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N. M. Bogoliubov

RANDOM ORDERED LATTICE PATHS GENERATED BY OPERATORS SATISFYING THE CUNTZ ALGEBRA

Abstract. The technique based on operators satisfying the Cuntz algebra is used for the enumeration of Dyck, Motzkin and Łukasiewicz lattice paths. It is shown that the weighted paths may be considered as the generators of master fields of the quantum field theory.

§1. Introduction

The discrete mathematics of the directed lattice paths are widely studied. For the enumeration of these paths many methods are often used. In literature, Dyck, Motzkin and Łukasiewicz paths are the most often considered (see e.g.) [1–10].

In the papers [11, 12] the master field in a number of cases including QCD_2 was explicitly constructed. The master field was generated by a collection of creation and annihilation operators satisfying the *Cuntz* algebra.

We shall demonstrate that the vacuum expectations of the generators constructed by the elements of Cunst algebra give an alternative approach to the calculation of the directed lattice paths. These weighted lattice paths allow to construct the master fields and to find the coefficients of the Voiculescu polynomials [13].

The paper is organized as follows. We begin with some preliminaries in Section 2. In particular, we introduce the directed lattice paths , simple one dimensional lattice steps, Fock space and the Cuntz algebra acting in this space. The technique developed in the previous Section will be applied to enumerate directed lattice paths in Section 3. The interpretation of one-matrix master fields in terms of lattice paths is the purpose of Section 4.

§2. Lattice paths and the Cuntz algebra

An N-step lattice path or walk is a sequence of a step set

$$\mathcal{S} = \{(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)\}, \qquad \mathcal{S} \subset \mathbb{Z}.$$

Key words and phrases: directed lattice paths, generating functions, master field, Cuntz algebra.

We restrict our attention to directed paths, which are defined by the fact, that for each step $(x, y) \in \mathcal{S}$, one has $x \geqslant 0$. Moreover, we will consider the case, where every element in the step set $\mathcal{S}$ is of the form $(1, b)$ which means that these paths constantly move one step to the right. Thus, they are essentially one-dimensional objects and can be seen as walks on the integers along the y-axis.

The spatial coordinates on the y-axis $\mathbb{Z}_0^+ = \{0, 1, \ldots, n, \ldots\}$ can be thought of as vector states in the Fock space:

$$\mathcal{F}_{\mathcal{B}} = \{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\} \tag{1}$$

The Fock states $|n\rangle$ can be created from the vacuum state $|0\rangle$ by operating by the operators $\phi, \phi^\dagger$:

$$|n\rangle = \left(\phi^\dagger\right)^n |0\rangle\,, \quad 0 \leqslant n\,, \tag{2}$$

and

$$\phi^\dagger |n\rangle = |n+1\rangle\,, \quad \phi|n\rangle = |n-1\rangle\,, \quad \phi|0\rangle = 0\,. \tag{3}$$

The introduced operator ϕ is "one-sided unitary" or an isometric; although

$$\phi\phi^\dagger = I \tag{4}$$

one has

$$\phi^\dagger\phi = I - \pi\,, \tag{5}$$

in which π is the vacuum projector $\pi = |0\rangle\langle 0|$. This implies that $[\phi, \phi^\dagger] = \pi$. The algebra ϕ and the $\phi^\dagger$ is called the *Cuntz* algebra [11].

The operators $\phi, \phi^\dagger$ with the number operator $\widehat{N}$

$$\widehat{N}|n\rangle = n|n\rangle$$

give the *phase algebra* [14] characterized by the commutation relations

$$[\phi, \phi^\dagger] = \pi\,, \qquad [\widehat{N}, \phi] = -\phi\,, \qquad [\widehat{N}, \phi^\dagger] = \phi^\dagger\,. \tag{6}$$

The matrix representation of operators ϕ and $\phi^\dagger$

$$\phi^\dagger \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & \ldots \\ 1 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 1 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \phi \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & 0 & \ldots & \ldots \\ 0 & 0 & 0 & 1 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ddots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7}$$

were introduced and studied by V. Fock in [15].

§3. Ordered lattice paths

3.1. Dyck paths. A Dyck path is a lattice path of $2N$ in the first quadrant $\mathbb{N}^2$ with up steps $U = (1,1)$, rises, and down steps $D = (1,-1)$, falls, that starts at the origin $(0,0)$, ends at $(2N,0)$, and never passes below the x-axis (see Fig. 1).

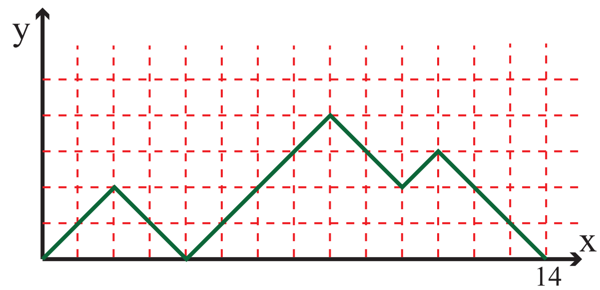

Figure 1. Example of a Dyck path in 14 steps corresponding to the word $UUDDUUUUDDUDDD$.

Let us associate the Dyck steps to the following operators

$$\phi^\dagger \longrightarrow (1,1) \equiv U\,; \qquad \phi \longrightarrow (1,-1) \equiv D\,. \tag{8}$$

Then the number of Dyck paths in $2N$ steps may be written in the form

$$\mathcal{D}_{2N} \equiv \langle 0|\widehat{H}_{\mathcal{D}}^{2N}|0\rangle\,, \quad \mathcal{D}_0 = 1\,, \tag{9}$$

where

$$\widehat{H}_{\mathcal{D}} = \phi + \phi^\dagger\,. \tag{10}$$

Applying the commutation relation

$$\phi \widehat{H}_{\mathcal{D}} = \widehat{H}_{\mathcal{D}}\phi + \pi$$

and the property that $\langle 0|\phi^\dagger = 0$, we can derive the expression

$$\begin{aligned}\langle 0|\widehat{H}_{\mathcal{D}}^{2(N+1)}|0\rangle &= \langle 0|\phi\widehat{H}_{\mathcal{D}}\widehat{H}_{\mathcal{D}}^{2N}|0\rangle = \langle 0|\widehat{H}_{\mathcal{D}}\phi\widehat{H}_{\mathcal{D}}^{2N}|0\rangle + \langle 0|\widehat{H}_{\mathcal{D}}^{2N}|0\rangle \\ &= \langle 0|\widehat{H}_{\mathcal{D}}^{3}\phi\widehat{H}_{\mathcal{D}}^{2(N-2)}|0\rangle + \langle 0|\widehat{H}_{\mathcal{D}}^{2}\pi\widehat{H}_{\mathcal{D}}^{2(N-2)}|0\rangle + \langle 0|\widehat{H}_{\mathcal{D}}^{2N}|0\rangle\,. \end{aligned} \tag{11}$$

The condition $\langle 0|\widehat{H}_{\mathcal{D}}^{2n+1}|0\rangle = 0\,, n \geqslant 0$ leads then to the quadratic recurrence on number of Dyck paths for $n \geqslant 1$:

$$\begin{aligned} \langle 0|\widehat{H}_{\mathcal{D}}^{2(N+1)}|0\rangle &= \sum_{k=0}^{N} \langle 0|\widehat{H}_{\mathcal{D}}^{2k}|0\rangle \langle 0|\widehat{H}_{\mathcal{D}}^{2(N-k)}|0\rangle\,, \\ \mathcal{D}_{2(N+1)} &= \sum_{k=0}^{N} \mathcal{D}_{2k}\mathcal{D}_{2(N-k)}\,. \end{aligned} \tag{12}$$

The Catalan numbers are given by the explicit formula $C_N = \frac{1}{N+1}\binom{2N}{N}$ and satisfy the numeric recurrence

$$C_{N+1} = \sum_{k=0}^{N} C_k C_{N-k}\,, \quad C_0 = 1\,. \tag{13}$$

From (12) and (13) it follows that

$$\mathcal{D}_{2N} = C_N\,. \tag{14}$$

The operator (10) was introduced in [16] in connection with the study of the phase problem in quantum mechanics and is known as the cosine operator. Consider the eigenvalue equation

$$\widehat{H}_{\mathcal{D}}|\cos\theta\rangle = E|\cos\theta\rangle\,. \tag{15}$$

The coefficients in decomposition

$$|\cos\theta\rangle = \sum_{n=0}^{\infty} S_n|n\rangle \tag{16}$$

satisfy the difference relations

$$\begin{aligned} ES_0 =& S_1\,, \\ ES_{n+1} =& S_n + S_{n+2}\,. \end{aligned} \tag{17}$$

The solution of these relations gives the answer to eigenvalue problem of the equation (15):

$$\begin{aligned} E =& 2\cos\theta\,, \\ |\cos\theta\rangle =& \sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty} \sin\big((n+1)\theta\big)|n\rangle\,. \end{aligned} \tag{18}$$

The eigenstates satisfy the conditions

$$\langle\cos\theta|\cos\theta'\rangle = \delta(\theta-\theta')\,,$$
$$\int_0^\pi d\theta|\cos\theta\rangle\langle\cos\theta| = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1\,. \tag{19}$$

This mentioned approach gives the analytical proof that $\mathcal{D}_{2N} = C_N$. Really

$$\begin{aligned}\mathcal{D}_{2N} = \langle 0|\widehat{H}_{\mathcal{D}}^{2N}|0\rangle &= \frac{2^{2N+1}}{\pi}\int_0^\pi \sin^2(\theta)\cos^{2N}(\theta)d\theta \\ &= \frac{2^{2N+1}}{\pi}\int_0^\pi \cos^{2N}(\theta)d\theta - \frac{2^{2N+1}}{\pi}\int_0^\pi \cos^{2(N+1)}(\theta)d\theta \\ &= 2\binom{2N}{N} - \frac{1}{2}\binom{2N+2}{N+1} = \frac{1}{N+1}\binom{2N}{N} = C_N\,.\end{aligned} \tag{20}$$

The generating function of the Dyck paths is defined as

$$\mathcal{D}(z) = \langle 0|\frac{1}{1-z\widehat{H}_{\mathcal{D}}}|0\rangle = \sum_{n=0}^{\infty} z^{2n}\langle 0|\widehat{H}_{\mathcal{D}}^{2n}|0\rangle = \sum_{n=0}^{\infty} z^{2n}C_n\,. \tag{21}$$

It is straightforward to derive that

$$\mathcal{D}(z) = \frac{2}{\pi}\int_0^\pi \frac{\sin^2\theta}{1-2z\cos\theta}d\theta = \frac{1-\sqrt{1-4z^2}}{2z^2}\,. \tag{22}$$

The integral in this expression was calculated in [17].

3.2. Łukasiewicz paths. A Łukasiewicz path of length N is a lattice path in $\mathbb{N}^2$ starting at the origin $(0,0)$, ending on the x-axis at $(N,0)$, which are made using steps into the set $D=(1,-1)$, $F=(1,0)$ and $U_k=(1,k)$ for $k\geqslant 1$ (see Fig. 2).

The introduced set of steps correspond to the following operators:

$$(\phi^\dagger)^k \longrightarrow (1,k)\equiv U_k\,;\qquad \phi\longrightarrow(1,-1)\equiv D\,;\qquad I\longrightarrow(1,0)\equiv F\,. \tag{23}$$

The number of Łukasiewicz paths in N steps is given by

$$\mathcal{L}_N \equiv \langle 0|\widehat{H}_{\mathcal{L}}^N|0\rangle\,,\quad \mathcal{L}_0 = 1\,, \tag{24}$$

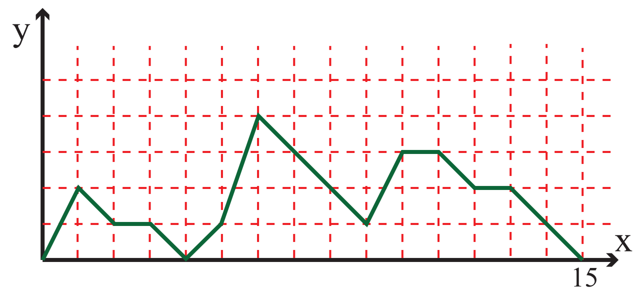

Figure 2. A Łukasiewicz path in 15 steps corresponding to the word $U_2DFDU_1U_3DDDU_2FDFDD$.

where

$$\widehat{H}_{\mathcal{L}} = \phi + \sum_{k\geqslant 0} \left(\phi^\dagger\right)^k \equiv \phi + \Phi^\dagger\,. \tag{25}$$

The commutation relations

$$\phi\Phi^\dagger = \Phi^\dagger\phi + \Phi^\dagger\pi, \quad \phi\widehat{H}_{\mathcal{L}} = \widehat{H}_{\mathcal{L}}\phi + \Phi^\dagger\pi = \widehat{H}_{\mathcal{L}}\left(\phi+\pi\right), \tag{26}$$

allows us to derive the following recurrence relation

$$\begin{aligned}\langle 0|\widehat{H}_{\mathcal{L}}^{N+1}|0\rangle &= \langle 0|(1+\phi)\widehat{H}_{\mathcal{L}}^{N}|0\rangle \\ &= \langle 0|\widehat{H}_{\mathcal{L}}^{N}|0\rangle + \langle 0|\widehat{H}_{\mathcal{L}}\phi\widehat{H}_{\mathcal{L}}^{N-1}|0\rangle + \langle 0|\widehat{H}_{\mathcal{L}}\pi\widehat{H}_{\mathcal{L}}^{N-1}|0\rangle \\ &= \langle 0|\widehat{H}_{\mathcal{L}}^{N}|0\rangle + \langle 0|\widehat{H}_{\mathcal{L}}\pi\widehat{H}_{\mathcal{L}}^{N-1}|0\rangle + \langle 0|\widehat{H}_{\mathcal{L}}^{2}\pi\widehat{H}_{\mathcal{L}}^{N-2}|0\rangle + \langle 0|\widehat{H}_{\mathcal{L}}^{2}\phi\widehat{H}_{\mathcal{L}}^{N-2}|0\rangle.\end{aligned} \tag{27}$$

This expression yields the following quadratic recurrence

$$\begin{aligned}\langle 0|\widehat{H}_{\mathcal{L}}^{N+1}|0\rangle &= \sum_{k=0}^{N} \langle 0|\widehat{H}_{\mathcal{L}}^{k}|0\rangle\langle 0|\widehat{H}_{\mathcal{L}}^{N-k}|0\rangle\,, \\ \mathcal{L}_{N+1} &= \sum_{k=0}^{N} \mathcal{L}_k\mathcal{L}_{N-k}\,.\end{aligned} \tag{28}$$

Comparing the obtained recurrence with the (13) we can derive that the number of Łukasiewicz paths in N steps is given by the Catalan number:

$$\mathcal{L}_N = C_N\,. \tag{29}$$

The generating function of the Łukasiewicz paths is

$$\mathcal{L}(z)=\langle 0|\frac{1}{1-z\widehat{H}_{\mathcal{L}}}|0\rangle=\sum_{n=0}^{\infty}z^n\langle 0|\widehat{H}_{\mathcal{L}}^n|0\rangle=\sum_{n=0}^{\infty}z^n\mathcal{L}_n=\frac{1-\sqrt{1-4z}}{2z}. \quad (30)$$

3.3. Motzkin paths. A Motzkin path is a lattice path of $2N$ in the first quadrant $\mathbb{N}^2$ with up steps $U=(1,1)$, down steps $D=(1,-1)$, and forward steps $F=(1,0)$ that starts at the origin $(0,0)$, ends at $(2N,0)$, and never passes below the x-axis (see Fig. 3).

The set of Motzkin steps are given by

$$(\phi^\dagger)\longrightarrow(1,1)\equiv U\,;\qquad \phi\longrightarrow(1,-1)\equiv D\,;\qquad I\longrightarrow(1,0)\equiv F\,. \quad (31)$$

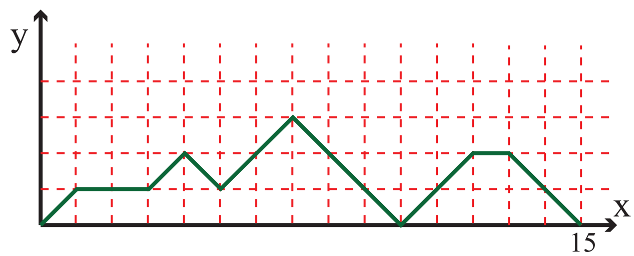

Figure 3. A Motzkin path in 15 steps corresponding to the word UFFUDUUDDDUUFDD.

The number of Motzkin paths in N steps is

$$\mathcal{M}_N=\langle 0|\widehat{H}_{\mathcal{M}}^N|0\rangle\,,\quad \mathcal{M}_0=1\,, \quad (32)$$

where

$$\widehat{H}_{\mathcal{M}}=\phi+\phi^\dagger+I\equiv\widehat{H}_{\mathcal{D}}+I\,. \quad (33)$$

Using the above definition we can write

$$\langle 0|\widehat{H}_{\mathcal{M}}^N|0\rangle=\sum_{k=0}^{[N/2]}\binom{N}{2k}\langle 0|\widehat{H}_{\mathcal{D}}^k|0\rangle\,, \quad (34)$$

and hence the number of Motzkin paths in N steps is equal to

$$\mathcal{M}_N=\sum_{k=0}^{[N/2]}\binom{N}{2k}C_k\,. \quad (35)$$

The eigenvalue difference equations of the operator (33) are similar to (17):

$$\begin{aligned}(E-1)S_0 =& S_1\,,\\ (E-1)S_{n+1} =& S_n + S_{n+2}\,.\end{aligned} \tag{36}$$

With these eigenvalue equations the generating function of the Motzkin paths is

$$\mathcal{M}(z) = \frac{2}{\pi}\int\limits_0^{\pi} \frac{\sin^2\theta}{(1-z)-2z\cos\theta}d\theta = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}\,. \tag{37}$$

§4. Master field

The operator construction of the master field for the one-matrix model was constructed in [11,13]. This operator is expandable in a basis of Cuntz operators ϕ and $\phi^\dagger$:

$$\widehat{M}(\phi,\phi^\dagger) = \phi + \sum_{k=0}^{\infty} M_k(\phi^\dagger)^k \tag{38}$$

with an appropriate choice of the scalar coefficients M_k.

Applying the representation of operators (7) one obtains the matrix form of $\widehat{M}(\phi,\phi^\dagger)$:

$$\widehat{M}(\phi,\phi^\dagger) = \begin{pmatrix} M_0 & 1 & 0 & 0 & \dots & \dots \\ M_1 & M_0 & 1 & 0 & \dots & \dots \\ M_2 & M_1 & M_0 & 1 & \dots & \dots \\ M_3 & M_2 & M_1 & M_0 & \ddots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{39}$$

The operator (38) may be considered as the generator of the weighted Łukasiewicz paths (25) – the step $(\phi^\dagger)^k$ carries the weight M_k ($k \geqslant 0$). One can compute Voiculescu polynomials [13] of the N-th order enumerating the weighted Łukasiewicz paths in N steps. The first such polynomials are of the form (See Fig. 4)

$$
\begin{aligned}
\langle 0|\widehat{M}(\phi,\phi^\dagger)|0\rangle &= M_0\,,\\
\langle 0|\widehat{M}^2(\phi,\phi^\dagger)|0\rangle &= M_0^2 + M_1\,,\\
\langle 0|\widehat{M}^3(\phi,\phi^\dagger)|0\rangle &= M_0^3 + 3M_0M_1 + M_2\,,\\
\langle 0|\widehat{M}^4(\phi,\phi^\dagger)|0\rangle &= M_0^4 + 6M_0^2M_1 + 2M_1^2 + 4M_2M_0 + M_3\,. \qquad (40)
\end{aligned}
$$

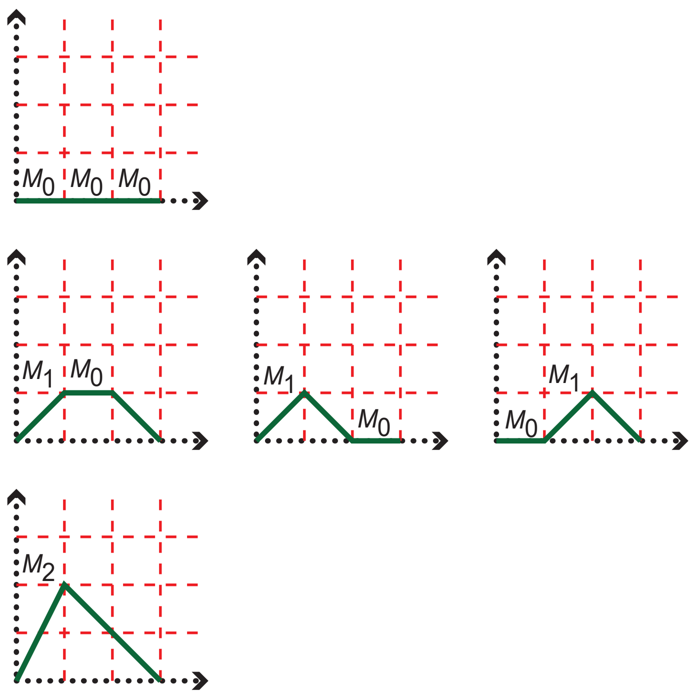

Figure 4. Łukasiewicz paths and the graphical representation of Voiculescu polynomial: $\langle 0|\widehat{M}^3(\phi,\phi^\dagger)|0\rangle = M_0^3 + 3M_0M_1 + M_2$.

The explicit operator of the Gaussian master field [11,12]

$$\widehat{M}_G(\phi,\phi^\dagger) = \phi + \alpha\phi^\dagger\,. \qquad (41)$$

can be made explicitly Hermitian by a similarity transformation:

$$S\widehat{M}_G(\phi, \phi^\dagger)S^{-1} = \sqrt{\alpha}(\phi + \phi^\dagger) = \sqrt{\alpha}\widehat{H}_D\,, \qquad (42)$$

using $S = \exp(-\frac{\widehat{N}}{2}\log\alpha)$, where $\widehat{N}$ is the the number operator (6). The coefficients

$$\langle 0|\widehat{M}_G^{2N}(\phi, \phi^\dagger)|0\rangle = \alpha^N \langle 0|\widehat{H}_D^{2N}|0\rangle = \alpha^N C_N\,, \qquad (43)$$

were C_N are the Catalan numbers. The Gaussian generating function is equal to generating function of the Dyck paths (22).

The "Motzkin" master field may be defined as

$$\widehat{M}_M(\phi, \phi^\dagger) = \phi + \alpha\phi^\dagger + \sqrt{\alpha}I\,, \qquad (44)$$

which is operator (33) after the similarity transformation

$$S\widehat{M}_M(\phi, \phi^\dagger)S^{-1} = \sqrt{\alpha}\widehat{H}_M\,,$$

and hence

$$\langle 0|\widehat{M}_M^N(\phi, \phi^\dagger)|0\rangle = \alpha^{\frac{1}{2}}\mathcal{M}_N\,,$$

where $\mathcal{M}_N$ are the Motzkin numbers (35).

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St.Petersburg Department
of V. A. Steklov Mathematical Institute, RAS
Fontanka 27, St.-Petersburg 191023, Russia

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