Записки научных
семинаров ПОМИ
Том 531, 2024 г.

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SUBGROUPS GENERATED BY A PAIR OF 2-TORI IN $\mathrm{GL}(4,K)$. I

Abstract. This paper is the third one in the series of the works dedicated to the geometry of 2-tori, i.e. subgroups conjugate to the diagonal subgroup of the form $\{\mathrm{diag}(\varepsilon,\varepsilon,1,\dots,1),\ \varepsilon\in K^*\}$, in the general linear group $\mathrm{GL}(n,K)$ over the field K. In the first one we proved a reduction theorem establishing that a pair of 2-tori is conjugate to such a pair in $\mathrm{GL}(6,K)$, and classified such pairs that cannot be embedded in $\mathrm{GL}(5,K)$. In the second we describe the orbits and spans of 2-tori in $\mathrm{GL}(5,K)$, that cannot be embedded in $\mathrm{GL}(4,K)$. Here we consider the most difficult case of $\mathrm{GL}(4,K)$ and classify the orbits of $\mathrm{GL}(4,K)$ acting by simultaneous conjugation on pairs of 2-tori.

In memory of N. A. Vavilov

Introduction

In the present paper we move on to the last step of the description of orbits and spans for pairs of 2-tori in $\mathrm{GL}(n,K)$. Namely, we classify the orbits of $\mathrm{GL}(4,K)$ acting by simultaneous conjugation on such pairs. Because of a large amount of calculations the description of spans for pairs of 2-tori will be considered in the next paper.

Recall that 2-tori in $\mathrm{GL}(n,K)$ are the subgroups conjugate to the diagonal subgroup of the following form

$$\{\mathrm{diag}(\varepsilon,\varepsilon,1,\dots,1),\ \varepsilon\in K^*\}.$$

From the general theory viewpoint 2-tori are microweight tori corresponding to the fundamental weight $\overline{\omega}_2$ in the extended Chevalley group of type A_{n-1}.

This paper is the third in the series of works dedicated to the geometry of microweight tori. In the previous paper [4] we proved a reduction theorem establishing that a pair of 2-tori is conjugate to such a pair in $\mathrm{GL}(6,K)$ and classified such pairs that cannot be embedded in $\mathrm{GL}(5,K)$. In the

Key words and phrases: general linear group, unipotent root subgroups, semisimple root subgroups, m-tori, diagonal subgroup.

paper and [9] we described the orbits and spans of 2-tori in $\mathrm{GL}(5, K)$, that cannot be embedded in $\mathrm{GL}(4, K)$. Note also that the orbits of 2-tori in $\mathrm{GL}(3, K)$ coincides with the orbits 1-tori and are described in [6] (see also Lemma 1 [4]).

Thus it remains to study the most difficult case of a pair 2-tori when it is embedded in $\mathrm{GL}(4, K)$ and cannot be embedded in $\mathrm{GL}(3, K)$.

Since the present paper is a sequel of the works listed before we do not discuss here context of this problem and related question. The reader can find all of it and many references in the survey [8] and in the detailed introduction of [9]. Here we only mention the paper [1] and surveys [5] and [7].

The idea of this cycle of papers was suggested by N. A. Vavilov more than ten years ago. The starting point of this research is his work [6]. The next three papers were written by N. A. Vavilov jointly with the first author. To our deep regret prof. N. A. Vavilov can not finish this project. The authors express their most sincere thanks to him for setting the problem and numerous inspiring discussions.

§1 Notation

All our notations are the same as in [4] and [9], but for reader's convenience we cite them briefly here.

Let K be a field and $K^* = K\backslash\{0\}$ be the multiplicative group of it. Further, $G = \mathrm{GL}(n, K)$ is the general linear group of degree n over K. By $D = \mathrm{D}(n, K)$ we denote the subgroup of diagonal matrices in G, and $N = N(n, K)$ denotes the subgroup of monomial matrices in G.

The quotient group N/D is isomorphic to S_n, the symmetric group on n letters. Denote by $W = W_n$ the group of permutation matrices in G. We identify S_n and W_n via the isomorphism $\pi \mapsto w_\pi$, where w_π is the matrix whose entry in the position (i, j) is $\delta_{i,\pi j}$.

Let $V = K^n$ be the *right* vector space of columns of height n over K. Usually we identify a matrix $g \in G$ with the corresponding linear map of the space K^n. Here g acts *on the left*. To stress that we are using this geometric viewpoint, in such cases we call elements of G *transformations*.

By $e_1, \dots, e_n$ we denote the standard base of K^n. Here e_i is the column, whose i-th component equals 1, whereas all other components are equal to 0. The dual space $V^* = {}^nK$ is *left* vector space of rows of length n. By $f_1, \dots, f_n$ we denote the standard base of ${}^nK$. It is dual to $e_1, \dots, e_n$ with respect to the standard pairing, $V^* \times V \longrightarrow K$.

Denote by e_{ij} a standard matrix unit, i.e. the matrix whose entry in the position (i,j) is 1 and all the remaining entries are zeroes. Next, $x_{ij}(\xi) = e + \xi e_{ij}$ for $\xi \in K$ and $1 \leqslant i \neq j \leqslant n$ denotes *elementary transvection*. For given $i \neq j$ we consider the corresponding unipotent *root subgroup* $X_{ij} = \{x_{ij}(\xi), \xi \in K\}$. The subgroup $E(n,K)$ of G, generated by all X_{ij}, $1 \leqslant i \neq j \leqslant n$, is called the *elementary* subgroup of G. In case of the field, it coincides with the special linear group $\mathrm{SL}(n,K)$.

Similarly, by $d_i(\varepsilon) = e + (\varepsilon - 1)e_{ii}$ we denote an *elementary pseudo-reflection*. For a given i we consider the corresponding 1**-torus**

$$Q_i = \{d_i(\varepsilon),\ \varepsilon \in K^*\}.$$

Clearly, $\mathrm{GL}(n,K)$ is generated by $E(n,K)$ and Q_1.

The largest subspace $W \leqslant V$ such that $g|_W = \mathrm{id}$ is called the *axis* of g. Similarly, the subspace $U = \{gv - v \mid v \in K^n\}$ is called the *centre* of g. Clearly, $\dim U = m$ and $\dim W = n - m$. Many useful properties of it can be found in [2].

The most important individual elements of $\mathrm{GL}(n,K)$ are the 1-dimensional tranformations, which plays the main role in studying linear groups. The general form of an 1-dimensional transformation is

$$x_{vu}(\xi) = e + v\xi u, \qquad v \in K^n, \quad u \in {}^nK, \quad \xi \in K.$$

In this case the centre of $x_{vu}(\xi)$ is the space generated by v, whereas its axis is the hyperplane orthogonal to u. Let $uv = \delta$. If $\delta = 0$, the transformation $x_{vu}(\xi)$ is a *transvection* for all $\xi \in K$. If $\delta \neq 0$, $x_{vu}(\xi)$ is a *pseudo-reflection* for all $\xi \in K \setminus \{-1\}$.

As we noted, the orbits and spans by a pair of 1-tori are described in [6]. This description is also reproduced in paper [4].

The elementary 2-torus $Q = Q_{U_0,W_0} = \{\mathrm{diag}(\varepsilon, \varepsilon, 1, \ldots, 1), \varepsilon \in K^*\}$ is defined by the subspaces $U_0 = \langle e_1, e_2 \rangle$ and $W_0 = \langle f_1, f_2 \rangle$. It means, that elements of it are

$$d_0(\varepsilon) = e + e_1(\varepsilon - 1)f_1 + e_2(\varepsilon - 1)f_2, \qquad \varepsilon \in K^*.$$

It is clear that

$$gQ_{UW}g^{-1} = Q_{gU,Wg^{-1}}, \qquad g \in \mathrm{GL}(n,K).$$

Therefore any 2-torus (see [4]) is conjugated to the elementary 2-torus Q. The elements of an arbitrary 2-torus are the elements of the following form

$$d(\varepsilon) = e + u_1(\varepsilon - 1)v_1 + u_2(\varepsilon - 1)v_2, \qquad \varepsilon \in K^*,$$

where $u_i = ge_i$, $v_i = f_i g^{-1}$, $1 \leqslant i \leqslant 2$, for some matrix $g \in \mathrm{GL}(n,K)$. Thus each 2-torus is completely determined by the subspaces $U = \langle u_1, u_2\rangle$ and $W = \langle v_1, v_2\rangle$.

The subspace U is precisely the *centre* of Q_{UW}, in the sense of being the centre of every $d(\varepsilon) \in Q_{UW}$, $\varepsilon \neq 1$. Similarly, the subspace $W^{\perp}$ orthogonal to $W \leqslant {}^nK$ with respect to the canonical pairing ${}^nK \times K^n \longrightarrow K$, is precisely the *axis* of Q_{UW}, in the above sense. Oftentimes we loosely refer to W itself as the axis of Q_{UW}.

Consider a pair of 2-tori X and Y with centers U_1 and U_2 and with axes W_1 and W_2, respictively. In paper [4] we introduce the following invariants for a pair of m-tori.

- $r = r(X,Y) = \dim(U_1 + U_2)$,
- $s = s(X,Y) = \dim(W_1 + W_2)$.
- $p = p(X,Y) = \dim(U_1 \cap W_2^{\perp})$,
- $q = q(X,Y) = \dim(U_2 \cap W_1^{\perp})$.
- $t = t(X,Y) = \max\Big(\dim\big((U_1+U_2)\cap(W_1+W_2)^{\perp}\big), \dim\big({}^{\perp}(U_1+U_2)\cap (W_1+W_2)\big)\Big)$.

Clearly that in our case $2 \leqslant r, s \leqslant 4$, $0 \leqslant p, q \leqslant 2$ and $0 \leqslant t \leqslant 2$.

In [4] we proved the reduction theorem for the pairs of m-tori. It follows from it that any pair of 2-tori (X,Y) can be embedded in $\mathrm{GL}(6,K)$ by simultaneous conjugation. We call an orbit of a pair of 2-tori (X,Y) the orbit in $\mathrm{GL}(n,K)$, if the pair X, Y is embedded in $\mathrm{GL}(n,K)$ by simultaneous conjugation and it can not be embedded in $\mathrm{GL}(n-1,K)$. It follows from the reduction theorem that n can take values 3, 4, 5 or 6.

Taking into account the paper [9] it remains to consider the orbits in $\mathrm{GL}(4,K)$. Indeed this is the most difficult and general case. In this paper we classify the orbits for a pair of 2-tori in $\mathrm{GL}(4,K)$. The next paper will be dedicated to calculation of their spans.

§2. The bases of orbits

Our aim is to describe orbits of pairs of 2-tori in $\mathrm{GL}(4,K)$ under simultaneous conjugation

$$(X,Y) \mapsto (gXg^{-1}, gYg^{-1}), \qquad g \in G.$$

Let X, Y be 2-tori in $\mathrm{GL}(4,K)$. Denote by U_1, U_2 and W_1, W_2 their centers and axes, respectively. We fix some bases in these subspaces

$$U_1 = \langle u_1, u_2\rangle, U_2 = \langle u_3, u_4\rangle.$$

$$W_1 = \langle w_1, w_2\rangle, W_2 = \langle w_3, w_4\rangle.$$

Let

$$U_1 + U_2 \leqslant \langle e_1, e_2, e_3, e_4\rangle,$$

$$W_1 + W_2 \leqslant \langle f_1, f_2, f_3, f_4\rangle.$$

Lemma 1. *Let X and Y be 2-tori in* $\mathrm{GL}(4,K)$. *Assume that at least one of r and s is 2, then the orbit (X,Y) is determined by the following bases*

For $r = 2$, $s = 2$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1,\\ u_2 &= e_2, w_2 = f_2,\\ u_3 &= e_1, w_3 = f_1,\\ u_4 &= e_2, w_4 = f_2. \end{aligned} \tag{r2s2a}$$

For $r = 2$, $s = 3$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + f_3,\\ u_2 &= e_2, w_2 = f_2 + \lambda f_3,\\ u_3 &= e_1, w_3 = f_1,\\ u_4 &= e_2, w_4 = f_2, \end{aligned} \tag{r2s3a}$$

where $\lambda = 0, 1$.

For $r = 2$, $s = 4$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + f_3,\\ u_2 &= e_2, w_2 = f_2 + f_4,\\ u_3 &= e_1, w_3 = f_1,\\ u_4 &= e_2, w_4 = f_2. \end{aligned} \tag{r2s4a}$$

Proof. Assume that $r = 2$, there are three possible values of s. That is, either $s = 2$, or $s = 3$, or $s = 4$.

We can immediately assume that $u_1 = e_1$, $u_2 = e_2$, then $u_3 = e_1$, $u_4 = e_2$. According to Lemma 2 in [4], if at least one of r or s is 2, then

$p = q = 0$. Conjugating by X_{13}, X_{14}, X_{23}, X_{24}, we can further obtain that $w_3 = f_1$, $w_4 = f_2$. Thus, we assume from the beginning that

$$w_1 = f_1 + \alpha_3 f_3 + \alpha_4 f_4,$$
$$w_2 = f_2 + \beta_3 f_3 + \beta_4 f_4.$$

• If $s = 2$, then $\alpha_3 = \alpha_4 = \beta_3 = \beta_4 = 0$. We have the base (r2s2a).

• If $s = 3$, then the vectors (α_3, α_4) and (β_3, β_4) are linearly dependent and non-zero, it follows that

$$\begin{vmatrix} \alpha_3 & \alpha_4 \\ \beta_3 & \beta_4 \end{vmatrix} = 0.$$

Since at least one of α_i, β_i, $i = 3, 4$ is non-zero, due to conjugation by suitable element of the Weyl group, we can consider that $\alpha_3 \neq 0$. And conjugation by $d_3(\alpha_3)$ leads to $\alpha_3 = 1$. As a result of conjugation by the element from X_{34}, we get $\alpha_4 = 0$, $\beta_4 = 0$. Therefore, we get the following base

$$u_1 = e_1, w_1 = f_1 + f_3,$$
$$u_2 = e_2, w_2 = f_2 + \beta f_3,$$
$$u_3 = e_1, w_3 = f_1,$$
$$u_4 = e_2, w_4 = f_2,$$

where $\beta \in K$.

Suppose that $\beta \neq 0$. Due to conjugation by $d_1(\beta)d_3(\beta)$, we may consider that $\beta = 1$. Finally, we have the base (r2s3a).

• Let $s = 4$, then the vectors (α_3, α_4) and (β_3, β_4) are linearly independent and non-zero. Since at least one of α_i, β_i, $i = 3, 4$ is non-zero, conjugating by permutation matrix ω_{34}, we can suppose that $\beta_4 \neq 0$. And conjugation by $d_4(\beta_4)$ leads to $\beta_4 = 1$. As a result of conjugation by the element from X_{43}, we get $\beta_3 = 0$. Therefore α_3 can not be equal to zero. Acting by the elements of $d_3(\alpha_3)$, $x_{34}(\alpha_4)$, we can obtain that $\alpha_3 = 1$, $\alpha_4 = 0$. Finally, we have the base (r2s4a).

It remains to note that if $s = 2$ we just need to interchange the rows and columns of the bases (r2s3a) and (r2s4a). So we have analogous bases (r3s2) and (r4s2), respectively. □

Lemma 2. *Let X and Y be 2-tori in* $\mathrm{GL}(4, K)$. *Assume that $p = 0$, at least one of r and s is 3 and $r, s \geqslant 3$, then the orbit (X, Y) is determined by the following bases*

For $r = 3$, $s = 3$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + f_4, \\ u_2 &= e_2, w_2 = f_2 + \beta f_4, \\ u_3 &= e_1 + e_3, w_3 = f_1, \\ u_4 &= e_2 + \lambda e_3, w_4 = f_2, \end{aligned} \tag{r3s3a}$$

where $\beta \in K$, $\lambda = 0, 1$.

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + \alpha f_3, \\ u_2 &= e_2, w_2 = f_2 + \beta f_3, \\ u_3 &= e_1 + e_3, w_3 = f_1, \\ u_4 &= e_2 + \lambda e_3, w_4 = f_2, \end{aligned} \tag{r3s3b}$$

where $\lambda = 0, 1$, $\alpha \in K^*$, $\beta \in K$.

For $r = 3$, $s = 4$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + \alpha f_3 + f_4, \\ u_2 &= e_2, w_2 = f_2 + \beta f_3, \\ u_3 &= e_1 + e_3, w_3 = f_1, \\ u_4 &= e_2 + \lambda e_3, w_4 = f_2, \end{aligned} \tag{r3s4a}$$

where $\lambda = 0, 1$, $\alpha \in K$, $\beta \in K^*$.

For $r = 4$, $s = 3$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + f_3, \\ u_2 &= e_2, w_2 = f_2 + \lambda f_3, \\ u_3 &= e_1 + \alpha e_3 + e_4, w_3 = f_1, \\ u_4 &= e_2 + \beta e_3, w_4 = f_2, \end{aligned} \tag{r4s3a}$$

where $\lambda = 0, 1$, $\alpha \in K$, $\beta \in K^*$.

Proof. Suppose that $r = 3$. Since $p = 0$ we can assume that $u_1 = e_1$, $u_2 = e_2$, $w_3 = f_1$, $w_4 = f_2$. As in Lemma 1 in case of $s = 3$ we get

$$u_3 = e_1 + e_3, u_4 = e_2 + \lambda e_3,$$

where $\lambda = 0$, 1.

Next, we have

$$\begin{aligned} w_1 &= f_1 + \alpha_3 f_3 + \alpha_4 f_4, \\ w_2 &= f_2 + \beta_3 f_3 + \beta_4 f_4. \end{aligned}$$

• Let $s = 3$. If $\alpha_3 = \beta_3 = 0$, we can consider that $\alpha_4 \neq 0$. Conjugating by $d_4(\alpha_4)$, we even have that $\alpha_4 = 1$. Put $\beta = \beta_4$, we have the base (r3s3a).

Suppose $\alpha_3 \neq 0$ and $\alpha_4 \neq 0$, then we act by the element from X_{34} and so that $\alpha_4 = 0$. Therefore we have $\beta_4 = 0$. Then we obtain the base

$$w_1 = f_1 + \alpha f_3,$$
$$w_2 = f_2 + \beta f_3,$$

where $\alpha \in K^*$, $\beta \in K$.

This is the base (r3s3b).

• Let $s = 4$. We may assume that $\beta_3 \neq 0$. Conjugation by suitable element from X_{34} leads to $\beta_4 = 0$, then α_4 can not be zero. Conjugating by diagonal matrix $d_4(\alpha_4)$, we get $\alpha_4 = 1$. Then we obtain the base (r3s4a).

• Suppose that $r = 4$, $s = 3$. We just need to interchange the rows and columns of the base (r3s4a), which gives the base (r4s3a). □

It is easily seen that in the case of the base (r3s3a) $W_1^{\perp} = \langle e_1 + \beta e_2 - e_4, e_3\rangle$ and $q = 0$. In the cases of the bases (r3s3b), (r3s4a) and (r4s3a) the intersection of U_2 and $W_1^{\perp}$ has dimension 1 only if $\alpha + \lambda\beta + 1 = 0$. In all other cases $q = 0$.

Lemma 3. *Let X and Y be 2-tori in* $\mathrm{GL}(4, K)$. *Assume that $p = 0$, $r = s = 4$, then the orbit (X, Y) is determined by the following bases*

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + \alpha f_3 + f_4,\\ u_2 &= e_2, w_2 = f_2 + \gamma f_3 + \delta f_4,\\ u_3 &= e_1 + e_3, w_3 = f_1,\\ u_4 &= e_2 + e_4, w_4 = f_2, \end{aligned} \tag{r4s4a}$$

where α, γ, $\delta \in K$ and $\gamma \neq \alpha\delta$.

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + \alpha f_3,\\ u_2 &= e_2, w_2 = f_2 + \delta f_4,\\ u_3 &= e_1 + e_3, w_3 = f_1,\\ u_4 &= e_2 + e_4, w_4 = f_2, \end{aligned} \tag{r4s4b}$$

where α, $\delta \in K^*$.

Proof. Since $p = 0$ we can assume at first that

$$u_1 = e_1, w_1 = f_1 + \alpha_3 f_3 + \alpha_4 f_4,$$
$$u_2 = e_2, w_2 = f_2 + \beta_3 f_3 + \beta_4 f_4,$$
$$u_3 = e_1 + \gamma_3 e_3 + \gamma_4 e_4, w_3 = f_1,$$
$$u_4 = e_2 + \delta_3 e_3 + \delta_4 e_4, w_4 = f_2.$$

It follows from $r = 4$ that

$$\begin{vmatrix} \gamma_3 & \gamma_4 \\ \delta_3 & \delta_4 \end{vmatrix} \neq 0.$$

Conjugating by permutation matrix ω_{34}, we can suppose that $\delta_4 \neq 0$. And conjugation by $d_4(\delta_4^{-1})$ leads to $\delta_4 = 1$. Then acting by the element from X_{34}, we get $\delta_3 = 0$, hence γ_3 can not be zero. After conjugating by $d_3(\gamma_3^{-1})$, $x_{43}(-\gamma_4)$, we obtain that $\gamma_3 = 1$, $\gamma_4 = 0$. Thus, we have

$$u_3 = e_1 + e_3, u_4 = e_2 + e_4.$$

Since $s = 4$, then the coefficients α_3, α_4, β_3, β_4 should satisfy the condition

$$\begin{vmatrix} \alpha_3 & \alpha_4 \\ \beta_3 & \beta_4 \end{vmatrix} \neq 0.$$

That is, $\alpha_3\beta_4 \neq \alpha_4\beta_3$. Thus, we obtain the following base

$$u_1 = e_1, w_1 = f_1 + \alpha f_3 + \beta f_4,$$
$$u_2 = e_2, w_2 = f_2 + \gamma f_3 + \delta f_4,$$
$$u_3 = e_1 + e_3, w_3 = f_1,$$
$$u_4 = e_2 + e_4, w_4 = f_2,$$

where α, β, γ, $\delta \in K$ and $\alpha\delta \neq \beta\gamma$.

If one of β and γ is different from zero, conjugating by suitable elements from the Weyl group, we may assume that $\beta \neq 0$. Due to conjugation by $d_2(\beta)d_4(\beta)$, we get $\beta = 1$. Finally, we have the base (r4s4a).

If $\beta = \gamma = 0$, then α, $\delta \neq 0$. Thus, we have the base (r4s4b). □

In the case of the base (r4s4a) one has $W_1^\perp = \langle e_1 + \delta e_2 - e_4, \alpha e_1 + \gamma e_2 - e_3\rangle$. Direct calculations show that $q = 1$ if $\gamma = (\alpha + 1)(\delta + 1)$. In all other cases $q = 0$. In the case (r4s4b) $W_1^\perp = \langle \alpha e_1 - e_3, \delta e_2 - e_4\rangle$ and we have $q = 2$ if $\alpha = \delta = -1$, $q = 1$ if $\alpha = -1$, $\delta \neq -1$ or $\delta = -1$, $\alpha \neq -1$ and $q = 0$ in other cases.

Further, by interchanging X and Y here, we additionally obtain the following orbits.

Lemma 4. *Let X and Y be 2-tori in* $\mathrm{GL}(4,K)$*. Assume that $q=0$ and $p\neq 0$. Then the orbit (X,Y) is determined by the following bases*

$$\begin{aligned} &u_1=e_1+e_3, w_1=f_1,\\ &u_2=e_2+\lambda e_3, w_2=f_2,\\ &u_3=e_1, w_3=f_1-(1+\lambda\beta)f_3,\\ &u_4=e_2, w_4=f_2+\beta f_3, \end{aligned} \tag{r3s3b'}$$

where $\beta\in K$, $\beta\neq -1$, $\lambda=0,1$.

$$\begin{aligned} &u_1=e_1+e_3, w_1=f_1,\\ &u_2=e_2+\lambda e_3, w_2=f_2,\\ &u_3=e_1, w_3=f_1-(1+\lambda\beta)f_3+f_4,\\ &u_4=e_2, w_4=f_2+\beta f_3, \end{aligned} \tag{r3s4a'}$$

where $\beta\in K^*$, $\lambda=0,1$.

$$\begin{aligned} &u_1=e_1-(1+\lambda\beta)e_3+e_4, w_1=f_1,\\ &u_2=e_2+\beta e_3, w_2=f_2,\\ &u_3=e_1, w_3=f_1+f_3,\\ &u_4=e_2, w_4=f_2+\lambda f_3, \end{aligned} \tag{r4s3a'}$$

where $\beta\in K^*$, $\lambda=0,1$.

$$\begin{aligned} &u_1=e_1+e_3, w_1=f_1,\\ &u_2=e_2+e_4, w_2=f_2,\\ &u_3=e_1, w_3=f_1+\alpha f_3+f_4,\\ &u_4=e_2, w_4=f_2+\gamma f_3+\delta f_4, \end{aligned} \tag{r4s4a'}$$

where α, γ, $\delta\in K$ and $\gamma\neq\alpha\delta$, $\gamma=(\alpha+1)(\delta+1)$.

$$\begin{aligned} &u_1=e_1+e_3, w_1=f_1,\\ &u_2=e_2+e_4, w_2=f_2,\\ &u_3=e_1, w_3=f_1+\alpha f_3,\\ &u_4=e_2, w_4=f_2+\delta f_4, \end{aligned} \tag{r4s4b'}$$

where $\alpha,\delta\in K^*$ and at least one of them is equal to -1.

Thus it remains to consider the pairs (X,Y) when the both invariants p and q are different from zero.

Lemma 5. *Let X and Y be 2-tori in* GL$(4, K)$. *Assume that $q = p = 1$, then the orbit (X, Y) is determined by the following bases*

For $r = 3$, $s = 4$, we have

$$\begin{aligned} &u_1 = e_1, w_1 = f_1 + \lambda_1 f_3 + f_4, \\ &u_2 = e_2, w_2 = f_2 + \lambda_1\lambda_2 f_3, \\ &u_3 = e_1 + \lambda_2 e_2, w_3 = f_1, \\ &u_4 = e_3, w_4 = f_3, \end{aligned} \tag{p1q1a}$$

where $\lambda_{1,2} = 0, 1$.

For $r = 4$, $s = 3$, we have

$$\begin{aligned} &u_1 = e_1, w_1 = f_1 + \lambda_1 f_2, \\ &u_2 = e_3, w_2 = f_3, \\ &u_3 = e_1 + \lambda_2 e_3 + e_4, w_3 = f_1, \\ &u_4 = e_2 + \lambda_1\lambda_2 e_3, w_4 = f_2, \end{aligned} \tag{p1q1b}$$

where $\lambda_{1,2} = 0, 1$.

For $r = 4$, $s = 4$, we have

$$\begin{aligned} &u_1 = e_1, w_1 = f_1 - f_4, \\ &u_2 = e_2, w_2 = f_2, \\ &u_3 = e_1 + e_4, w_3 = f_1, \\ &u_4 = e_3, w_4 = f_3. \end{aligned} \tag{p1q1c}$$

Proof. Let a non-zero vector u lie in the intersection $U_1 \cap W_2^{\perp}$, due to suitable conjugation we may assume that $u = e_2$. It follows that the coefficients at f_2 of w_3 and w_4 must be equal to zero. Then conjugating by the elements from X_{13}, X_{14}, X_{34}, we can further assume that $u_1 = e_1$, $u_2 = e_2$, $w_3 = f_1$, $w_4 = f_3$. Next, we have

$$u_3 = e_1 + \gamma_2 e_2 + \gamma_4 e_4,\ w_1 = f_1 + \alpha_3 f_3 + \alpha_4 f_4.$$

$$u_4 = \delta_2 e_2 + e_3 + \delta_4 e_4,\ w_2 = f_2 + \beta_3 f_3 + \beta_4 f_4.$$

• Suppose that $r = 3$, then $\gamma_4 = \delta_4 = 0$. Conjugating by $x_{23}(-\delta_2)$, we get that $\delta_2 = 0$, so

$$u_3 = e_1 + \gamma e_2,\ u_4 = e_3,$$

where $\gamma \in K$.

Next consider w_1 and w_2. Due to the conjugation by $x_{24}(\beta_4)$, we can assume that $\beta_4 = 0$. Since our orbit is in $\mathrm{GL}(4, K)$, then $\alpha_4 \neq 0$, thus

$$w_1 = f_1 + \alpha f_3 + \beta f_4, w_2 = f_2 + \delta f_3,$$

where α, $\delta \in K$, $\beta \in K^*$.

Next we have $q = 1$. Due to direct calculations it follows that $\delta = \alpha\gamma$. Hence we obtain the following base

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + \alpha f_3 + \beta f_4, \\ u_2 &= e_2, w_2 = f_2 + \alpha\gamma f_3, \\ u_3 &= e_1 + \gamma e_2, w_3 = f_1, \\ u_4 &= e_3, w_4 = f_3, \end{aligned}$$

where γ, $\alpha \in K$, $\beta \neq 0$.

If $\gamma = 0$, $\alpha = 0$, conjugation by $d_4(\beta)$ leads to $\beta = 1$.

If $\gamma = 0$, $\alpha \neq 0$, conjugation by $d_3(\alpha)d_4(\beta)$ leads to $\alpha = 1$, $\beta = 1$.

If $\gamma \neq 0$, $\alpha \neq 0$, conjugation by $d_1(\gamma)d_3(\alpha\gamma)d_4(\beta\gamma)$ leads to $\alpha = 1$, $\beta = 1$, $\gamma = 1$.

Finally, we obtain the base (p1q1a).

• For $r = 4$, $s = 3$, we interchange the rows and columns of the base (p1q1a), which give us the base (p1q1b).

• For $r = s = 4$, one of coefficients γ_4 and δ_4 is different from zero, we assume that $\gamma_4 \neq 0$. Conjugation by $d_4(\gamma_4^{-1})$ leads to $\gamma_4 = 1$, then conjugating by $x_{43}(-\delta_4)$, $x_{24}(-\gamma_2)$, $x_{23}(-\delta_2)$, we get $\delta_4 = 0$, $\gamma_2 = 0$, $\delta_2 = 0$. Thus, we have

$$u_3 = e_1 + e_4, u_4 = e_3.$$

Since $s = 4$, then at least one of α_4 and β_4 is different from zero, we assume $\alpha_4 \neq 0$. Due to the conjugation from X_{23}, we can assume that $\beta_3 = 0$. Thus, we obtain

$$w_1 = f_1 + \alpha f_3 + \beta f_4, w_2 = f_2 + \delta f_4,$$

where $\beta \in K^*$, α, $\delta \in K$.

We have $q = 1$. So direct calculations show that $\alpha = 0$, $\beta = -1$, $\delta = 0$. Thus we get the base (p1q1c). □

Lemma 6. *Let X and Y be 2-tori in* $\mathrm{GL}(4, K)$. *Suppose that one of p and q is 2, then $r = s = 4$, and the orbit (X, Y) is determined by the following bases*

For $p = 2$, $q = 0$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + \alpha f_3 + f_4,\\ u_2 &= e_2, w_2 = f_2 + \gamma f_3 + \delta f_4,\\ u_3 &= e_3, w_3 = f_3,\\ u_4 &= e_4, w_4 = f_4, \end{aligned} \tag{p2q0a}$$

where α, γ, $\delta \in K$ and $\alpha\delta \neq \gamma$.

For $p = 2$, $q = 1$, we have

$$\begin{aligned} u_1 &= e_1, w_1 = f_1 + \alpha f_3 + f_4,\\ u_2 &= e_2, w_2 = f_2 + \alpha\delta f_3 + \delta f_4,\\ u_3 &= e_3, w_3 = f_3,\\ u_4 &= e_4, w_4 = f_4, \end{aligned} \tag{p2q1a}$$

where α, $\delta \in K$.

For $p = 2$, $q = 2$, we have

$$\begin{aligned} u_1 &= e_1,\ w_1 = f_1,\\ u_2 &= e_2,\ w_2 = f_2,\\ u_3 &= e_3,\ w_3 = f_3,\\ u_4 &= e_4,\ w_4 = f_4. \end{aligned} \tag{p2q2a}$$

For $p = 0$, $q = 2$, we have

$$\begin{aligned} u_1 &= e_3, w_1 = f_3,\\ u_2 &= e_4, w_2 = f_4,\\ u_3 &= e_1, w_3 = f_1 + \alpha f_3 + f_4,\\ u_4 &= e_2, w_4 = f_2 + \gamma f_3 + \delta f_4, \end{aligned} \tag{p0q2a}$$

where α, γ, $\delta \in K$ and $\alpha\delta \neq \gamma$.

For $p = 1$, $q = 2$, we have

$$\begin{aligned} u_1 &= e_3, w_1 = f_3,\\ u_2 &= e_4, w_2 = f_4,\\ u_3 &= e_1, w_3 = f_1 + \alpha f_3 + f_4,\\ u_4 &= e_2, w_4 = f_2 + \alpha\delta f_3 + \delta f_4, \end{aligned} \tag{p1q2a}$$

where α, $\delta \in K$.

Proof. Suppose that $p = 2$, then $U_1 = W_2^{\perp}$, and we have $U_1 = \langle e_1, e_2\rangle$, $W_2 = \langle f_3, f_4\rangle$. One has

$$u_1 = e_1,\ w_1 = f_1 + \alpha_3 f_3 + \alpha_4 f_4,$$
$$u_2 = e_2,\ w_2 = f_2 + \beta_3 f_3 + \beta_4 f_4,$$
$$u_3 = e_3 + \gamma_1 e_1 + \gamma_2 e_2,\ w_3 = f_3,$$
$$u_4 = e_4 + \delta_1 e_1 + \delta_2 e_2,\ w_4 = f_4,$$

where γ_i, δ_i, α_j, $\beta_j \in K$, $i = 1, 2$, $j = 3, 4$. It is clear that $r = s = 4$. Conjugating by the elements from X_{13}, X_{23}, X_{14}, X_{24} we get $\gamma_i = \delta_i = 0$, $i = 1, 2$.

Next, consider the intersection of the subspaces $W_1^{\perp}$ and U_2. The vector lying in $W_1^{\perp}$ has the form $(-\alpha_3 t - \alpha_4 s, -\beta_3 t - \beta_4 s, t, s)^t$, where t, $s \in K$, the vector of U_2 has the form $(0, 0, k_1, k_2)^t$, where k_1,$k_2 \in K$. Thus the values of q depend on the dimension of the solution subspace of the linear system $Ax = 0$, $x = (t, s, 0, 0)^t$, where

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

and $\alpha = -\alpha_3$, $\beta = -\alpha_4$, $\gamma = -\beta_3$, $\delta = -\beta_4$.

• Suppose that $A = 0$, then $q = 2$. Thus, we have the base (p2q2a).

Now let one of the coefficients α, β, γ or δ be different from zero. Conjugating by permutation matrix, we may assume that $\beta \neq 0$. Conjugation by $d_4(\beta)$ leads to $\beta = 1$.

• Suppose that $\alpha\delta = \beta\gamma$, it means $\operatorname{rank} A = 1$. Then $q = 1$ and we have the base (p2q1a).

• Suppose $\alpha\delta \neq \beta\gamma$, $\operatorname{rank} A = 2$. Then the system has only zero solution, i.e. $q = 0$. We have the base (p2q0a).

Finally, by interchanging X and Y, we obtain the bases (p0q2a) and (p1q2a). □

§3. The classification of orbits

To finally figure out the orbits of pairs of 2-tori X and Y, we introduce additional invariants (see [6])

$$a = a(X,Y) = \frac{(w_1,u_3)(w_3,u_1)}{(w_1,u_1)(w_3,u_3)},\ b = b(X,Y) = \frac{(w_2,u_3)(w_3,u_2)}{(w_2,u_2)(w_3,u_3)},$$

$$c = c(X,Y) = \frac{(w_1,u_4)(w_4,u_1)}{(w_1,u_1)(w_4,u_4)},\ d = d(X,Y) = \frac{(w_2,u_4)(w_4,u_2)}{(w_2,u_2)(w_4,u_4)}.$$

At this time the parameters a, b, c, d are determined for each pair (X,Y) up to permutation by the elements (12)(34), (13)(24), (14)(23) of Vierergruppe.

Consider the set $\{(w_i, u_j), (w_j, u_i), i = 1, 2, j = 3, 4\}$. Since the torus does not determine the bases of the axis and center uniquely, the elements (w_i, u_j) can not be taken as invariants. But they preserve non-zero or zero value under conjugation. This allows us, for a pair of 2-tori X and Y, to introduce another invariant $e = e(X, Y)$ equal to the number of non-zero elements of a given set.

Theorem 1. *Let X, Y be 2-tori in $GL(4, K)$. Suppose that $r = 3$, $s = 3$ and one of invariants p or q is equal to zero, then the orbit (X,Y) is uniquely determined by the following Table 1.*

Table 1. For $r = 3$, $s = 3$.

	a	**d**	**e**	**p**	**q**	**base**
1	1	1	4	0	0	(r3s3a)
2	0	0	4	0	0	(r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta = -1$
3	0	$1+\beta$	5	0	0	(r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta \neq 0, -1$
4	$1+\alpha$	1	5	0	0	(r3s3b), $\alpha \neq 0, -1$; $\lambda = 0$, $\beta \neq 0$ or $\lambda = 1$, $\beta = 0$
5	$1+\alpha$	1	4	0	0	(r3s3b), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = 0$
6	$1+\alpha$	0	5	0	0	(r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta = -1$
7	$1+\alpha$	$1+\beta$	6	0	0	(r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1+\alpha+\beta \neq 0$
8	0	1	3	0	1	(r3s3b), $\alpha = -1$, $\lambda = \beta = 0$
9	0	1	4	0	1	(r3s3b), $\alpha = -1$; $\lambda = 0$, $\beta \neq 0$ or $\lambda = 1$, $\beta = 0$
10	$1+\alpha$	$-\alpha$	6	0	1	(r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1+\alpha+\beta = 0$
11	0	1	3	1	0	(r3s3b′), $\lambda = 0$, $\beta = 0$
12	0	1	4	1	0	(r3s3b′), $\lambda = 0$, $\beta \neq 0, -1$ or $\lambda = 1$, $\beta = 0$
13	$-\beta$	$1+\beta$	6	1	0	(r3s3b′), $\lambda = 1$, $\beta \neq 0, -1$

In all cases $b = c = 0$.

Proof. From lemma 2, we have two bases (r3s3a) and (r3s3b) for the case $r = 3$, $s = 3$. Now we calculate specified invariants for these two bases. Direct calculations show that $q = 0$, except for base (r3s3b) at $1 + \alpha + \beta\lambda = 0$. Next, we calculate these invariants in detail.

For the base (r3s3a), direct calculations give row 1 on the Table 1.

For the case (r3s3b), we have

$$a = \frac{1 \cdot (1 + \alpha)}{1 \cdot 1} = 1 + \alpha, \quad b = \frac{0 \cdot \beta}{1 \cdot 1} = 0,$$

$$c = \frac{0 \cdot \alpha\lambda}{1 \cdot 1} = 0, \quad d = \frac{1 \cdot (1+\beta\lambda)}{1 \cdot 1} = 1 + \beta\lambda,$$

thus α, λ and β determine the value of e. Furthermore, if $1+\alpha+\beta\lambda = 0$, then $p = 0$, $q = 1$. Finally, we obtain 2–10 on the table 1.

For the case (r3s3b'), we have

$$a = -\beta\lambda, b = 0, \quad c = 0, \quad d = 1 + \beta\lambda,$$

thus λ and β determine the value of e. Finally, we obtain 11–13 on the Table 1. □

Theorem 2. *Let X, Y be 2-tori in $GL(4, K)$. Suppose that at least one of r and s is 3 and r, $s \geqslant 3$, and one of invariants p or q is equal to zero, then the orbit (X,Y) is uniquely determined by the following Table 2 and 3.*

Table 2. For $r = 3$, $s = 4$.

	a	**d**	**e**	**p**	**q**	**base**
1	0	1	4	0	1	(r3s4a), $\alpha = -1$, $\lambda = 0$, $\beta \neq 0$
2	1	0	4	0	1	(r3s4a), $\lambda = 1$, $\alpha = 0$, $\beta = -1$
3	$1+\alpha$	$-\alpha$	6	0	1	(r3s4a), $\lambda=1$, $\alpha\neq0,-1$, $\beta\neq0,-1$, $1+\alpha+\beta\lambda=0$
4	1	1	5	0	0	(r3s4a), $\alpha = 0$, $\lambda = 0$, $\beta \neq 0$
5	$1+\alpha$	1	5	0	0	(r3s4a), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = -1$
6	1	$1+\beta$	5	0	0	(r3s4a), $\alpha = 0$, $\lambda = 1$, $\beta \neq 0, -1$
7	0	0	4	0	0	(r3s4a), $\alpha = -1$, $\lambda = 1$, $\beta = -1$
8	0	$1+\beta$	5	0	0	(r3s4a), $\alpha = -1$, $\lambda = 1$, $\beta \neq 0, -1$
9	$1+\alpha$	0	5	0	0	(r3s4a), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta = -1$
10	$1+\alpha$	$1+\beta$	6	0	0	(r3s4a), $\alpha\neq0,-1$, $\beta\neq0,-1$, $\lambda=1$, $1+\alpha+\beta\lambda\neq0$
11	0	1	4	1	0	(r3s4a′), $\lambda = 0$, $\beta \neq 0$
12	1	0	4	1	0	(r3s4a′), $\beta = -1$,$\lambda = 1$
13	$-\beta$	$1+\beta$	6	1	0	(r3s4a′), $\beta \neq 0, -1$, $\lambda = 1$

In all cases $b = c = 0$.

Proof. From Lemma 2, for the case $r = 3$, $s = 4$, we have the base (r3s4a). Direct calculations show that $q = 0$, except for base (r3s4a) at $1+\alpha+\beta\lambda=0$. Next, we calculate these invariants in detail.

For the base (r3s4a), we have

$$a = 1 + \alpha, b = 0, c = 0, d = 1 + \beta\lambda,$$

thus α, λ and β determine the value of e. Furthermore, if $1 + \alpha + \delta \neq 0$, then $p = 0$, $q = 1$. Finally, we obtain 1–10 on the Table 2.

For the base (r3s4a'), we obtain

$$a = -\beta\lambda, b = 0, c = 0, d = 1 + \beta\lambda,$$

thus λ and β determine the value of e. Finally, we obtain 11–13 on the Table 2.

For the case $r = 4$, $s = 3$, by interchanging X and Y, we can get a similar table 3 that corresponds to Table 2 just changing the positions of p and q. □

Theorem 3. *Let X, Y be 2-tori in $GL(4, K)$. Suppose that $r = 4$, $s = 4$ and at least one of invariants p or q is equal to zero, then the orbit (X,Y) is uniquely determined by the following Table 4.*

Table 3. For $r = 4$, $s = 3$.

	a	**d**	**e**	**p**	**q**	**base**
1	0	1	4	1	0	(r4s3a), $\alpha = -1$, $\lambda = 0$, $\beta \neq 0$
2	1	0	4	1	0	(r4s3a), $\lambda = 1$, $\alpha = 0$, $\beta = -1$
3	$1{+}\alpha$	$-\alpha$	6	1	0	(r4s3a), $\lambda{=}1$, $\alpha{\neq}0, -1$, $\beta{\neq}0, -1$, $1{+}\alpha{+}\beta\lambda{=}0$
4	1	1	5	0	0	(r4s3a), $\alpha = 0$, $\lambda = 0$, $\beta \neq 0$
5	$1 + \alpha$	1	5	0	0	(r4s3a), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = -1$
6	1	$1 + \beta$	5	0	0	(r4s3a), $\alpha = 0$, $\lambda = 1$, $\beta \neq 0, -1$
7	0	0	4	0	0	(r4s3a), $\alpha = -1$, $\lambda = 1$, $\beta = -1$
8	0	$1 + \beta$	5	0	0	(r4s3a), $\alpha = -1$, $\lambda = 1$, $\beta \neq 0, -1$
9	$1 + \alpha$	0	5	0	0	(r4s3a), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta = -1$
10	$1{+}\alpha$	$1{+}\beta$	6	0	0	(r4s3a), $\alpha{\neq}0, -1$, $\beta{\neq}0, -1$,$\lambda{=}1$,$1{+}\alpha{+}\beta\lambda{\neq}0$
11	0	1	4	0	1	(r4s3a′), $\lambda = 0$, $\beta \neq 0$
12	1	0	4	0	1	(r4s3a′), $\beta = -1$, $\lambda = 1$
13	$-\beta$	$1 + \beta$	6	0	1	(r4s3a′), $\beta \neq -1$, $\lambda = 1$

In all cases $b = c = 0$.

Proof. For the base (r4s4a), we have

$$a = 1 + \alpha, b = 0, c = 0, d = 1 + \delta,$$

thus α, δ and γ determine the value of e. Furthermore, if $1 + \alpha + \delta \neq 0$, then $p = 0$, $q = 1$.

For the base (r4s4b), we have

$$a = 1 + \alpha, b = 0, c = 0, d = 1 + \delta,$$

thus α and δ determine the value of e. Furthermore, if one of α and δ is equal to -1, then $p = 0$, $q = 1$. If $\alpha = \delta = -1$, then $p = 0$, $q = 2$.

For the base (r4s4a'), we obtain

$$a = 1 + \alpha, b = 0, c = 0, d = 1 + \delta,$$

thus α and δ determine the value of e.

For the base (r4s4b'), if one of α and δ is equal to -1, then $p = 1$, $q = 0$. If $\alpha = \delta = -1$, then $p = 2$, $q = 0$. □

From Lemma 5, when $p = q = 1$, direct calculations give us the following theorem.

Theorem 4. *Let X, Y be 2-tori in $GL(4, K)$. Suppose that $p = q = 1$, then the orbit (X, Y) is uniquely determined by the following Table 5.*

Table 4. For $r = 4$, $s = 4$.

	a	d	e	p	q	base
1	1	0	5	0	0	(r4s4a), $\alpha = -1$, $\gamma \neq 0$, $\delta = 0$
2	1	$1{+}\delta$	6	0	0	(r4s4a), $\alpha = 0$, $\gamma \neq 0$, $\delta \neq 0, -1$, $\gamma \neq (\alpha{+}1)(\delta{+}1)$
3	1	1	6	0	0	(r4s4a), $\alpha = 0$, $\gamma \neq 0, 1$, $\delta = 0$
4	0	0	4	0	0	(r4s4a), $\alpha = -1$, $\gamma \neq 0, 1$, $\delta = -1$
5	0	$1{+}\delta$	5	0	0	(r4s4a), $\alpha \neq 0, -1$, $\gamma \neq 0$, $\delta = -1$
6	$1{+}\alpha$	$1{+}\delta$	5	0	0	(r4s4a), $\alpha{\neq}0, -1$, $\gamma{=}0$, $\delta{\neq}0, -1$
7	$1{+}\alpha$	$1{+}\delta$	6	0	0	(r4s4a), $\alpha{\neq}0, -1$, $\gamma{\neq}0$, $\delta{\neq}0, -1$, $\gamma \neq \alpha\delta$, $\gamma \neq (\alpha{+}1)(\delta{+}1)$
8	0	0	3	0	1	(r4s4a), $\alpha = -1$, $\gamma = 0$, $\delta = -1$
9	0	$1{+}\delta$	4	0	1	(r4s4a), $\alpha = -1$, $\gamma = 0$, $\delta \neq 0, -1$
10	1	1	6	0	1	(r4s4a), $\alpha = 0$, $\gamma = 1$, $\delta = 0$
11	1	$1{+}\delta$	6	0	1	(r4s4a), $\alpha = 0$, $\gamma \neq 0, 1$, $\delta \neq 0, -1$, $\gamma = (\alpha{+}1)(\delta{+}1)$
12	$1{+}\alpha$	$1{+}\delta$	6	0	1	(r4s4a), $\alpha{\neq}0, -1$, $\gamma{\neq}0$, $\delta{\neq}0, -1$, $\gamma{\neq}\alpha\delta$, $\gamma = (\alpha{+}1)(\delta{+}1)$
13	0	0	3	1	0	(r4s4a$'$), $\alpha = -1$, $\gamma = 0$, $\delta = -1$
14	0	$1{+}\delta$	4	1	0	(r4s4a$'$), $\alpha = -1$, $\gamma = 0$, $\delta \neq 0, -1$
15	1	1	6	1	0	(r4s4a$'$), $\alpha = 0$, $\gamma = 1$, $\delta = 0$
16	1	$1{+}\delta$	6	1	0	(r4s4a$'$), $\alpha = 0$, $\gamma \neq 0, 1$, $\delta \neq 0, -1$, $\gamma = (\alpha{+}1)(\delta{+}1)$
17	$1{+}\alpha$	$1{+}\delta$	6	1	0	(r4s4a$'$), $\alpha{\neq}0, -1$, $\gamma{\neq}0$, $\delta{\neq}0, -1$, $\gamma \neq \alpha\delta$, $\gamma = (\alpha{+}1)(\delta{+}1)$
18	$1{+}\alpha$	$1 + \delta$	4	0	0	(r4s4b), $\alpha \neq 0, -1$, $\delta \neq 0, -1$
19	0	$1 + \delta$	3	0	1	(r4s4b), $\alpha = -1$, $\delta \neq 0, -1$
20	0	0	2	0	2	(r4s4b), $\alpha = -1$, $\delta = -1$
21	$1{+}\alpha$	0	3	1	0	(r4s4b$'$), $\delta \neq -1$, $\alpha \neq 0, -1$
22	0	0	2	2	0	(r4s4b$'$), $\alpha = -1$, $\delta = -1$

In all cases $b = c = 0$.

From lemma 6, when one of p and q is 2, direct calculations give us the following theorem.

Table 5. For $p = q = 1$.

	a	d	e	r	s	base
1	1	0	2	3	4	(p1q1a), $\lambda_1 = 0$, $\lambda_2 = 0$
2	1	0	3	3	4	(p1q1a), $\lambda_1 = 0$, $\lambda_2 = 1$ or $\lambda_1 = 1$, $\lambda_2 = 0$
3	1	1	5	3	4	(p1q1a), $\lambda_1 = 1$, $\lambda_2 = 1$
4	1	0	2	4	3	(p1q1b), $\lambda_1 = 0$, $\lambda_2 = 0$
5	1	0	3	4	3	(p1q1b), $\lambda_1 = 0$, $\lambda_2 = 1$ or $\lambda_1 = 1$, $\lambda_2 = 0$
6	1	1	5	4	3	(p1q1b), $\lambda_1 = 1$, $\lambda_2 = 1$
7	0	0	1	4	4	(p1q1c)

In all cases $b = c = 0$.

Table 6

	a	d	e	p	q	base
1	0	0	2	2	0	(p2q0a), $\alpha = \delta = 0$, $\gamma \neq 0$
2	0	0	3	2	0	(p2q0a), α, $\delta \neq 0$, $\gamma = 0$ or α, $\gamma \neq 0$, $\delta = 0$ or δ, $\gamma \neq 0$, $\alpha = 0$
3	0	0	4	2	0	(p2q0a), α, δ, $\gamma \neq 0$, $\alpha\delta \neq \gamma$
4	0	0	2	0	2	(p0q2a), $\alpha = \delta = 0$, $\gamma \neq 0$
5	0	0	3	0	2	(p0q2a), α, $\delta \neq 0$, $\gamma = 0$ or α, $\gamma \neq 0$, $\delta = 0$ or δ, $\gamma \neq 0$, $\alpha = 0$
6	0	0	4	0	2	(p0q2a),α, δ, $\gamma \neq 0$, $\alpha\delta \neq \gamma$
7	0	0	1	2	1	(p2q1a), $\alpha = 0$, $\delta = 0$
8	0	0	2	2	1	(p2q1a), $\alpha = 0$, $\delta \neq 0$ or $\alpha \neq 0$, $\delta = 0$
9	0	0	4	2	1	(p2q1a), $\alpha \neq 0$, $\delta \neq 0$
10	0	0	1	1	2	(p1q2a), $\alpha = 0$, $\delta = 0$
11	0	0	2	1	2	(p1q2a), $\alpha = 0$, $\delta \neq 0$ or $\alpha \neq 0$, $\delta = 0$
12	0	0	4	1	2	(p1q2a), $\alpha \neq 0$, $\delta \neq 0$
13	0	0	0	2	2	(p2q2a)

In all cases $b = c = 0$.

Theorem 5. *Let X, Y be 2-tori in $GL(4, K)$. Suppose that one of invariants p or q is* 2 *and p, $q \neq 0$, then the orbit (X, Y) is uniquely determined by the Table* 6.

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Поступило 19 июня 2024 г.

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