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CORRELATION FUNCTIONS OF TWO 3-DIMENSIONAL TRANSVERSE POTENTIALS WITH POWER SINGULARITIES

Abstract. We study convolutions of two localized transverse potentials with $-5/2$-power singularities with the Green function of the Laplace operator in the 3-dimensional space. These potentials correspond to the electromagnetic field with $-1/2$-power singularities which resides at a minimum distance to the domain of the quadratic form of the Laplacian, but does not belong to the latter. The discussed correlation functions can be used as the Nevanlinna functions for the closable extensions of quadratic form of the Laplace operator for the electromagnetic field with $-1/2$-power singularities, and in this way they are important for studying of perturbed Hamiltonians.

§1. Introduction

Electromagnetic fields with $-1/2$-power singularities at isolated points $\boldsymbol{x}_n$ of the 3-dimensional space

$$A_l(\boldsymbol{x})=\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_n|^{1/2}}\Big(C_{nl}+C^1_{nlj}\frac{(x^j-x^j_n)}{|\boldsymbol{x}-\boldsymbol{x}_n|}+C^2_{nljk}\frac{(x^j-x^j_n)(x^k-x^k_n)}{|\boldsymbol{x}-\boldsymbol{x}_n|^2}\Big) +\mathcal{O}(1), \quad \boldsymbol{x}\to\boldsymbol{x}_n, \tag{1}$$

represent examples of fields residing at minimum distance to the domain of the functional of the potential energy

$$H(A)=\int\limits_{\mathbb{R}^3}\big(\partial_k A_l(\boldsymbol{x})\big)^2\,d^3x.$$

Indeed, the derivative of the field with a weaker singularity

$$A^\epsilon_l(\boldsymbol{x})=\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_n|^{1/2-\epsilon}}\Big(C_{nl}+C^1_{nlj}\frac{(x^j-x^j_n)}{|\boldsymbol{x}-\boldsymbol{x}_n|}+C^2_{nljk}\frac{(x^j-x^j_n)(x^k-x^k_n)}{|\boldsymbol{x}-\boldsymbol{x}_n|^2}\Big) + \mathcal{O}(1), \quad \boldsymbol{x}\to\boldsymbol{x}_n,$$

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after being squared gives $2\epsilon - 3$-power singularity

$$\big(\partial_k A_l^\epsilon(\boldsymbol{x})\big)^2 = \mathcal{O}(|\boldsymbol{x} - \boldsymbol{x}_n|^{2\epsilon-3}), \quad \boldsymbol{x} \to \boldsymbol{x}_n,$$

and this expression is integrable in vicinity of the points $\boldsymbol{x}_n$ or in the whole space $\mathbb{R}^3$ for a field appropriately vanishing at infinity

$$H(A^\epsilon) = \int\limits_{\mathbb{R}^3} \big(\partial_k A_l^\epsilon(\boldsymbol{x})\big)^2 \, d^3x < \infty, \quad \epsilon > 0. \tag{2}$$

But when ϵ is zero, then the integral (2) diverges logarithmically and thus we can say that the field (1) resides close to the boundary of the domain of H.

The theory of extensions of symmetric operators and quadratic forms [1, 2] allows us to extend the functional $H(A)$ to a set of fields in the vicinity of its domain. From the physical point of view, this corresponds to introduction of interaction with some singular potential into the free electromagnetic field model. In the literature, see, e.g., [3–5], one can find discussion of the interaction with the δ-potentials, but this is not the case of our interest: in 3 dimensions fields generated by the presence of the δ-interaction have -1-power singularities and thus are too far from the domain of $H(A)$. Extensions of the quadratic form of Laplacian to the fields close to its domain, that is of type (1), are used for construction of solutions for the eigenfunctional equation of the quantum field theory in the Schrödinger representation [6].

An important component of the extension theory is calculation of convolutions of singular potentials with the Green function $R(\mu)$ of the unperturbed operator. These integrals represent examples of the Nevanlinna (Herglotz, Pick)[1] functions [7] and they are meromorphic with respect to μ on the complex plane without the positive semi-axis. We call them "correlators" in the sense of integrals of Green functions enveloped with δ-potentials.

Closable extension H_κ of the quadratic form H is defined by the set, in the index j and points $\boldsymbol{x}_n$, of singular potentials $V_{n,l}^j(\boldsymbol{x})$ and the matrix of extension parameters κ. Domain of H_κ is represented by a direct sum

[1] More precisely, the Nevanlinna function is the value of quadratic form of convolution matrix of singular potentials calculated with some (any) vector.

of the domain of H and the linear span of vectors $R(\rho)V^j_{n,l}$

$$\mathcal{D}(H_\kappa) = \mathcal{D}(H) \dotplus \Big\{ \sum_{n,j} \alpha_{nj} R(\rho) V^j_{n,l} \Big\}, \quad \alpha_{nj} \in \mathbb{C}. \tag{3}$$

Here, we imply that the functions $R(\rho)V^j_{n,l}(\boldsymbol{x})$ are square-integrable, so that, due to the resolvent identity, the domain $\mathcal{D}(H_\kappa)$ does not depend on the choice of ρ,

$$R(\rho_1)V^j_{n,l} - R(\rho_2)V^j_{n,l} = (\rho_1 - \rho_2)R(\rho_1)R(\rho_2)V^j_{n,l} \in \mathcal{D}(H).$$

The action of H_κ on functions from $\mathcal{D}(H)$ coincides with the action of H,

$$H_\kappa(A) = H(A) = \int_{\mathbb{R}^3} \big(\partial_k A_l(\boldsymbol{x})\big)^2 \, d^3x, \quad A \in \mathcal{D}(H),$$

and the action of H_κ on the vectors $R(\rho)V^j_{n,l}$ is defined in some way by the matrix of extension parameters κ. The latter connection is not important in what follows so we will not dwell on it.

It is easy to see that for the vectors $R(\rho)V^j_{n,l}$ having at the points $\boldsymbol{x}_n$ expansions of type (1), the potentials $V^j_{n,l}$ should have $-5/2$-power singularities,

$$V^j_{n,l}(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_n|^{5/2}} \Big(C_{njl} + C^1_{njlk} \frac{(x^k - x^k_n)}{|\boldsymbol{x}-\boldsymbol{x}_n|} + C^2_{nljkk'} \frac{(x^k - x^k_n)(x^{k'} - x^{k'}_n)}{|\boldsymbol{x}-\boldsymbol{x}_n|^2} \Big) + \mathcal{O}(|\boldsymbol{x}-\boldsymbol{x}_n|^{-3/2}), \quad \boldsymbol{x} \to \boldsymbol{x}_n. \tag{4}$$

The matrix generating the Nevanlinna functions for the extension H_κ is defined as the matrix of correlation functions of the potentials $V^j_{n,l}(\boldsymbol{x})$ for the points $\boldsymbol{x}_n$ and $\boldsymbol{x}_m$ with the Green function (resolvent of the Laplace operator) $R(\mu)$,

$$\Gamma^{mk}_{nj}(\mu) = \int_{\mathbb{R}^3} \overline{V^j_{n,l}(\boldsymbol{x})} R(\mu, \boldsymbol{x}-\boldsymbol{y}) V^k_{m,l}(\boldsymbol{y}) \, d^3x \, d^3y. \tag{5}$$

Calculation of such functions for potentials of type (4) satisfying the transversality condition

$$\frac{\partial}{\partial x^l} V^j_{n,l}(\boldsymbol{x}) = 0,$$

is the goal of the present work.

One may note that finiteness of the integral in (5) for $n = m$ and $j = k$ implies finiteness of the integral

$$\Gamma^{nj}_{nj}(\rho) = -\int\limits_{\mathbb{R}^3} (\overline{R(\rho)V^j_{n,l}})(\boldsymbol{x})(\partial^2 + \rho)(R(\rho)V^j_{n,l})(\boldsymbol{x})\, d^3x, \quad \rho < 0,$$

and the latter means that the vectors $R(\rho)V^j_{n,l}$ belong to $\mathcal{D}(H)$, which contradicts the condition (1). Thus for the case $n = m$ and $j = k$ it is necessary to calculate first the integral

$$\Gamma^{nj}_{nj}(\mu) - \Gamma^{nj}_{nj}(\nu) = \int\limits_{\mathbb{R}^3} \overline{V^j_{n,l}(\boldsymbol{x})}(R(\mu, \boldsymbol{x}-\boldsymbol{y}) - R(\nu, \boldsymbol{x}-\boldsymbol{y}))V^j_{n,l}(\boldsymbol{y})\, d^3x\, d^3y, \quad (6)$$

and next fix the function $\Gamma^{nj}_{nj}(\mu)$ choosing the separation point.

§2. Correlation functions for potentials localized in one point

In order to calculate integrals (5) and (6) we will use Fourier transforms of the functions A and potentials $V^j_{n,l}$,

$$\widehat{A}(\boldsymbol{p}) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} A(\boldsymbol{x}) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\, d^3x.$$

In this case the Laplace operator acts as multiplication by the square of the modulus of $\boldsymbol{p}$,

$$L : \widehat{A}(\boldsymbol{p}) \to p^2 \widehat{A}(\boldsymbol{p}),$$

and its resolvent — as multiplication by $(p^2 - \mu)^{-1}$,

$$R(\mu) : \widehat{A}(\boldsymbol{p}) \to R(\mu)\widehat{A}(\boldsymbol{p}) = \frac{1}{p^2 - \mu}\widehat{A}(\boldsymbol{p}).$$

The transverse singular potential $V^j_{n,l}$ with the expansion (4) corresponds to the Fourier preimage of the following function:

$$\widehat{V}^j_{n,l}(\boldsymbol{p}) = \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_n}}{p^{1/2}} \Big(\delta_{lj} - \frac{p_l p_j}{p^2}\Big).$$

Indeed, the function $\widehat{V}^j_{n,l}(\boldsymbol{p})$ satisfies the transversality condition

$$p_l \widehat{V}^j_{n,l}(\boldsymbol{p}) = p_l \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_n}}{p^{1/2}} \Big(\delta_{lj} - \frac{p_l p_j}{p^2}\Big) = 0,$$

and its Fourier preimage has $-5/2$-power singularity

$$
\begin{aligned}
V^j_{n,l}(\boldsymbol{x}) &= \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} \widehat{V}^j_{n,l}(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\, d^3p \\
&= \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} \Big(\delta_{lj} - \frac{p_l p_j}{p^2}\Big) e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_n)} \frac{d^3p}{p^{1/2}} \\
&= \frac{\delta_{lj}}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_n)} \frac{d^3p}{p^{1/2}} + \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^j} \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_n)} \frac{d^3p}{p^{5/2}} \\
&= -\frac{\delta_{lj}}{2|\boldsymbol{x}-\boldsymbol{x}_n|^{5/2}} + \frac{5}{2}\frac{(x^j - x^j_n)(x^l - x^l_n)}{|\boldsymbol{x}-\boldsymbol{x}_n|^{9/2}}.
\end{aligned}
$$

Let us calculate integral (6) for potentials localized around the same point $\boldsymbol{x}_n$, and we also include the case $j \neq k$. After the Fourier transform the convolutions in (6) turn into multiplications and we can write

$$
\begin{aligned}
\Gamma^{nj}_{nk}(\mu) - \Gamma^{nj}_{nk}(\lambda) &= \int\limits_{\mathbb{R}^3} \overline{\widehat{V}^j_{n,l}} \Big(\frac{1}{p^2-\mu} - \frac{1}{p^2-\lambda}\Big) \widehat{V}^k_{n,l}\, d^3p \\
&= \int\limits_{\mathbb{R}^3} \Big(\delta_{lj} - \frac{p_l p_j}{p^2}\Big)\Big(\delta_{lk} - \frac{p_l p_k}{p^2}\Big)\Big(\frac{1}{p^2-\mu} - \frac{1}{p^2-\lambda}\Big)\frac{d^3p}{p} \\
&= \int\limits_{\mathbb{R}^3} \Big(\delta_{jk} - \frac{p_j p_k}{p^2}\Big)\Big(\frac{1}{p^2-\mu} - \frac{1}{p^2-\lambda}\Big)\frac{d^3p}{p} \\
&= \delta_{jk} \int\limits_{\mathbb{R}^3} \Big(\frac{\sum_{j'\neq j} p_{j'} p_{j'}}{p^2}\Big)\Big(\frac{1}{p^2-\mu} - \frac{1}{p^2-\lambda}\Big)\frac{d^3p}{p} \\
&= \frac{2}{3}\delta_{jk} \int\limits_{\mathbb{R}^3} \Big(\frac{1}{p^2-\mu} - \frac{1}{p^2-\lambda}\Big)\frac{d^3p}{p} \\
&= \frac{4\pi}{3}\delta_{jk}(\ln(-\lambda) - \ln(-\mu)),
\end{aligned}
$$

where the branching of the logarithm goes along the negative semi-axis. Finally, we choose the main branch of the logarithm

$$
\Im(\ln\rho) = 0, \quad \rho > 0,
$$

and choose a positive separation constant $\tilde{\kappa}$ to define $\Gamma^{nj}_{nk}(\mu)$,

$$\Gamma^{nj}_{nk}(\mu) = \frac{4\pi}{3}\delta_{jk} \ln \frac{\tilde{\kappa}}{-\mu}. \qquad (7)$$

Here, we can see that the logarithmic behavior of $\Gamma^{nj}_{nk}(\mu)$ coincides up to a coefficient with the Nevanlinna function of an interaction of the 2-dimensional particle with the δ-potential (correlation function of the 2-dimensional δ-potentials localized at the same point) [8].

§3. Correlation functions for potentials localized at different points

Let us turn to the integral (5) for potentials localized at different spatial points. To simplify the notation we will assume that the first potential is localized at the coordinate origin and the second one is attached to the point $\boldsymbol{x}$. We get the following two terms for the function $\Gamma^{2k}_{1j}(\mu)$:

$$\Gamma^{2k}_{1j}(\mu) = \int\limits_{\mathbb{R}^3} \overline{\widehat{V}^j_{1,l}} \frac{1}{p^2-\mu} \widehat{V}^k_{2,l}\, d^3p = \int\limits_{\mathbb{R}^3} (\delta_{jk} - \frac{p_j p_k}{p^2}) \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}}{p^2-\mu} \frac{d^3p}{p}. \qquad (8)$$

The first term is δ_{jk} multiplied by the integral which depends only on the product μx^2,

$$\mathrm{J}(\mu x^2) = \int\limits_{\mathbb{R}^3} \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}}{p^2-\mu} \frac{d^3p}{p}.$$

Our goal is to write the latter integral in terms of special functions and further use it to re-express the second term in (8), which evidently will also depend on coordinates of $\boldsymbol{x}$.

Let us calculate the integral over the angular variable between the vectors $\boldsymbol{p}$ and $\boldsymbol{x}$,

$$\mathrm{J}(\mu x^2) = \int\limits_{\mathbb{R}^3} \frac{-e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{p^2-\mu} \frac{d^3p}{p} = 4\pi \int\limits_0^\infty \frac{\sin px}{x} \frac{dp}{p^2-\mu}.$$

Using the table expression [9, eq. 3.723.8], we get

$$\mathrm{J}(\mu x^2) = \frac{4\pi}{\sqrt{\mu}x}\big(\cos\sqrt{\mu}x \;\mathrm{Si}\,\sqrt{\mu}x - \sin\sqrt{\mu}x\,\widetilde{\mathrm{Ci}}\sqrt{\mu}x\big), \qquad (9)$$

where the integral sine and cosine are defined as follows

$$\mathrm{Si}(\sqrt{\mu}x) = \int\limits_0^{\sqrt{\mu}x} \frac{\sin t}{t} dt, \tag{10}$$

$$\widetilde{\mathrm{Ci}}(\sqrt{\mu}x) = \boldsymbol{\gamma} + \frac{1}{2}\ln(-\mu x^2) - \int\limits_0^{\sqrt{\mu}x} \frac{1-\cos t}{t} dt, \tag{11}$$

and $\boldsymbol{\gamma}$ is the Euler constant. One may note that the expression (9) is a combination of even functions and the logarithm, thus it is not important how the square root of μ is defined.

Now let us calculate the second term in the right-hand side of (8),

$$\begin{aligned}
-\int\limits_{\mathbb{R}^3} \frac{p_j p_k}{p^2} \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}}{p^2-\mu} \frac{d^3 p}{p} &= \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \int\limits_{\mathbb{R}^3} \frac{(e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} - 1)}{(p^2-\mu)p^3} \frac{d^3 p}{p^3} \\
&= 4\pi \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \int\limits_0^\infty \Big(\frac{\sin px}{x} - p\Big) \frac{dp}{(p^2-\mu)p^2} \\
&= 4\pi \frac{\partial}{\partial x_j} \frac{x_k}{x} \int\limits_0^\infty \frac{\partial}{\partial x}\Big(\frac{\sin px}{x}\Big) \frac{dp}{(p^2-\mu)p^2} \\
&= 4\pi \Big(\frac{\delta_{jk}}{x} - \frac{x_j x_k}{x^3}\Big) \int\limits_0^\infty \frac{\partial}{\partial x}\Big(\frac{\sin px}{x}\Big) \frac{dp}{(p^2-\mu)p^2} \\
&\quad + 4\pi \frac{x_j x_k}{x^2} \int\limits_0^\infty \frac{\partial^2}{\partial x^2}\Big(\frac{\sin px}{x}\Big) \frac{dp}{(p^2-\mu)p^2}.
\end{aligned}$$

Here, in order to regularize the formal divergence at the coordinate origin we have added the term independent of $\boldsymbol{x}$ to the right-hand side of the first line and terminate it at the third line.

Further, we can write the first derivative with respect to x in the integral as

$$\frac{\partial}{\partial x}\Big(\frac{\sin px}{x}\Big) = p\frac{\cos px}{x} - \frac{\sin px}{x^2}$$

and group it together with the second-order derivative,

$$\frac{\partial^2}{\partial x^2}\Big(\frac{\sin px}{x}\Big) - \frac{1}{x}\frac{\partial}{\partial x}\Big(\frac{\sin px}{x}\Big) = 3\Big(\frac{\sin px}{x^3} - p\frac{\cos px}{x^2}\Big) - p^2\frac{\sin px}{x}$$
$$= -\frac{3}{x}\frac{\partial}{\partial x}\Big(\frac{\sin px}{x}\Big) - p^2\frac{\sin px}{x}.$$

Now one can observe that

$$\frac{1}{p}\frac{\partial}{\partial x}\frac{\sin px}{px} = \frac{1}{x}\frac{\partial}{\partial p}\frac{\sin px}{px}$$

and use this relation to transform the integral with the first derivative

$$\frac{4\pi}{x}\int_0^\infty \frac{\partial}{\partial x}\Big(\frac{\sin px}{x}\Big)\frac{dp}{(p^2-\mu)p^2} = \frac{4\pi}{x^2}\int_0^\infty \frac{\partial}{\partial p}\Big(\frac{\sin px}{px}\Big)\frac{dp}{p^2-\mu}$$
$$= 4\pi\Big(\frac{\sin px}{px^3}\Big)\frac{1}{p^2-\mu}\Big|_{p=0}^{\infty} + \frac{4\pi}{x^2}\int_0^\infty \frac{\sin px}{px}\frac{2p\,dp}{(p^2-\mu)^2}$$
$$= \frac{4\pi}{\mu x^2} + \frac{8\pi}{x^2}\frac{\partial}{\partial\mu}\int_0^\infty \frac{\sin px}{x}\frac{dp}{p^2-\mu} = \frac{4\pi}{\mu x^2} + 2\mathrm{J}'(\mu x^2).$$

Gathering the above equations we obtain the following expression:

$$-\int_{\mathbb{R}^3} \frac{p_j p_k}{p^2}\frac{-e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{p^2-\mu}\frac{d^3p}{p} = -4\pi\frac{x_j x_k}{x^2}\int_0^\infty \frac{\sin px}{x}\frac{dp}{p^2-\mu}$$
$$+ 4\pi\Big(\frac{\delta_{jk}}{x} - 3\frac{x_j x_k}{x^3}\Big)\int_0^\infty \frac{\partial}{\partial x}\Big(\frac{\sin px}{x}\Big)\frac{dp}{(p^2-\mu)p^2}$$
$$= -\frac{x_j x_k}{x^2}\mathrm{J}(\mu x^2) + \Big(\delta_{jk} - 3\frac{x_j x_k}{x^2}\Big)\Big(\frac{4\pi}{\mu x^2} + 2\mathrm{J}'(\mu x^2)\Big).$$

Adding the first term of (8), we get

$$\Gamma^{2k}_{1j}(\mu) = \Big(\delta_{jk} - \frac{x_j x_k}{x^2}\Big)\mathrm{J}(\mu x^2) + \Big(\delta_{jk} - 3\frac{x_j x_k}{x^2}\Big)\Big(\frac{4\pi}{\mu x^2} + 2\mathrm{J}'(\mu x^2)\Big).$$

This equation together with (7) can be combined in the following 6×6 matrix generating the Nevanlinna functions for singular potentials $V^j_{1,l}$ and

$V^k_{2,l}$ localized at the points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, such that $\boldsymbol{x}_2 - \boldsymbol{x}_1 = \boldsymbol{x}$,

$$\Gamma^{mk}_{nj}(\mu) = \begin{pmatrix} \frac{4\pi}{3}\delta_{jk}\ln\frac{\kappa}{-\mu} & P^{\boldsymbol{x}}_{jk}\mathrm{J}(\mu x^2)+Q^{\boldsymbol{x}}_{jk}(\frac{4\pi}{\mu x^2}+2\mathrm{J}'(\mu x^2)) \\ P^{\boldsymbol{x}}_{jk}\mathrm{J}(\mu x^2)+Q^{\boldsymbol{x}}_{jk}(\frac{4\pi}{\mu x^2}+2\mathrm{J}'(\mu x^2)) & \frac{4\pi}{3}\delta_{jk}\ln\frac{\kappa}{-\mu} \end{pmatrix}.$$

Here $m, n = 1, 2$ and we denote by $P^{\boldsymbol{x}}_{jk}$ and $Q^{\boldsymbol{x}}_{jk}$ matrices which depend on the coordinates of the vector $\boldsymbol{x}$

$$P^{\boldsymbol{x}}_{jk} = \delta_{jk} - \frac{x_j x_k}{x^2}, \quad Q^{\boldsymbol{x}}_{jk} = \delta_{jk} - 3\frac{x_j x_k}{x^2}.$$

The Nevanlinna functions themselves are represented by the quadratic form of the matrix $\Gamma^{mk}_{nj}(\mu)$ calculated with 6-dimensional vectors, corresponding to particular extension subspaces.

§4. Conclusion

We have calculated the matrix generating the Nevanlinna functions for extensions of the quadratic form of the transverse Laplace operator to functions with $-1/2$-power singularities in isolated points of 3-dimensional space. Matrix elements have logarithmic branching across the positive semi-axis, and squares of distances between the potentials enter the non-diagonal terms in dimensionless combinations with the spectral parameter.

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