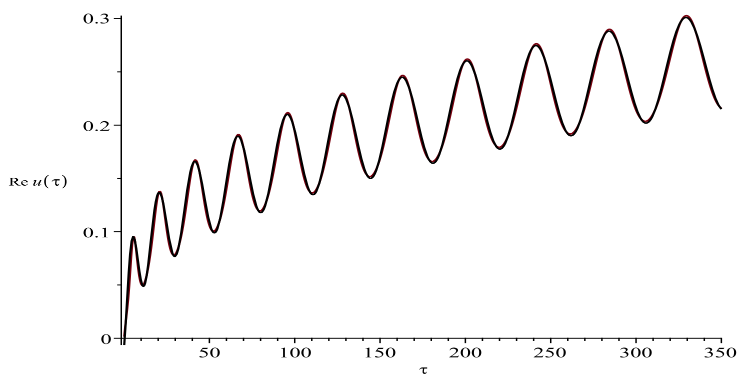

Figure 5. The brown and black plots (virtually coincident) are, respectively, the real parts of the numeric and large-τ asymptotic values of the function $u(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.23) for $c_2 = 1 + 2\mathrm{i}$.

By varying the values of the parameters c_1 and c_2, one can obtain more interesting large-τ behaviours (cf. Theorems C.2–C.4 in [21]) of the solutions that are studied in this appendix. Our numerical experiments described at the beginning of this appendix demonstrate that the solutions are calculated with an error that does not exceed 10^{-12}, so that the discrepancy between the numerics and the asymptotics visible in Figs. 1–8 is related to the accuracy of the approximation of the solution by the leading term of its large-τ asymptotics. As delineated in our work [21], the convergence of the leading term of the large-τ asymptotics to the corresponding

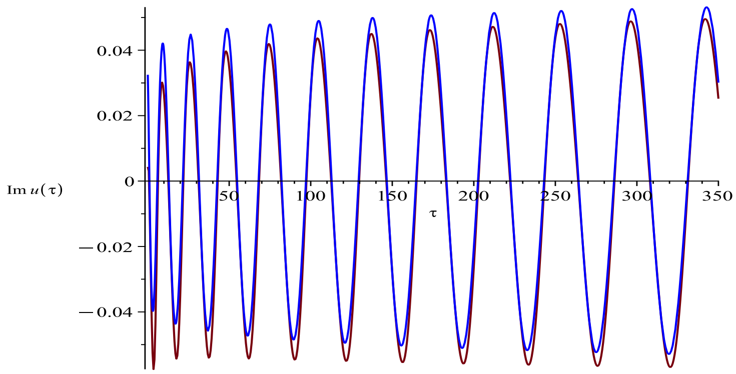

Figure 6. The brown (lower extrema) and blue plots are, respectively, the imaginary parts of the numeric and large-τ asymptotic values of the function $u(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.23) for $c_2 = 1 + 2\mathrm{i}$.

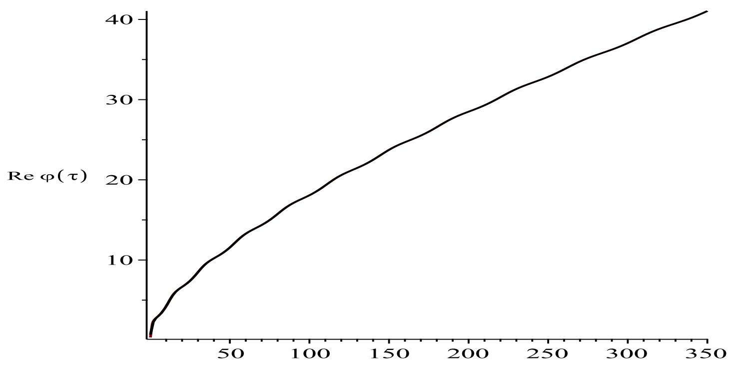

Figure 7. The brown and black plots (virtually coincident) are, respectively, the real parts of the numeric and large-τ asymptotic values of the function $\varphi(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.24) for $c_2 = 1 + 2\mathrm{i}$.

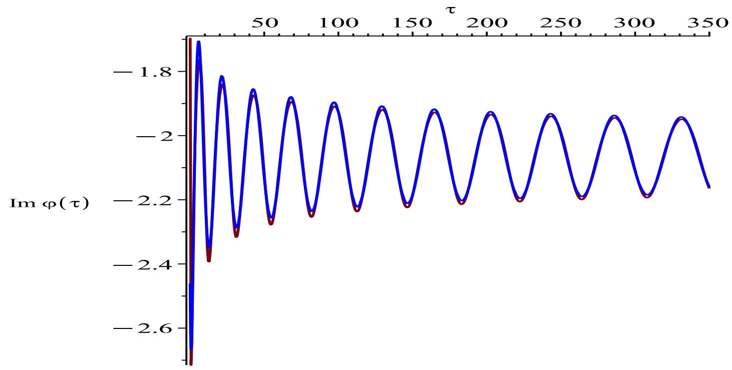

Figure 8. The brown (slightly lower extrema) and blue plots are, respectively, the imaginary parts of the numeric and large-τ asymptotic values of the function $\varphi(\tau)$ for $\tau \geqslant 0.1$ corresponding to the initial values defined by the small-τ asymptotics (3.24) for $c_2 = 1 + 2\mathrm{i}$.

numerical solution is very slow. In most, but not all, cases, this approximation can be improved with the help of the correction terms which are given in Appendix C of [21].

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