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SEMI-INFINITE HEISENBERG XX0 CHAIN AND RANDOM WALKS

Abstract. Heisenberg XX0 chain on semi-infinte interval enables modelling of random walks restricted by presence of impenetrable wall. The state vectors of the Hamiltonian are represented in terms of symplectic Schur functions. The transition amplitudes of the model are obtained in the integral form and are estimated in the case of unlimited increasing of the number of steps of random walks.

§1. Introduction

Exactly solvable lattice systems are playing an important role in the development of statistical mechanics, enumerative combinatorics and representation theory in modern mathematics. The $XX0$ chain is the zero anisotropy limit of the prominent Heisenberg XXZ model, and it also may be considered as a special free fermion case [1,2]. Connection between the $XX0$ chain and the low-energy QCD, as well as a possibility of third order phase transition in the spin chain, are discussed in [3–6].

One of the most interesting properties of the model under consideration is that the answers can be obtained exactly. In particular dynamical correlators, transition amplitudes, off-shell wave functions are represented in the determinantal form. Mathematical methods used are based on the theory of Schur functions, of plane partitions, of Young diagrams and of random walks. The scalar products of the state-vectors, of the generating functions may be naturally modelled as vicious (non-intersecting) random walkers on the two-dimensional square lattice [7–15]. Vicious walkers describe the situation in which two or more walkers arriving at the same lattice site annihilate one another [16].

In this paper, we show how some results from the theory of symmetric functions can be used to examine the Heisenberg $XX0$ model on a semi-infinte chain and describe the more difficult case of walkers in the presence of impenetrable wall. We shall demonstrate that the exchange matrix of the

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model can be represented as the sum of two matrices that are generators of the *Cuntz algebra* [17]. In turn, this algebra can be considered as a special case of the *phase algebra* [18]. The two standard topologies of interest for random walks are that of a *star* and a *random turns*. The state vectors are written in terms of symplectic Schur functions and are described as sets of stars. The transition amplitudes of the model are obtained in the integral form and are expressed as random turns. The answers are estimated in case of unlimited increasing of the number of steps.

Organization of the paper is as follows. After introductory Section 1, we present in Section 2 outline of the Heisenberg $XX0$ model on semi-infinite chain. In Section 3 we introduce the even symplectic Schur function, discuss the combinatorial description of the symmetric functions and the eigenfunctions of the model. In Section 4 we define the generating function of random vicious walks and discuss its asymptotical behavior in Section 5. The asymptotics calculated for the infinite chain in the Section 6 allows us to compare it with the result obtained for the impenetrable wall. Section 7 concludes the paper.

§2. Outline

Open ends Heisenberg $XX0$ model describing $\frac{1}{2}$-spins on sites of a *semi-imfinite* chain is described by the Hamiltonian

$$\hat{H} = \sum_{n,m=0}^{\infty} \Delta_{nm} \sigma_n^- \sigma_m^+ = \sum_{n=0}^{\infty} \sigma_n^- \sigma_{n+1}^+ + \sigma_{n+1}^- \sigma_n^+ \,, \tag{1}$$

where the local spin operators $\sigma_n^\pm = \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)$ and σ_n^z depend on the lattice argument $n \in \bar{\mathbb{N}}$, and $\bar{\mathbb{N}}$ is a union of zero and of all natural numbers: $\bar{\mathbb{N}} \equiv \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$. The commutation relations are valid:

$$[\sigma_n^+, \sigma_m^-] = \sigma_n^z \delta_{nm} \,, \quad [\sigma_n^z, \sigma_m^\pm] = \pm 2\sigma_n^\pm \delta_{nm} \,. \tag{2}$$

The coupling of spins in (1) is expressed by the entries Δ_{nm} constituting the *exchange matrix* $\boldsymbol{\Delta}$:

$$\Delta_{nm} = \delta_{|n-m|,1} \,, \tag{3}$$

where $\delta_{n,l} (\equiv \delta_{nl})$ is the Kronecker symbol.

Spin "up" and "down" states on n^{th} site, $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$, are defined so that the rising/lowering operators $\sigma_n^\pm$ act on them as follows:

$$\sigma_n^+ \, |\downarrow\rangle_n = |\uparrow\rangle_n \,, \qquad \sigma_n^- \, |\uparrow\rangle_n = |\downarrow\rangle_n \,, \qquad \sigma_n^- \, |\downarrow\rangle_n = \sigma_n^+ \, |\uparrow\rangle_n = 0 \,. \tag{4}$$

The ferromagnetic state with all spins "up",

$$|\Uparrow\rangle \equiv \bigotimes_{n\in\bar{\mathbb{N}}} |\uparrow\rangle_n \equiv \bigotimes_{n\in\bar{\mathbb{N}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n , \tag{5}$$

is an eigen-vector of σ_m^z: $\sigma_m^z|\Uparrow\rangle = |\Uparrow\rangle$, and it is chosen as the reference state (i.e., pseudovacuum). The state (5) is annihilated by σ_m^+, $\sigma_m^+|\Uparrow\rangle = 0$, and, therefore, it is annihilated by the Hamiltonian (1):

$$\hat{H}\,|\Uparrow\rangle = 0\,.$$

The state (5) is normalized $\langle\Uparrow|\Uparrow\rangle = 1$.

The exchange matrix $\boldsymbol{\Delta}$ is of the form:

$$\boldsymbol{\Delta} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \end{pmatrix} ,$$

and a couple of special matrices,

$$\mathsf{S} \equiv \begin{pmatrix} 0 & 0 & 0 & \dots & \dots \\ 1 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \dots \end{pmatrix}$$

and its transpose S^{T}, enables to express $\boldsymbol{\Delta}$:

$$\boldsymbol{\Delta} = \mathsf{S} + \mathsf{S}^{\mathrm{T}}\,. \tag{6}$$

Let us identify a spin "down" state on j^{th} site $|\,j\rangle$ as the "coordinate" column,

$$|\,j\rangle \sim \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} (0) \\ \vdots \\ (j-1) \\ (j) \\ (j+1) \\ \vdots \end{matrix} ,$$

whereas a row $\langle j\,|$ as transpose $|\,j\rangle$. Then, the relations are valid for S and S^{T} (6):

$$\mathsf{S}\,|\,j\rangle = \delta_{r-j,1}\,|\,r\rangle\,, \quad r \geqslant j \geqslant 0\,,$$

$$\mathsf{S}^{\mathrm{T}}\,|\,j\rangle = \delta_{j-r,1}\,|\,r\rangle\,, \quad j \geqslant r \geqslant 0\,.$$

In other words, S looks like "creation operator", S^{T} is analogous to "annihilation operator", and $|\,0\rangle$ is "vacuum vector" since $\mathsf{S}^{\mathrm{T}}\,|\,0\rangle = \langle 0\,|\mathsf{S} = 0$ and $\mathsf{S}\,|\,0\rangle = |\,1\rangle$. The algebra given by S and S^{T} is called the *Cuntz algebra* [17]:

$$\mathsf{S}^{\mathrm{T}}\,\mathsf{S} = \mathbb{I}\,, \qquad \mathsf{S}\,\mathsf{S}^{\mathrm{T}} = \mathbb{I} - \mathbb{P}\,, \tag{7}$$

where $\mathbb{I}$ is the identity operator, and $\mathbb{P}$ is the vacuum projector:

$$\mathbb{P} \equiv \begin{pmatrix} 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \dots \end{pmatrix},$$

since $\mathbb{P}|\,j\rangle = \delta_{j0}|\,j\rangle$, $j \geqslant 0$. The matrices S and S^{T} were introduced and studied by V. Fock in [19]. Due to the property (7), the exchange matrix $\boldsymbol{\Delta}$ under consideration significantly differs from that defined on a ring, which is expressed analogously to (6), but in terms of the *circulant matrices* [13, 14].

Let us introduce *strict partition*, i.e., N-tuple $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)$ consisting of strictly decreasing integers $\mu_k \in \bar{\mathbb{N}}$, $1 \leqslant k \leqslant N$ called *parts* of $\boldsymbol{\mu}$, where

$$\mu_1 > \mu_2 > \dots > \mu_N \geqslant 0\,. \tag{8}$$

Since the operators $\sigma_n^{\pm}$ act on $|\!\uparrow\rangle_n$ and $|\!\downarrow\rangle_n$ according to (4), we define the state $|\boldsymbol{\mu}\rangle$ corresponding to N *flipped* (i.e., "down") spins ("particles") on the sites labelled by the parts ("coordinates") μ_k, and the corresponding conjugate state $\langle\boldsymbol{\nu}|$:

$$|\boldsymbol{\mu}\rangle \equiv |\mu_1, \mu_2, \dots, \mu_N\rangle \equiv \left(\prod_{k=1}^{N} \sigma_{\mu_k}^{-}\right) |\!\Uparrow\rangle\,, \tag{9}$$

$$\langle\boldsymbol{\nu}| = \langle\nu_1, \nu_2, \dots, \nu_N|\, \equiv \langle\Uparrow\!|\, \left(\prod_{k=1}^{N} \sigma_{\nu_k}^{+}\right), \tag{10}$$

where $|\Uparrow\rangle$ is given by (5). The states (9), (10) provide a complete orthogonal base:

$$\langle \boldsymbol{\nu} | \boldsymbol{\mu} \rangle = \boldsymbol{\delta}_{\boldsymbol{\nu}\boldsymbol{\mu}} \equiv \prod_{n=1}^{N} \delta_{\nu_n \mu_n} \,.$$

Let us introduce the state $|\Psi(\mathbf{p}_N)\rangle$ as the linear combination of $|\boldsymbol{\mu}\rangle$ (9) with the coefficients $\chi_{\boldsymbol{\mu}}(e^{i\mathbf{p}_N})$:

$$|\Psi(\mathbf{p}_N)\rangle = \sum_{\{\boldsymbol{\mu},\, \mu_N \geqslant 0\}} \chi_{\boldsymbol{\mu}}(e^{i\mathbf{p}_N})\, |\boldsymbol{\mu}\rangle \,. \tag{11}$$

Under the exponential parameterization:

$$\chi_{\boldsymbol{\mu}}(e^{i\mathbf{p}_N}) \equiv \chi_{\boldsymbol{\mu}}(e^{\pm ip_1}, e^{\pm ip_2}, \dots, e^{\pm ip_N}) = \det_{1\leqslant j,k \leqslant N} \big(\sin(\mu_k + 1)p_j\big)\,, \tag{12}$$

where $e^{\pm ip_k}$ is a shorthand notation for e^{ip_k}, e^{-ip_k}, and $p_k \in [0, \pi] \subset \mathbb{R}$, $\forall k$. Solving the eigenvalue problem

$$\hat{H} \mid \Psi(\mathbf{p}_N)\rangle = E_N(\mathbf{p}_N) \mid \Psi(\mathbf{p}_N)\rangle$$

leads to the following identity

$$\sum_{k=1}^{N} \big(\chi_{\boldsymbol{\mu}+\mathbf{e}_k}(e^{\pm ip_1}, e^{\pm ip_2}, \dots, e^{\pm ip_N}) + \chi_{\boldsymbol{\mu}-\mathbf{e}_k}(e^{\pm ip_1}, e^{\pm ip_2}, \dots, e^{\pm ip_N})\big)$$
$$= \sum_{k=1}^{N} \big(e^{ip_k} + e^{-ip_k}\big) \times \chi_{\boldsymbol{\mu}}(e^{\pm ip_1}, e^{\pm ip_2}, \dots, e^{\pm ip_N})\,.$$

Here $\mathbf{e}_k$, $1 \leqslant k \leqslant N$, are N-tuples consisting of zeros except of a unity at k^{th} place, say, from the left. The state $|\Psi(\mathbf{p}_N)\rangle$ (11) provides us N-particle eigen-vector of the Hamiltonian (1) with

$$E_N(\mathbf{p}_N) = \sum_{k=1}^{N} \big(e^{ip_k} + e^{-ip_k}\big) = 2\sum_{k=1}^{N} \cos p_k \,. \tag{13}$$

§3. Star of lattice paths

Let us discuss the combinatorial interpretation of the function introduced in (12). Consider an r-tuple $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ consisting of weakly decreasing non-negative integers:

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0\,. \tag{14}$$

The number of non-trivial parts of $\boldsymbol{\lambda}$ is the *length* of partition $l(\boldsymbol{\lambda}) = r$. The *Ferrers board* of $\boldsymbol{\lambda}$ is an array of cells with $l(\boldsymbol{\lambda})$ left-justified rows and

λ_i cells in row i. The conjugate of $\boldsymbol{\lambda}$ is the partition $\boldsymbol{\lambda}' = (\lambda'_1, \lambda'_2, \dots, \lambda'_N)$, where λ'_j is the length of j^{th} column in the Ferrers board of $\boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda}$ be a partition of length $r \leqslant N$. A tableau T of shape $\boldsymbol{\lambda}$ is called a $2N$-symplectic tableau [20] which is filled with entries in $(1 < \bar{1} < 2 < \bar{2} < \dots < N < \bar{N})$ such that entries are weakly increasing along rows, strictly increasing along columns, and obeys the additional constraint $T_{ij} \geqslant i$ (T_{ij} denotes the entry in cell (i, j) of T). For the symplectic case, let $\mathbf{x}_N = (x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_N^{\pm 1})$ be a sequence of $2N$ variables, where $x_k^{\pm 1}$ is a shorthand notation for x_k, x_k^{-1} ($x_k \in \mathbb{C}, \forall k$), [21]. The weight of a symplectic tableau is defined as

$$\mathbf{x}_N^T \equiv \prod_{i=1}^{N} x_i^{\#\, of\, i's\, in\, T} \left(x_i^{-1}\right)^{\#\, of\, \bar{i}'s\, \, in\, T} .$$

The even symplectic Schur function associated to $\boldsymbol{\lambda}$ is defined by

$$\mathtt{sp}_{\boldsymbol{\lambda}}(\mathbf{x}_N) \equiv \sum_T \mathbf{x}_N^T \,, \tag{15}$$

where the sum is over all symplectic tableau T of shape $\boldsymbol{\lambda}$.

The even symplectic Schur functions (15) can be expressed as [20]:

$$\mathtt{sp}_{\boldsymbol{\lambda}}(\mathbf{x}_N) = \frac{\det\limits_{1 \leqslant j,k \leqslant N} \left(x_j^{\lambda_k + N - k + 1} - x_j^{-(\lambda_k + N - k + 1)} \right)}{\det\limits_{1 \leqslant j,k \leqslant N} \left(x_j^{N - k + 1} - x_j^{-(N - k + 1)} \right)} . \tag{16}$$

This function is symmetric with respect to the variables $x_1, x_2, \dots, x_N$ and is invariant under the exchange $x_i \longleftrightarrow x_i^{-1}$, $1 \leqslant i \leqslant N$. The denominator in this expression is an analog of the Vandermonde identity calculated in [22]:

$$\det_{1 \leqslant j,k \leqslant N} \left(x_j^{N-k+1} - x_j^{-(N-k+1)} \right) = \prod_{j=1}^{N} \left(x_j - x_j^{-1} \right) \prod_{1 \leqslant k < j \leqslant N} \left(x_k + x_k^{-1} - x_j - x_j^{-1} \right) . \tag{17}$$

The r^{th} order *elementary symmetric function* $e_r = e_r(\mathbf{x}_N)$ of N variables, $\mathbf{x}_N = (x_1, x_2, \dots, x_N)$, is defined by

$$e_r(\mathbf{x}_N) \equiv \sum_{1 \leqslant i_1 < i_2 < \dots < i_r \leqslant N} x_{i_1} x_{i_2} \dots x_{i_r} .$$

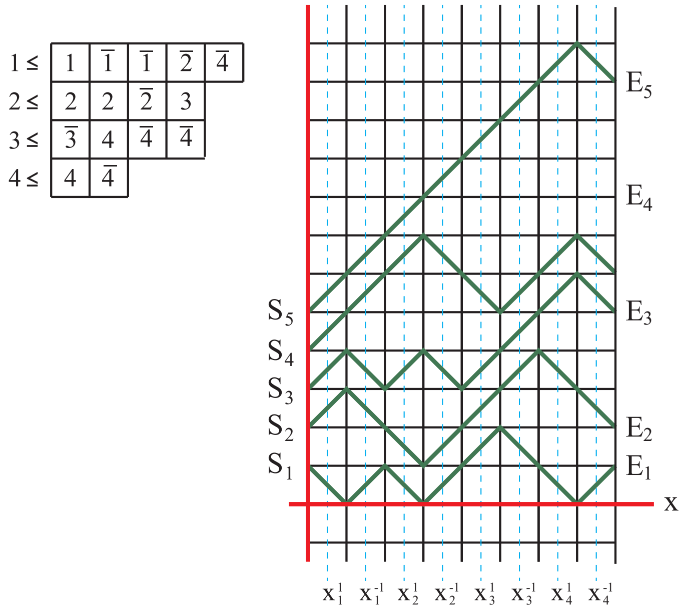

Figure 1. A symplectic tableau of shape $\boldsymbol{\lambda} = (5,4,4,2)$ (left), and the corresponding nest of lattice paths with weight $\mathbf{x}_4^T = x_1(x_1^{-1})^2 x_2^2 (x_2^{-1})^2 x_3 x_3^{-1} x_4^2 (x_4^{-1})^4 = x_1^{-1} x_4^{-2}$.

The identity for the symplectic character was proved in [23]:

$$\mathtt{sp}_{\boldsymbol{\lambda}}(\mathbf{x}_N) = \det_{1\leqslant i,j\leqslant \lambda_1} \left(e_{\lambda'_i - i + j}(\mathbf{x}_N) - e_{\lambda'_i - i - j}(\mathbf{x}_N) \right) ,$$

where $\boldsymbol{\lambda}'$ is the partition conjugate to $\boldsymbol{\lambda}$.

A symplectic tableau may be expressed by a nest of lattice paths [21]. A nest of lattice paths is called *non-intersecting* if no two paths in the nest to cross the same lattice site. A nest of lattice paths associated with the shape $\boldsymbol{\lambda}$ is a collection of non-intersecting lattice paths from starting points $S_i = (0, i)$ to $E_i = (2N, e_i)$, $i = 1, 2, \ldots, \lambda_1$; $0 < e_i \leqslant 2N + \lambda_1$, which are characterized by a set $\{(1,1), (1,-1)\}$ of up and down steps, and which never go below x-axis. The weight of a nest is the product of the weights of its steps. The weight of an up-step is unity, while either x_n or x_n^{-1} is the weight of a down-step, provided that either $(2n-1)^{\mathrm{st}}$ or $(2n)^{\mathrm{th}}$ step of a path is respectively considered. This type of collection of lattice paths is known as a *star* (Figure 1). The number of stars with fixed end points and with wall restriction is given by [21]:

$$\begin{aligned}\mathtt{sp}_{\boldsymbol\lambda}(\mathbf{1}) &\equiv \mathtt{sp}_{\boldsymbol\lambda}(1,\dots,1)\\ &= \prod_{1\leqslant k<l\leqslant N} \frac{\lambda_k - k - \lambda_l + l}{l-k} \prod_{1\leqslant k\leqslant l\leqslant N} \frac{\lambda_k + \lambda_l + N - k - l + 2}{N-k-l+2}\end{aligned} \tag{18}$$

The relationship between the parts of $\boldsymbol\mu$ (8) and $\boldsymbol\lambda$ (14) is expressed as

$$\lambda_j = \mu_j + j - N\,, \qquad 1\leqslant j \leqslant N\,, \tag{19}$$

or $\boldsymbol\lambda = \boldsymbol\mu - \boldsymbol\delta_N$, where $\boldsymbol\delta_N$ is the *staircase* partition

$$\boldsymbol\delta_N \equiv (N-1, N-2, \dots, 0)\,. \tag{20}$$

The *volume* $|\boldsymbol\lambda|$ of partition $\boldsymbol\lambda$ is the sum of its parts: $|\boldsymbol\lambda| \equiv \sum_{i=1}^N \lambda_i$. The volumes of $\boldsymbol\mu$, $\boldsymbol\lambda$, and $\boldsymbol\delta$ are connected: $|\boldsymbol\mu| = |\boldsymbol\lambda| + \frac{N}{2}(N-1)$.

The eigenfunction (11), (12) is related to the symplectic Schur function (16) under the exponential parametrization provided that (19) holds:

$$\begin{aligned}\chi_{\boldsymbol\mu}(e^{i\mathbf{p}_N}) &= \mathtt{sp}_{\boldsymbol\lambda}(e^{\pm ip_1}, e^{\pm ip_2}, \dots, e^{\pm ip_N}) \det_{1\leqslant j,k\leqslant N}(\sin(N-k+1)p_j)\\ &\equiv \mathtt{sp}_{\boldsymbol\lambda}(\mathbf{p}_N) \det_{1\leqslant j,k\leqslant N}(\sin(N-k+1)p_j)\,.\end{aligned} \tag{21}$$

This representation gives the enumerative interpretation of eigenfunctions (11), (12), (16) in terms of stars. The asymptotic behavior of stars when N tends to infinity was studied in [21].

§4. The generating function

We shall consider the generating function $G(\boldsymbol\mu;\boldsymbol\nu\,|K)$, which provides the number of nests of non-intersecting lattice paths of N random turns vicious walkers performing K steps [7]. It is given by N-particle *transition amplitude* between the states $|\boldsymbol\mu\rangle$ (9) and $\langle\boldsymbol\nu|$ (10) parameterized by strict partitions $\boldsymbol\mu$ and $\boldsymbol\nu$ (see (8)), which are interpreted as initial and final positions of random walks [7, 14] (Figure 2):

$$G(\boldsymbol\mu;\boldsymbol\nu\,|K) \equiv \langle\boldsymbol\nu \mid \hat H^K \mid \boldsymbol\mu\rangle\,, \tag{22}$$

where $\hat H$ is the Hamiltonian (1). This generating function is interpreted in the following way.

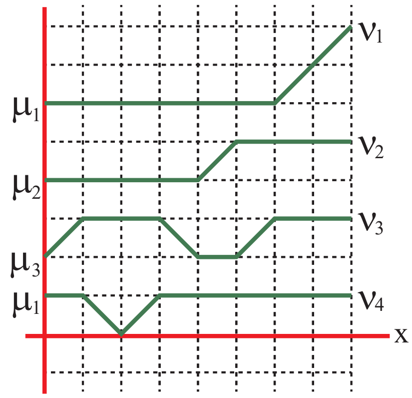

Figure 2. A nest of lattice paths of random turns vicious walkers for $N = 4$, $K = 8$

Really, applying the commutation relation (2), one obtains at $N = 1$:

$$\begin{aligned} G(\mu;\nu\,|K) &\equiv \langle\Uparrow|\,\sigma_\nu^+ \hat{H}^K \sigma_\mu^-\,|\Uparrow\rangle \\ &= \langle\Uparrow|\,\sigma_\nu^+ \hat{H}^{K-1}[-\hat{H},\sigma_\mu^-]\,|\Uparrow\rangle = \langle\Uparrow|\,\sigma_\nu^+ \hat{H}^{K-1}\sum_{n_1}\Delta_{n_1\mu}\sigma_{n_1}^-\,|\Uparrow\rangle \\ &= \langle\Uparrow|\,\sigma_\nu^+ \sum_{n_1,n_2,\dots,n_K}\Delta_{n_K n_{K-1}}\cdots\Delta_{n_2 n_1}\Delta_{n_1\mu}\sigma_{n_K}^-\,|\Uparrow\rangle \\ &= \sum_{n_1,n_2,\dots,n_{K-1}}\Delta_{\nu n_{K-1}}\cdots\Delta_{n_2 n_1}\Delta_{n_1\mu} = \left(\mathbf{\Delta}^K\right)_{\nu\mu} = \left((\mathsf{S}+\mathsf{S}^{\mathrm{T}})^K\right)_{\nu\mu}. \end{aligned} \tag{23}$$

The position of a single particle on a lattice is labeled by the spin down state, while the spin up states correspond to the empty sites. The particle starts at randomly chosen initial site l, at first step it moves to one of the sites defined by the matrix element Δ_{nm} (3), namely one step up $(\mathsf{S})_{nm}$ or down $(\mathsf{S}^{\mathrm{T}})_{nm}$: $\{(1,1),(1,-1)\}$. The difference equation of the generating function (22) for $N = 1$ follows from the equation (23):

$$G(\mu,\nu\,|K+1) = G(\mu+1;\nu\,|K) + G(\mu-1;\nu\,|K) \tag{24}$$

for the fixed ν and a similar equation for the fixed μ. Equation (24) is supplied with the condition $G(\mu;0\,|K) = G(0;\nu\,|K) = 0$.

After K steps all admissible paths of the particle starting from the site l and ending at j are given by the matrix product (23). Using the commutation relation

$$[\hat{H}, \sigma^-_{\mu_1}\sigma^-_{\mu_2}...\sigma^-_{\mu_N}] = \sum_{k=1}^{N} \sigma^-_{\mu_1}...\sigma^-_{\mu_{k-1}}[\hat{H}, \sigma^-_{\mu_k}]\sigma^-_{\mu_{k+1}}...\sigma^-_{\mu_N}\,, \tag{25}$$

we see that the average (22) is equal to the number of configurations that have the N random turns walkers being initially located on the lattice sites $\mu_1 > \mu_2 > \ldots > \mu_N \geqslant 0$ and after K steps arrived at the positions $\nu_1 > \nu_2 > \ldots > \nu_N \geqslant 0$. The vicious walk condition, the condition that paths does not touch each other up to K steps, is guaranteed by the property of the Pauli matrices $(\sigma^{\pm}_k)^2 = 0$. The additive form of the equation (25) means that under the action of Hamiltonian the only one walker jumps up or down $\{(1,1),(1,-1)\}$, while the rest are staying $\{(0,1)\}$.

Taking into account the equation (25), we shall obtain the difference equation for the generating function (22):

$$G(\boldsymbol{\mu}, \boldsymbol{\nu}\,|K+1) = \sum_{k=1}^{N} G(\boldsymbol{\mu} + \mathbf{e}_k; \boldsymbol{\nu}\,|K) + G(\boldsymbol{\mu} - \mathbf{e}_k; \boldsymbol{\nu}\,|K) \tag{26}$$

for the fixed $\boldsymbol{\mu}$, and a similar equation is found for the fixed $\boldsymbol{\nu}$. The non-intersection condition means that $G(\boldsymbol{\mu}, \boldsymbol{\nu}\,|K) = 0$ if $\mu_k = \mu_p$ or $\nu_k = \nu_p$ for any $1 \leqslant k, p \leqslant N$, and the boundary conditions are $G(\boldsymbol{\mu}, \boldsymbol{\nu}\,|K) = 0$ if $\mu_k = 0$ or $\nu_k = 0$ for any $1 \leqslant k \leqslant N$.

§5. Asymptotics

5.1. First way. Knowing the eigen-functions (12), (21) of the Hamiltonian (1), we may write the solution of the difference equation (26) in the following form:

$$\begin{aligned} G(\boldsymbol{\mu}; \boldsymbol{\nu}\,|K) &= \frac{1}{\pi^N N!} \prod_{n=1}^{N} \int_{-\pi}^{\pi} \mathrm{d}p_n \Big(2\sum_{k=1}^{N} \cos p_k\Big)^K \chi_{\boldsymbol{\mu}}(e^{i\mathbf{p}_N}) \chi_{\boldsymbol{\nu}}(e^{i\mathbf{p}_N}) \\ &= \frac{1}{\pi^N N!} \prod_{n=1}^{N} \int_{-\pi}^{\pi} \mathrm{d}p_n \Big(2\sum_{k=1}^{N} \cos p_k\Big)^K \\ &\times \det_{1\leqslant j,k\leqslant N} (\sin p_j \bar{\mu}_k) \det_{1\leqslant j,k\leqslant N} (\sin p_j \bar{\nu}_k)\,. \end{aligned} \tag{27}$$

Here the notations $\bar{\mu}_k \equiv \mu_k + 1$, $\bar{\nu}_k \equiv \nu_k + 1$ are introduced for the parts of auxiliary partitions $\bar{\boldsymbol{\mu}} = \boldsymbol{\mu} + \mathbf{1}$, where $\boldsymbol{\mu}$ is given by (8) and $\mathbf{1}$ is N-tuple $(1, 1, \ldots, 1)$:

$$\bar{\mu}_1 > \bar{\mu}_2 > \ldots > \bar{\mu}_N \geqslant 1\,. \tag{28}$$

Using (21), one obtains:

$$
\begin{aligned}
G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \;&=\; \frac{1}{\pi^N N!}\prod_{n=1}^{N}\int_{-\pi}^{\pi}\mathrm{d}p_n\Big(2\sum_{k=1}^{N}\cos p_k\Big)^K \\
&\times\;\Big(\det_{1\leqslant j,k\leqslant N}(\sin kp_j)\Big)^2\,\mathtt{sp}_{\boldsymbol{\lambda}^L}(\mathbf{p}_N)\,\mathtt{sp}_{\boldsymbol{\lambda}^R}(\mathbf{p}_N)\,, \qquad (29)
\end{aligned}
$$

where $\boldsymbol{\lambda}^R = \boldsymbol{\nu} - \boldsymbol{\delta}_N$, $\boldsymbol{\lambda}^L = \boldsymbol{\mu} - \boldsymbol{\delta}_N$ (see (19)). The determinant in the integrand of (29) is obtained using the identity (17) (see [21]):

$$
\begin{aligned}
\det_{1\leqslant r,s\leqslant N}(\sin s\theta_r) \;&=\; 2^{N(N-1)}\prod_{r=1}^{N}\sin\theta_r \\
&\times\;\prod_{1\leqslant j<k\leqslant N}\sin\left(\frac{\theta_j-\theta_k}{2}\right)\sin\left(\frac{\theta_j+\theta_k}{2}\right). \qquad (30)
\end{aligned}
$$

We are interested here in the large K limit with $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ kept fixed. In order to apply the standard saddle-point approximation, we express the first factor of integrand in the above equations (29) in the form

$$
\exp\{K\log\sum_k\cos p_k\},
$$

and thereby obtain the following system of saddle point-equations [9]:

$$
\frac{\sin p_r}{\sum_{k=1}^{N}\cos p_k} = 0\,, \qquad r\in\{1,2,\dots,N\}\,. \qquad (31)
$$

It is obvious that the solutions to this system of equations satisfy $\sin p_r = 0$ $(1\leqslant r\leqslant N)$ with the restriction that $\sum_k\cos p_k\neq 0$. Requiring that the matrix of second derivatives

$$
\frac{\partial^2}{\partial p_r\partial p_s}\log\left(\sum_{k=1}^{N}\cos p_k\right) = \frac{\cos p_r}{\sum_{k=1}^{N}\cos p_k}\,\delta_{rs} - \frac{\sin p_r\,\sin p_s}{\Big(\sum_{k=1}^{N}\cos p_k\Big)^2}
$$

for the solution of (31) is a negative definite matrix, we find that the steepest descent corresponds to the solution for which $\cos p_r = 1$, $1\leqslant r\leqslant N$, i.e., the main contribution to the integrals in (29) comes from near the points $p_r = 0$, $1\leqslant r\leqslant N$. Therefore, we may replace the first factor of the integrand in (29) by its approximation

$$
\Big(2\sum_{k=1}^{N}\cos p_k\Big)^K \propto (2N)^K\exp\Big\{-\frac{K}{2N}\sum_{k=1}^{N}p_k^2\Big\}\,. \qquad (32)
$$

As $K \to \infty$, the main contributions to the integrals in (29) come from near the origin of the integration variables. Using (30) under integration in (29), we find in the leading order:

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \simeq \mathtt{sp}_{\boldsymbol{\lambda}^L}(\mathbf{1})\mathtt{sp}_{\boldsymbol{\lambda}^R}(\mathbf{1})\,\frac{(2N)^K}{\pi^N N!} \times \prod_{j=1}^{N}\left(\int_{-\infty}^{\infty} \mathrm{d}p_j\, p_j^2\right) e^{-(K/2N)\sum_{k=1}^N p_k^2} \prod_{1\leqslant i<l\leqslant N} \left(p_i^2 - p_l^2\right)^2 , \quad (33)$$

where the symplectic Schur polynomials $\mathtt{sp}_{\boldsymbol{\lambda}^{L,R}}(\mathbf{1})$ are associated with the combinatorial interpretation of the eigen-functions (18). The integral (33) is the Mehta integral [24], which is evaluated:

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) = \mathtt{sp}_{\boldsymbol{\lambda}^L}(\mathbf{1})\mathtt{sp}_{\boldsymbol{\lambda}^R}(\mathbf{1})\,\frac{(2N)^K \prod_{m=1}^N (2m)!}{(2\pi)^{\frac{N}{2}} N!}\left(\frac{N}{K}\right)^{\frac{N(2N+1)}{2}}, \quad (34)$$

where $\boldsymbol{\lambda}^R = \boldsymbol{\nu} - \boldsymbol{\delta}_N$, $\boldsymbol{\lambda}^L = \boldsymbol{\mu} - \boldsymbol{\delta}_N$.

5.2. Second way. Making use of the symmetry of the integrand with respect to permutations of $p_1, p_2, \ldots, p_N$, the determinant in (27) can be transformed as

$$\frac{1}{N!} \det_{1\leqslant j,k\leqslant N}\big(\sin(p_j\bar{\mu}_k)\big) \det_{1\leqslant j,k\leqslant N}\big(\sin(p_j\bar{\nu}_k)\big) \longrightarrow \det_{1\leqslant j,k\leqslant N}\big(\sin(p_j\bar{\mu}_k)\big) \prod_{k=1}^{N} \sin(p_k\bar{\nu}_k) \longrightarrow \det_{1\leqslant j,k\leqslant N}\big(\sin(p_j\bar{\mu}_k)\sin(p_j\bar{\nu}_j)\big) . \quad (35)$$

Using (35) we get an alternative expression for the generating function (29):

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) = \frac{2^K}{\pi^N} \prod_{n=1}^{N} \int_{-\pi}^{\pi} \mathrm{d}p_n \left(\sum_{m=1}^{N} \cos p_m\right)^K \times \det_{1\leqslant j,k\leqslant N}\big(\sin(p_j\bar{\mu}_k)\sin(p_j\bar{\nu}_j)\big) , \quad (36)$$

where the partitions (28) are used.

The answer for large K may be obtained started from (36). Indeed, applying (32) we obtain:

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \backsimeq \frac{(2N)^K}{(2\pi)^N}\prod_{j=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}p_j\,\exp\bigl(\tfrac{-K}{2N}\textstyle\sum_{k=1}^{N}p_k^2\bigr) \times \det_{1\leqslant j,k\leqslant N}\bigl\{\cos\bigl(p_j(\bar\mu_k-\bar\nu_j)\bigr)-\cos\bigl(p_j(\bar\mu_k+\bar\nu_j)\bigr)\bigr\}\,. \quad (37)$$

The matrix Gaussian integral (37) is evaluated explicitly, and one finds:

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \simeq \frac{(2N)^K}{(2\pi)^{N/2}}\Bigl(\frac{N}{K}\Bigr)^{N/2} q^{(|\bar{\boldsymbol{\mu}}|^2+|\bar{\boldsymbol{\nu}}|^2)/2}\det_{1\leqslant j,k\leqslant N}\bigl(q^{-\bar\mu_j\bar\nu_k}-q^{\bar\mu_j\bar\nu_k}\bigr)\,, \quad (38)$$

where $q\equiv e^{-N/K}$, and $|\bar{\boldsymbol{\mu}}|^2\equiv\sum_{j=1}^{N}\bar\mu_j^2$ (the same for $|\bar{\boldsymbol{\nu}}|^2$).

Knowing the definition of the symplectic Schur function (16) we re-express (38):

$$G(\boldsymbol{\mu};\boldsymbol{\nu}) \simeq \frac{(2N)^K}{(2\pi)^{N/2}}\Bigl(\frac{N}{K}\Bigr)^{N/2} q^{(|\bar{\boldsymbol{\mu}}|^2+|\bar{\boldsymbol{\nu}}|^2)/2} \times \mathsf{sp}_{\boldsymbol{\lambda}^L}(q^{-\bar\nu_1},q^{-\bar\nu_2},\dots,q^{-\bar\nu_N})\det_{1\leqslant r,s\leqslant N}\bigl(q^{-\bar\nu_s\bar\delta_r}-q^{\bar\nu_s\bar\delta_r}\bigr)\,, \quad (39)$$

where $\boldsymbol{\lambda}^L\equiv\boldsymbol{\lambda}^L_N=\boldsymbol{\mu}_N-\boldsymbol{\delta}_N$, and the parts $\bar\delta_i=N-i+1$ of the partition $\bar{\boldsymbol{\delta}}_N=\boldsymbol{\delta}_N+\mathbf{1}$ (see $\boldsymbol{\delta}_N$ (20)) are used. Another representation equivalent to (39) is due to the identity

$$\mathsf{sp}_{\boldsymbol{\lambda}^R}(q^{-N},q^{-(N-1)},\dots,q) = \frac{\det_{1\leqslant r,s\leqslant N}\bigl(q^{-\bar\nu_s\bar\delta_r}-q^{\bar\nu_s\bar\delta_r}\bigr)}{\det_{1\leqslant j,k\leqslant N}\bigl(q^{-jk}-q^{jk}\bigr)}\,, \quad (40)$$

where $\boldsymbol{\lambda}^R\equiv\boldsymbol{\lambda}^R_N=\boldsymbol{\nu}_N-\boldsymbol{\delta}_N$, and

$$\det_{1\leqslant r,s\leqslant N}\bigl(q^{-\bar\delta_s\bar\delta_r}-q^{\bar\delta_s\bar\delta_r}\bigr)=\det_{1\leqslant r,s\leqslant N}\bigl(q^{-sr}-q^{sr}\bigr)$$

is accounted for. One obtains:

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \simeq \frac{(2N)^K}{(2\pi)^{N/2}}\Bigl(\frac{N}{K}\Bigr)^{N/2} q^{(|\bar{\boldsymbol{\mu}}|^2+|\bar{\boldsymbol{\nu}}|^2)/2}\det_{1\leqslant j,k\leqslant N}\bigl(q^{-jk}-q^{jk}\bigr) \times \mathsf{sp}_{\boldsymbol{\lambda}^L}(q^{-\bar\nu_1},q^{-\bar\nu_2},\dots,q^{-\bar\nu_N})\,\mathsf{sp}_{\boldsymbol{\lambda}^R}(q^{-N},q^{-(N-1)},\dots,q)\,. \quad (41)$$

Equations (39) and (41) provide “exact” expressions for the generating function $G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K)$.

Let us obtain using (17) the asymptotics at $q \approx 1 - N/K$:

$$\det_{1\leqslant r,s\leqslant N}\bigl(q^{-\bar\nu_s\bar\delta_r} - q^{\bar\nu_s\bar\delta_r}\bigr) \approx 2^N\Bigl(\frac{N}{K}\Bigr)^{N^2}\prod_{i=1}^{N}\bar\nu_i\prod_{1\leqslant k<l\leqslant N}(\bar\nu_l^2 - \bar\nu_k^2)\,. \qquad (42)$$

Consider the parts $\bar\delta_i = N - i + 1$, $1 \leqslant i \leqslant N$:

$$\prod_{i=1}^{N}\bar\delta_i = N\,!\,, \qquad \prod_{1\leqslant k<l\leqslant N}(\bar\delta_l^2 - \bar\delta_k^2) = \frac{\prod_{k=1}^{N}(2k-1)\,!}{N\,!}\,,$$

and, therefore, (42) at $\bar{\boldsymbol\nu} \longrightarrow \bar{\boldsymbol\delta}$, i.e., $\boldsymbol\nu \longrightarrow \boldsymbol\delta$ results in

$$\begin{aligned}\det_{1\leqslant r,s\leqslant N}\bigl(q^{-rs} - q^{rs}\bigr) &\approx 2^N\Bigl(\frac{N}{K}\Bigr)^{N^2}\prod_{k=1}^{N}(2k-1)\,! \\ &= \Bigl(\frac{N}{K}\Bigr)^{N^2}\frac{\prod_{k=1}^{N}(2k)\,!}{N\,!}\,. \qquad (43)\end{aligned}$$

The limiting form of (40) results from (42) and (43) in agreement with (18):

$$\mathsf{sp}_{\boldsymbol\lambda^R}(\mathbf{1}) = \Bigl(\prod_{k=1}^{N}(2k-1)\,!\Bigr)^{-1}\prod_{i=1}^{N}\bar\nu_i\prod_{1\leqslant k<l\leqslant N}(\bar\nu_l^2 - \bar\nu_k^2)\,.$$

Using the approximation of expression (42) we get the final answer for (39):

$$\begin{aligned}G(\boldsymbol\mu;\boldsymbol\nu\,|K) &\simeq \mathsf{sp}_{\boldsymbol\lambda^L}(\mathbf{1})\prod_{i=1}^{N}(1+\nu_i)\prod_{1\leqslant k<l\leqslant N}(\nu_l - \nu_k)\,(\nu_l + \nu_k + 2) \\ &\times (2N)^K\Bigl(\frac{2}{\pi}\Bigr)^{N/2}\Bigl(\frac{N}{K}\Bigr)^{N^2+N/2}. \qquad (44)\end{aligned}$$

When $\boldsymbol\nu \longrightarrow \boldsymbol\delta$, one obtains

$$G(\boldsymbol\mu;\boldsymbol\delta\,|K) \simeq \mathsf{sp}_{\boldsymbol\lambda^L}(\mathbf{1})\,\frac{\prod_{k=1}^{N}(2k)\,!}{(2\pi)^{N/2}N\,!}\times(2N)^K\Bigl(\frac{N}{K}\Bigr)^{N^2+N/2}. \qquad (45)$$

With the help of (43) the equation (41) acquires the form:

$$\begin{aligned}G(\boldsymbol\mu;\boldsymbol\nu\,|K) &\simeq \mathsf{sp}_{\boldsymbol\lambda^R}(\mathbf{1})\,\mathsf{sp}_{\boldsymbol\lambda^L}(\mathbf{1})\Bigl(\frac{2}{\pi}\Bigr)^{N/2}\prod_{k=1}^{N}(2k-1)\,! \\ &\times (2N)^K\Bigl(\frac{N}{K}\Bigr)^{N^2+N/2}. \qquad (46)\end{aligned}$$

This answer coincides with (34) and (44). Equation (46) is reduced at $\boldsymbol{\nu} \longrightarrow \boldsymbol{\delta}$ to (45) since $\mathsf{sp}_{\boldsymbol{\lambda}^R}(\mathbf{1}) = 1$.

§6. Infinite chain

Let us turn to it in the limit of infinite chain, where the analogue of the integral representation $G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K)$ (36) acquires the form [14] (notice that $\mu_j - \nu_k = \bar{\mu}_j - \bar{\nu}_k$):

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) = \frac{2^K}{(2\pi)^N} \prod_{n=1}^{N} \int_{-\pi}^{\pi} \mathrm{d}p_n \Big(\sum_{m=1}^{N} \cos p_m \Big)^K \det_{1\leqslant j,k\leqslant N} \big(e^{-i(\mu_j - \nu_k)p_j} \big) \,. \tag{47}$$

Provided that the approximation (32) valid at $K \gg 1$ is used in (47), one obtains:

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \simeq \frac{(2N)^K}{(2\pi)^{N/2}} \Big(\frac{N}{K}\Big)^{N/2} q^{(|\boldsymbol{\mu}|^2 + |\boldsymbol{\nu}|^2)/2} \det_{1\leqslant j,k\leqslant N} (q^{-\mu_j \nu_k})\,, \tag{48}$$

where $q \equiv e^{-N/K}$, and $|\boldsymbol{\mu}|^2 \equiv \sum_{j=1}^N \mu_j^2$ (the same for $|\boldsymbol{\nu}|^2$). Besides, the relation

$$\det_{1\leqslant j,k\leqslant N} (q^{(\mu_j - \nu_k)^2/2}) = q^{(|\boldsymbol{\mu}|^2 + |\boldsymbol{\nu}|^2)/2} \det_{1\leqslant j,k\leqslant N} (q^{-\mu_j \nu_k})\,,$$

valid both for (8) and (28), is taken into account. The latter is convenient to compare with the approximate expressions (38) and (48).

Let us remind the definition of the *Schur functions* $S_{\boldsymbol{\lambda}}$ [25]:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_N) \equiv S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) \equiv \frac{\det(x_j^{\lambda_k + N - k})_{1\leqslant j,k\leqslant N}}{\mathcal{V}(\mathbf{x}_N)}\,. \tag{49}$$

Here $\mathcal{V}(\mathbf{x}_N)$ is the Vandermonde determinant

$$\mathcal{V}(\mathbf{x}_N) \equiv \det(x_j^{N-k})_{1\leqslant j,k\leqslant N} = \prod_{1\leqslant m<l\leqslant N} (x_m - x_l)\,. \tag{50}$$

Using (49) we re-express (48) (compare with (39)):

$$G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \simeq \frac{(2N)^K}{(2\pi)^{N/2}} \Big(\frac{N}{K}\Big)^{N/2} q^{(|\boldsymbol{\mu}|^2 + |\boldsymbol{\nu}|^2)/2} \times \mathsf{S}_{\boldsymbol{\lambda}^L}(q^{-\nu_1}, q^{-\nu_2}, \dots, q^{-\nu_N}) \det_{1\leqslant r,s\leqslant N} \big(q^{-\nu_s(N-r)} \big)\,,$$

where $\boldsymbol{\lambda}^L \equiv \boldsymbol{\lambda}^L_N = \boldsymbol{\mu}_N - \boldsymbol{\delta}_N$.

Applying the asymptotics of (50) at $q \approx 1 - N/K$:

$$\det_{1\leqslant r,s\leqslant N}\big(q^{-\nu_s(N-r)}\big) \approx \Big(\frac{N}{K}\Big)^{N(N-1)/2} \prod_{1\leqslant k<l\leqslant N} (\nu_k - \nu_l)\,, \tag{51}$$

and the limiting value of the Schur functions (49) [26]:

$$\mathsf{S}_{\boldsymbol{\lambda}^L}(\mathbf{1}) = \frac{\prod_{1\leqslant k<l\leqslant N}(\mu_k - \mu_l)}{\prod_{k=1}^{N}(k-1)\,!}\,, \tag{52}$$

one obtains the dependence of the asymptotics of the generating function on the growing number of steps K in the case of infinite chain:

$$\begin{aligned} G(\boldsymbol{\mu};\boldsymbol{\nu}\,|K) \;&\simeq\; \frac{(2N)^K}{(2\pi)^{N/2}}\Big(\frac{N}{K}\Big)^{N^2/2}\mathsf{S}_{\boldsymbol{\lambda}^L}(\mathbf{1}) \prod_{1\leqslant k<l\leqslant N}(\nu_k-\nu_l) \\ &=\; \frac{\prod_{1\leqslant k<l\leqslant N}(\mu_l-\mu_k)(\nu_l-\nu_k)}{\prod_{k=1}^{N}(k-1)\,!}\,\frac{(2N)^K}{(2\pi)^{N/2}}\,\Big(\frac{N}{K}\Big)^{N^2/2}. \end{aligned} \tag{53}$$

The asymptotics (53) should be compared with that for the generating function (46) corresponding to impenetrable wall.

§7. Discussion

The number of nests of paths of N walkers is growing mainly as $(2N)^K$ provided that the number of steps K is increasing both in the cases of semi-infinite and infinite chains, however critical exponent $N^2 + \frac{N}{2}$ and $N^2/2$ are different respectively. As a result, the growth of the number of nests, governed mainly by $(2N)^K$, is slowly for the semi-infinite chain in comparison with that of infinite chain. The combinatorial factors are different in both cases.

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