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ON THE HIERARCHY OF CLASSICALITY AND SYMMETRY OF QUANTUM STATES

Abstract. The interrelation between classicality/quantumness and symmetry of states is discussed within the phase-space formulation of finite-dimensional quantum systems. We derive representations for classicality measures $\mathcal{Q}_N[H_\varrho]$ of states from the stratum of given symmetry type $[H_\varrho]$ for the Hilbert–Schmidt ensemble of qudits. The expressions for measures are given in terms of the permanents of matrices constructed from the vertices of the special Wigner function's positivity polytope. The supposition about the partial order of classicality indicators $\mathcal{Q}_N[H_\varrho]$ in accordance with the symmetry type of stratum is formulated.

§1. Introduction

Not all things admit to be ordered, but some do. It is remarkable that sometimes after their ordering is recognized, the other things, at first glance independent from the former, reveal the corresponding order as well, thereby showing their hidden interrelations with one another. In the present note we would like to draw attention to a similar situation occurring in statistical description of finite-dimensional quantum systems. Namely, we argue that if quantum states are ordered with respect to their "symmetry", then they exhibit also the ordering with respect to their "classicality" in a way that can be formulated as:

"The larger symmetry quantum states possess, the more classical they are!"

Below, attempting to alter the above sonorous utterance into the rigorous statement, we briefly recapitulate two issues – **the equivalence and partial order from unitary symmetry** and **classicality of states**:

- **Equivalence and partial order relations from the unitary symmetry** — the equivalence relation between quantum states of an N-level system related to the unitary group $SU(N)$ transformation. This equivalence results in the partition of a state space into

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a strata with the symmetry characterized by the partially ordered isotropy subgroups $H_\alpha \subseteq SU(N)$;

- **Classicality of states** — the notion of classical states based on the non-negativity of their quasiprobability distributions and the idea of geometric indicators of classicality $\mathcal{Q}_N[H_\alpha]$ of quantum states defined as the geometric probability to find a classical state on a stratum with symmetry type $[H_\alpha]$,

$$\mathcal{Q}_N[H_\alpha] = \frac{\text{Volume of classical states on stratum type } [H_\alpha]}{\text{Volume of all states on stratum type } [H_\alpha]}\,. \tag{1}$$

Bearing in mind the above underlying features of partial ordering of isotropy groups and the corresponding classification of strata in $\mathfrak{P}_N$, we pose the question about the order of the classicality measures (1). Based on our computations of $\mathcal{Q}_N[H_\alpha]$ for 3- and 4-dimensional systems we formulate the following conjecture.

The hierarchy conjecture: *Let us arrange the isotropy groups H_α in ascending order, starting from the maximal torus T_N up to the whole group $SU(N)$,*

$$T_N = H_{\min} < H_1 < \cdots < H_{\max} = SU(N)\,, \tag{2}$$

then the set of classicality indicators $\mathcal{Q}_N[H_\alpha]$ inherits the hierarchy,

$$\mathcal{Q}_N[T_N] < \mathcal{Q}_N[H_1] < \cdots < \mathcal{Q}_N[SU(N)] = 1\,. \tag{3}$$

In the present note we describe two methods of analytical calculations of measures (1) for an arbitrary N-level quantum system. For the readers convenience, before describing these technical tools, in the next section we start with the generic issues of the unitary symmetry representation in closed quantum systems putting an accent on geometrical features of phase space description of finite-dimensional quantum systems mainly following our recent publications [1, 2].

§2. Symmetry and geometry

Here we briefly summarise how the unitary symmetry of the underling Hilbert space $\mathbb{C}^N$ of N-dimensional quantum system[1] imposes certain constraints on geometric and statistical properties of its state space (for generic concepts see review [3] and references therein).

[1]For brevity, we will henceforth call N-level system "N-qudit", or simply "qudit", if a specific dimension is irrelevant.

The unitary symmetry, equivalence classes and partial order. The state space $\mathfrak{P}_N$ of an N-qudit can be identified with the subspace of $N\times N$ Hermitian, trace-one positive semidefinite matrices:

$$\mathfrak{P}_N = \{\, \varrho \in M_N(\mathbb{C}) \mid \varrho = \varrho^\dagger\,, \quad \varrho \geqslant 0\,, \quad \mathrm{tr}\,(\varrho) = 1 \,\}\,. \tag{4}$$

The $U(N)$ automorphism of $\mathbb{C}^N$ induces the adjoint $SU(N)$ transformations of density matrices $\varrho \in \mathfrak{P}_N$:

$$\varrho \mapsto \varrho' = \mathrm{Ad}_g \varrho\,, \qquad g \in SU(N) \tag{5}$$

and sets up the equivalence between points of the orbit $\mathcal{O}_\varrho = \{\mathrm{Ad}_g\varrho,\ g \in SU(N)\}$ through the state $\varrho \in \mathfrak{P}_N$. In a view of this equivalence the orbits provide partition of $\mathfrak{P}_N$, but being not locally finite (every non-empty open set intersects infinitely many orbits) it can not serve as decomposition of $\mathfrak{P}_N$. However, with this equivalence relation there is another kind of partition named the "orbit type", which is based on the notion of the *isotropy group (stabilizer)* $H_x \subset SU(N)$ of point $x \in \mathfrak{P}_N$,

$$H_x = \{\, g \in SU(N) \mid \mathrm{Ad}_g x = x \,\}\,.$$

Two points $x, y \in \mathfrak{P}_N$ are declared to be of the same type if their stabilizers are conjugate subgroups of $SU(N)$. If the stabilizer H_x of some/any point x in the orbit belongs to the conjugacy class of subgroup H in $SU(N)$, we say that *the type of the orbit* is $[H]$ and by $\mathfrak{P}_{[H_\alpha]}$ denote the set of points of $\mathfrak{P}_N$, whose stabilizer is conjugated to the subgroup H_α:

$$\mathfrak{P}_{[H_\alpha]} := \left\{\, x \in\ \mathfrak{P}_N \mid H_x \text{ is conjugate to } H_\alpha \,\right\}. \tag{6}$$

Here α is the set enumerating the conjugacy classes of the isotropy groups. The isotropy group of density matrix is determined by the algebraic degeneracy of its spectrum and therefore the number of conjugacy classes is equal to the number $P(N)$ of different representations of integer N as the sum of positive natural numbers, $\alpha = \{1, 2, \dots, p(N)\}$. The subsets $\mathfrak{P}_{[\mathrm{H}_\alpha]}$ are termed as *strata* and can be partially ordered in accordance with the partial order of the corresponding isotropy groups[2]. Hence we arrive at the orbit type decomposition of state space:

$$\mathfrak{P}_N = \bigcup_\alpha \mathfrak{P}_{[H_\alpha]}\,. \tag{7}$$

[2]If H and K are isotropy subgroups of G, we define a partial ordering on equivalence classes by writing $(H) < (K)$ if H is G-conjugate to a subgroup of K. This defines a partial ordering on the set of the isotropy types of orbits.

Each stratum in (7) can be described in terms of states with a fixed degeneracy as follows. Consider $(N-1)$-dimensional simplex C_{N-1} of ordered eigenvalues:

$$C_{N-1} := \{\, \boldsymbol{r} \in \mathbb{R}^N \,\Big|\, \sum_{i=1}^{N} r_i = 1\,, \quad 1 \geqslant r_1 \geqslant r_2 \geqslant \ldots \geqslant r_{N-1} \geqslant r_N \geqslant 0 \,\}\,. \tag{8}$$

For our further aims it is enough to restrict ourselves by considering only the positive density matrices of maximal rank, i.e. remove from the simplex the subset $\{r_1 = 1\} \cup \{r_N = 0\}$. This truncated simplex is a union of eigenvalues of non-singular density matrices of the fixed degeneracy $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$,

$$\mathfrak{P}_{\boldsymbol{k}} = \{\, \varrho \in \mathfrak{P}_N\,, k_i \in \mathbb{Z}_+ \mid \det(\varrho - \lambda) = \prod_{i=1}^{n} (r_i - \lambda)^{k_i}\,, \quad \sum_{i=1}^{n} k_i = N \,\}\,. \tag{9}$$

Finally, taking into account an admissible transposition of eigenvalues, we arrive at the decomposition of a given stratum:

$$\mathfrak{P}_{[H_\alpha]} = \bigcup_{\omega \in S_n} \mathfrak{P}_{\omega.\boldsymbol{k}}\,. \tag{10}$$

In (10) by $\omega \cdot \boldsymbol{k} = \{k_{\omega(1)}, k_{\omega(2)}, \ldots, k_{\omega(n)}\}$ we denote the action of a symmetric group S_n on a given partition of N into n natural numbers $k_1, k_2, \ldots, k_n$.

Unitary invariance of probability distributions on strata. Let us assume that the probability density function of the qudit ensemble is invariant under (5):

$$P(\varrho) = P(g\varrho g^\dagger)\,, \qquad \forall\; g \in SU(N)\,. \tag{11}$$

Due to the invariance property (11) one can get convinced that the probability density function on a given stratum $\mathfrak{P}_{[H_\alpha]}$ reduces to the following expressions:

$$P(\varrho) = \sum_{\omega \in S_n} P_{\omega\cdot\boldsymbol{k}}(r_1, \ldots, r_n)\; \mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_N \wedge \mathrm{d}\mu_{U(N)/H}\,, \tag{12}$$

which shows that the measure factorizes into the factor corresponding to the measure on subset $\mathfrak{P}_{\boldsymbol{k}}$ of the simplex $\mathcal{C}_{N-1}$ and the Haar measure on the coset $U(N)/H$.

The Hilbert-Schmidt ensemble of qudits on principal stratum. One of the widely used unitary invariant probability density function originates from the Hilbert–Schmidt (HS) metric on $\mathfrak{P}_N$:

$$\mathrm{g}_{\mathrm{HS}} \propto \mathrm{Tr}\,(\mathrm{d}\varrho \otimes \mathrm{d}\varrho)\ . \tag{13}$$

If a density matrix ϱ belongs to the principal stratum with maximal torus isotropy group, $\varrho \in \mathfrak{P}_{[T^{(N-1)}]}$, then the metric (13) defines the standard *Hilbert–Schmidt ensemble* of random full-rank N-qudits with the well-known joint probability distribution of distinct eigenvalues,

$$P(r_1,\dots,r_N) \propto\ \delta(1-\sum_{j=1}^{N} r_j)\prod_{j<k}^{N}(r_j-r_k)^2\,. \tag{14}$$

The Hilbert-Schmidt ensemble of qudits on degenerate strata. If the full-rank density matrix is degenerate with multiplicity of eigenvalues $\{k_1,k_2,\dots,k_n\}$, i.e., its isotropy group is $H = U(k_1)\times\cdots\times U(k_n)$, then the joint probability distribution of eigenvalues reads:

$$P_{k_1,\dots,k_s}(r_1,\dots,r_s) \propto \delta(1-\sum_{i=1}^{n} k_i r_i)\prod_{i<j}^{1\dots n}(r_i-r_j)^{2k_ik_j}\,. \tag{15}$$

§3. Classicality and geometry

In this section we formulate the notion of classicality of qudit states as an existence of a corresponding proper probability distributions. Namely, we relate the classicality with the Wigner function (WF) positivity and describe the underlying geometry of the state space. In our consideration we use the $(N-2)$-parametric family WFs given by the dual pairing of a density matrix ϱ and Stratonovich–Weyl (SW) matrix valued kernel $\Delta(\boldsymbol{z}|\boldsymbol{\nu})$ on the phase space Ω_N (cf. for details in [1, 2]):

$$W^{(\boldsymbol{\nu})}_{\varrho}(\boldsymbol{z}) = \mathrm{tr}(\varrho\Delta(\boldsymbol{z}|\boldsymbol{\nu}))\,, \qquad \boldsymbol{z}\in\Omega_N\,. \tag{16}$$

Classical states and WF positivity polytope. The "classical states" form the subset $\mathfrak{P}^{\mathrm{Cl}}_N \subset \mathfrak{P}_N$ of states whose Wigner function $W^{(\boldsymbol{\nu})}_{\varrho}(z)$ in a given representation with moduli parameters $\boldsymbol{\nu} = \{\nu_1,\nu_2,\dots,\nu_{N-2}\}$ is non-negative everywhere over the phase space:

$$\mathfrak{P}^{\mathrm{Cl}}_N = \{\,\varrho\in\mathfrak{P}_N\ \mid W^{(\boldsymbol{\nu})}_{\varrho}(z)\geqslant 0\,,\quad \forall z\in\Omega_N\,\}. \tag{17}$$

The "classical states on a fixed stratum" $\mathfrak{P}_{H_\alpha}$ are defined respectively as:

$$\mathfrak{P}^{\mathrm{Cl}}_{H_\alpha} = \mathfrak{P}^{\mathrm{Cl}}_N \cap \mathfrak{P}_{H_\alpha}\,. \tag{18}$$

In order to describe explicitly the classical states (17) and (18) one can consider the following linear functional $\mathfrak{P}_N \to \mathbb{R}$:

$$w[\varrho] := \inf_{g\in U(N)} W^{(\boldsymbol{\nu})}_{g\varrho g^\dagger}(z) \tag{19}$$

and exploit the following observation.

Proposition I. *The zero-level set of functional $w[\varrho]$,*

$$H_N: \ \{\varrho \in \mathfrak{P}_N \,|\, w[\varrho] = 0\}\,, \tag{20}$$

describes the supporting hyperplane

$$(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) = r_1\pi_N + r_2\pi_{N-1} + \cdots + r_N\pi_1 = 0 \tag{21}$$

of the convex set of classical states. The tuples $\boldsymbol{r}^\downarrow = \{r_1, r_2, \dots, r_N\}$ and $\boldsymbol{\pi}^\downarrow = \{\pi_1, \pi_2, \dots, \pi_N\}$ in (21) denote the eigenvalues of the density matrix ϱ and the SW kernel $\Delta(\boldsymbol{z}\,|\,\boldsymbol{\nu})$ respectively, both arranged in decreasing order. The SW kernel eigenvalues $\boldsymbol{\pi}$ satisfy the following equations:

$$\sum_i^N \pi_i = 1\,, \qquad \sum_i^N \pi_i^2 = N\,. \tag{22}$$

Proposition II. *The intersection of the hyperplane H with the simplex (8) defines the Wigner function's positivity polytope corresponding to the canonical projection $p : \mathfrak{P}_N \mapsto \mathfrak{P}_N/SU(N)$ of the classical states.*

The Propositions I and II follow from the results of [2], where the image of classical subsets $\mathfrak{P}^{(\mathrm{Cl})}_{[H_\alpha]}$ under the canonical quotient mapping where introduced:

$$\mathcal{C}^*_{N-1}(H_\alpha) = \{\, p(x) \ |\ x \in \mathfrak{P}^{\mathrm{Cl}}_{H_\alpha}\,\}\,. \tag{23}$$

The set of classicality measures. A knowledge of the WF positivity polytope allows one to extract information on the classicality/quantumness of states. Based on the definition of region of classical states (17), we can define sequence of classicality/quantumness indicators evaluating relative weight of the classical states. Namely one can consider the collection of different geometric probabilities of finding a classical state in a given unitary invariant statistical ensemble (12), among them [4, 5]:

(1) The *global indicator of classicality of ensemble*,

$$\mathcal{Q}_N = \frac{\int_{\mathfrak{P}_N^{\mathrm{Cl}}} \mathrm{d}\mu}{\int_{\mathfrak{P}_N} \mathrm{d}\mu}\,. \tag{24}$$

(2) The *indicator of classicality of a stratum ensemble*,

$$\mathcal{Q}_N[H_\alpha] = \frac{\int\limits_{\mathfrak{P}^{\mathrm{Cl}}_{[H_\alpha]}} \mathrm{d}\mu}{\int\limits_{\mathfrak{P}_{[H_\alpha]}} \mathrm{d}\mu}\,. \tag{25}$$

Here it is in order to make a few comments, in (24)-(25) the measure $\mathrm{d}\mu$ is assumed to be the unitary invariant of the form (12). In the subsequent section we will specify the measure corresponding to the ensemble of Hilbert-Schmidt qudits (13) and (15) for $N = 2, 3, 4$, i.e. qubit, qutrit, and quatrit respectively. Note that we expect that $\mathcal{Q}_N = \mathcal{Q}_N[H_0]$, since the principal stratum with $H_0 = U(1)^N$ differs from the whole space state $\mathfrak{P}_N$ by a measure-zero set only.

§4. Computing the indicators

According to (10), the stratum $\mathfrak{P}_{[H_\alpha]}$ consists from subsets of matrices with a certain degeneracy type. Due to the unitary invariance of probability distribution functions (11), any above introduced classicality indicator depends only on the joint probability distribution of eigenvalues of the density matrix and thus can be rewritten as:

$$\mathcal{Q}_N[H_\alpha] = \frac{\sum\limits_{\omega\in S_s}\int\limits_{\mathcal{C}^*_{N-1}(H_\alpha)} P_{k_\omega(1),\dots,k_\omega(s)}(r_1,\dots,r_s)\,\mathrm{d}r_1\wedge\dots\wedge\mathrm{d}r_s}{\sum\limits_{\omega\in S_s}\int\limits_{\mathcal{C}_{N-1}(H_\alpha)} P_{k_\omega(1),\dots,k_\omega(s)}(r_1,\dots,r_s)\,\mathrm{d}r_1\wedge\dots\wedge\mathrm{d}r_s}\,. \tag{26}$$

In (26) the integral in the denominator represents the volume of the orbit space of stratum $\mathfrak{P}_{[H_\alpha]}$. The integration in the nominator of (26) is over the WF positivity polytope $\mathcal{C}^*_{N-1}(H_\alpha)$:

$$\mathcal{C}^*_{N-1}(H_\alpha) = \left\{\boldsymbol{\pi}\in\mathbf{spec}\,(\Delta(\Omega_N))\mid(\boldsymbol{r}^{\downarrow},\boldsymbol{\pi}^{\uparrow})\geqslant 0,\ \forall\,\boldsymbol{r}\in\mathcal{C}_{N-1}(H_\alpha)\right\}. \tag{27}$$

Hence, for the Hilbert-Schmidt qudits with probability distribution functions (13) and (15) the evaluation of the classicality indicators reduces to the problem of integration of polynomials over the convex polytopes.

Simplicial decomposition. It is known that computation of the volume of polytopes of varying dimension is #P-hard and that even approximating the volume is hard [6]. Currently, the most powerful method for an efficient approximation of integrals in (26) over polytopes remains the Monte Carlo-type algorithms. However, often when the polytopes are functions of parameters (as in our case, when its structure depends on representation of SW kernel), an exact analytical calculations of the volume is requested, the situation becomes extremely complicated. In this case the computational methods stem from the observation that a convex polytope admits decomposition into a union of simplices, satisfying certain properties. Based on this idea of triangulation, the polytope volume might be computed either summing up volumes of simplices or using the signed decomposition methods if a given polytope is decomposed into signed simplices such that the signed sum of their volumes gives the volume of the polytope. Leaving aside the question of an efficient simplicial decomposition, below we describe two methods of evaluation of integrals from homogeneous polynomials over the simplex.

The 1st Lasserre–Avrachenkov (LA) method of integration over simplex. We are interested in calculation of the integral of the polynomial $p(x_1, x_2, \dots, x_n)$ over the n-simplex $C_n \in \mathbb{R}^n$ with vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n$

$$V\,(p;\, C_n) = \int\limits_{C_n} p(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{28}$$

with respect to the n-dimensional Lebesgue measure. With this aim we recall an elegant analytical method reducing calculation of integrals from homogeneous polynomials to the integration of the corresponding polarization form of those polynomials [7]. Briefly it can be stated as follows. Let $p(\boldsymbol{x})$ be q-homogeneous polynomial,

$$p : \mathbb{R}^n \to \mathbb{R}\,, \qquad p(t\boldsymbol{x}) = t^q p(\boldsymbol{x})\,, \quad \forall\, t \in \mathbb{R} \text{ and } \boldsymbol{x} \in \mathbb{R}^n\,, \tag{29}$$

and let $H_p(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_q)$ be the polarization of p, the mapping $(\mathbb{R}^n)^q \mapsto \mathbb{R}$, which is symmetric q-linear form such that[3]

$$H_p(\boldsymbol{x}, \boldsymbol{x}, \dots, \boldsymbol{x}) = p(\boldsymbol{x}). \tag{31}$$

According to the Lasserre-Avrachenkov theorem [7], the integration in (28) results in summation of the values of polarization $H_p(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_q)$

[3]The well-known formula,

$$H_p(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_q) = \frac{1}{q!}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\cdots\frac{\partial}{\partial t_q} p(t_1\boldsymbol{X}_1 + t_2\boldsymbol{X}_2 + \dots + t_q\boldsymbol{X}_q)\Big|_{\boldsymbol{t}=0}\,, \tag{30}$$

gives a compact representation for the polarization.

evaluated at the vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of a simplex:

$$V(p;\, C_n) = \frac{\mathrm{vol}(C_n)}{\binom{n+q}{n}} \sum_{\sum_0^n a_i = q} H_p(\overbrace{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_0}^{a_0}, \overbrace{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_1}^{a_1}, \ldots, \overbrace{\boldsymbol{v}_n, \ldots, \boldsymbol{v}_n}^{a_n})\,. \tag{32}$$

The 2nd Lasserre method of integration over simplex. Another important for us result has been noted by J. Lassere [8]. He proved that integrating a polynomial of degree q on an arbitrary simplex (with respect to Lebesgue measure) reduces to evaluating q-homogeneous polynomials of degree $j = 1, 2, \ldots, q$ each at a unique point $\boldsymbol{s}_j$ of the simplex. Bearing in mind that the integration over an arbitrary simplex can be reduced to the integration over the canonical simplex[4] by a certain affine transformation, we give the formulation of the method for the canonical simplex case. Namely, let the polynomial $p(\boldsymbol{x})$ of degree q be rewritten as $p(\boldsymbol{x}) = \sum\limits_{j=0}^{q} p_j(\boldsymbol{x})$, where $p_j(\boldsymbol{x}) = \sum\limits_{|\alpha|=j} p_\alpha \boldsymbol{x}^\alpha$ is a homogeneous polynomial of degree j. Then according to [8] the integration over the canonical n-dimensional simplex K_n gives

$$\int\limits_{K_n} p(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \mathrm{vol}(K) \left(\widehat{p}_0 + \sum_{j=1}^{q} \widehat{p}_j(\boldsymbol{s}_j) \right), \tag{33}$$

where $\boldsymbol{s}_j = \frac{1}{\sqrt[j]{(n+1)\ldots(n+j)}}(1, 1, \ldots, 1)$ and $\widehat{p}(\boldsymbol{x})$ stands for the associated "Bombieri" polynomial:

$$\widehat{p}(\boldsymbol{x}) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \alpha_1! \ldots \alpha_n!\, \boldsymbol{x}^\alpha\,, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\,. \tag{34}$$

Note that expression (33) differs from the well-known cubature formulae. In (33) instead of evaluating a single polynomial at several points, as it takes place in the case of cubature formulae, one evaluates polynomials of degree j at a single point only.

Applying methods to the Hilbert–Schmidt measure. Both the above mentioned methods of integration can be used analyzing the classicality indicators $\mathcal{Q}_N[H_\alpha]$ of the Hilbert–Schmidt ensembles of qudits. Here we outline the general scheme of calculation while in the next section considering low-dimensional systems $N = 2, 3, 4$, some principal technical details

[4]The canonical n-simplex $K_n \subset \mathrm{R}^n$ is defined as $K_n = \{\boldsymbol{x} \in \mathrm{R}^n_+ \mid x_1 + x_2 + \ldots + x_n \leqslant 1\}$.

will be elucidated. As a first step, we decompose the WF positivity polytope into the sum of simplices, i.e.,

$$Q_N[T_N] = \sum_{\text{simplices}} I_C(\boldsymbol{\pi}) \,, \tag{35}$$

where the typical element of the sum is integral over a certan n-simplex C_n given as the convex hull of n vertices $\mathcal{C}_n(\boldsymbol{\pi}) := \mathrm{conv}(v_0, v_1(\boldsymbol{\pi}), \dots, v_n(\boldsymbol{\pi}))$:

$$I_C(\boldsymbol{\pi}) \propto \int\limits_{C_n(\boldsymbol{\pi})} \mathrm{d}\boldsymbol{r}\, \delta(1 - \sum_i^{n+1} r_i) \prod_{i<j}^{n+1} (r_i - r_j)^2 \,. \tag{36}$$

Note that in the case we are interested in, the vertices $v_i(\boldsymbol{\pi})$ are rational functions of the SW kernel eigenvalues. Their exact form follows from the separating hyperplane equation (21). The integrand in (36) due to δ-function factor is not a homogeneous polynomial and thus the LA formula (32) is not applicable directly. But, using the map from the canonical (standard) simplex K_n to the simplex C_n :

$$K_n \mapsto C_n(\boldsymbol{\pi}) : \;\; \boldsymbol{r} = \boldsymbol{v}_0 + \sum_{\alpha=1}^{n} (\boldsymbol{v}_\alpha(\boldsymbol{\pi}) - \boldsymbol{v}_0) u_\alpha \,, \tag{37}$$

the integral reduces to the integral over the canonical n-simplex

$$I_C(\boldsymbol{\pi}) \propto \mathrm{vol}_{\mathrm{E}}(C_n(\boldsymbol{\pi})) \int\limits_{K_n} \mathrm{d}\boldsymbol{u}\, \mu(\boldsymbol{u}) \,, \tag{38}$$

where $\mathrm{vol}_{\mathrm{E}}(C_n(\boldsymbol{\pi}))$ denotes the Euclidean volume of the simplex $C_n(\boldsymbol{\pi})$[5] and $\mu(\boldsymbol{u})$ is homogeneous polynomial of order $q = n(n-1)$:

$$\mu(\boldsymbol{u}) = \prod_{i<j} \left(\sum_{\alpha=1}^{n} (v_\alpha^i(\boldsymbol{\pi}) - v_\alpha^j(\boldsymbol{\pi})\, u_\alpha) \right)^2 \,. \tag{39}$$

The polynomial (39) can be rewritten as

$$\mu(\boldsymbol{u}) = (-1)^n \prod_{s=1}^{q} (\boldsymbol{\alpha}_s, \boldsymbol{l}(\boldsymbol{u})) \,, \tag{40}$$

[5] The Euclidean volume of n-simplex in $\mathbb{R}^n$ in terms of $(n+1)$-vertices reads:

$$\mathrm{vol}_{\mathrm{E}}(C_n) := \frac{1}{n!} \left| \det \begin{pmatrix} \boldsymbol{v}_0 & \boldsymbol{v}_1 & \dots & \boldsymbol{v}_n \\ 1 & 1 & \dots & 1 \end{pmatrix} \right| .$$

where

$$\boldsymbol{l}(\boldsymbol{u})^i = \sum_{\alpha=1}^{n} v_\alpha^i u_\alpha \,, \tag{41}$$

and $\boldsymbol{\alpha}_s = \{\boldsymbol{e}_i - \boldsymbol{e}_j \,,\ i, j = 1, 2, \dots n\}$ are n-vectors constructed out of the standard unit n-dimensional vectors $\boldsymbol{e}_i$. Linearity of $\boldsymbol{l}(\boldsymbol{u})^i$ implies q-linearity of the associated to the polynomial $\mu(\boldsymbol{u})$ polarization form H,

$$H_\mu(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_q) := \frac{1}{q!} \sum_{\sigma \in S_q} \prod_{s=1}^{q} (\boldsymbol{\alpha}_s, \, \boldsymbol{l}(\boldsymbol{X}_{\sigma(s)}))\,. \tag{42}$$

The expression (42) shows that the polarization form corresponding to the Hilbert–Schmidt measure is given by the normalized permanent of $q \times q$ matrix,

$$H_p(\boldsymbol{X}_1 \, \boldsymbol{X}_2, \dots, \boldsymbol{X}_q) = \frac{1}{q!} \, \mathrm{perm} ||(\boldsymbol{\alpha}_i, \, \boldsymbol{l}(\boldsymbol{X}_j)||\,. \tag{43}$$

Hence, using the LA formula (32) and noting that $\boldsymbol{l}(\boldsymbol{e}_j) = \boldsymbol{v}_j$, we arrive at

$$I_C(\boldsymbol{\pi}) \propto \frac{\kappa(\boldsymbol{\pi})}{q!} \sum_{\sum\limits_{i=1}^{n} a_i = q} \mathrm{perm}\, ||\mathrm{M}(a_1, a_2, \dots, a_n)||\,, \tag{44}$$

where

$$\kappa(\boldsymbol{\pi}) = \frac{\mathrm{vol_E}(C_n(\boldsymbol{\pi}))\mathrm{vol_E}(K_n)}{\binom{n+q}{q}}\,, \tag{45}$$

and $q \times q$ matrices M :

$$\mathrm{M}_{st}(a_1, a_2, \dots, a_n) := (\boldsymbol{\alpha}_s, \, \boldsymbol{V}_t)\,, \quad s, t = 1, 2, \dots, q\,, \tag{46}$$

constructed out of tuples $V = \{\overbrace{\boldsymbol{v}_0, \dots, \boldsymbol{v}_0}^{a_0}, \overbrace{\boldsymbol{v}_1, \dots, \boldsymbol{v}_1}^{a_1}, \dots, \overbrace{\boldsymbol{v}_n, \dots, \boldsymbol{v}_n}^{a_n}\}$ for all admissible partitions of degree of homogeneity $\sum a_i = q$ in integers.

We finalize this paragraph noting that the above scheme of calculations is applicable to the evaluation of the classicality indicators $\mathcal{Q}[H_\alpha]$ for the lower-dimensional strata as well.

4.1. Qubit. The ordered eigenvalue simplex of a qubit represents the line segment in $\mathbb{R}^2$:

$$C_1 : \quad \{r_1 + r_2 = 1, 1 \geqslant r_1 \geqslant r_2 \geqslant 0\}\,.$$

This interval is convex hull of points $\boldsymbol{v}_0 = \{1/2, 1/2\}$ and $\boldsymbol{v}_1 = \{1, 0\}$. Among qubit states the maximally mixed state at vertex $\boldsymbol{v}_0$ has maximal symmetry, the $SU(2)$ isotropy group, while all the other states $\varrho \in \mathfrak{P}_2$

have the torus $\mathrm{T}^2 \in SU(2)$ as their isotropy group. Noting that for the maximally mixed state $\varrho_* = 1/2\,\mathbb{I}_2$ the Wigner function is positive, we can formally assign the value one to the classicality indicator, $\mathcal{Q}_2[SU(2)] = 1$. The indicator $\mathcal{Q}_{[\mathrm{T}^2]}$ for the principle stratum can be calculated along the methods described in previous section noting that the spectrum of SW kernel is uniquely determined from (22):

$$\pi_1 = \frac{1+\sqrt{3}}{2}\,, \qquad \pi_2 = \frac{1-\sqrt{3}}{2}\,, \tag{47}$$

and the supporting hyperplane $\pi_1 r_2 + \pi_2 r_1 = 0$ intersects the segment C_1 at $\boldsymbol{v}_1(\boldsymbol{\pi}) = \{\frac{1}{2}+\frac{1}{2\sqrt{3}}, \frac{1}{2}-\frac{1}{2\sqrt{3}}\}$. The integration over the intervals is trivial and as a result the qubit global indicator of classicality is

$$\mathcal{Q}_{[\mathrm{T}^2]} = \frac{1}{3\sqrt{3}}\,. \tag{48}$$

4.2. Qutrit.

Unitary strata of qutrit state space. The ordered eigenvalue simplex of qutrit is triangle in $\mathbb{R}^3$:

$$C_2: \quad \{r_1 + r_2 + r_3 = 1, \qquad 1 \geqslant r_1 \geqslant r_2 \geqslant r_3 \geqslant 0\}\,.$$

It is convex hull of three points $\boldsymbol{v}_0 = \{1/3, 1/3, 1/3\}$, $\boldsymbol{v}_1 = \{1/2, 1/2, 0\}$ and $\boldsymbol{v}_2 = \{1, 0, 0\}$. The possible multiplicity of eigenvalues are $\boldsymbol{k} = (1,1,1)$, $\boldsymbol{k} = (1,2)$ and $\boldsymbol{k} = (2,1)$, and there are three corresponding strata of $\mathfrak{P}_3$:

- the 8-dimensional principal stratum with isotropy class $[\mathrm{T}^3]$ consisting of matrices with a simple spectrum, $1 > r_1 \neq r_2 \neq r_3 > 0$,

$$\mathfrak{P}_{[\mathrm{T}^3]}: \quad \{\varrho \in \mathfrak{P}_3 \,|\, \mathrm{spec}(\varrho) := (r_1, r_2, r_3)\,, 1 > r_1 > r_2 > r_3 > 0\};$$

- the 5-dimensional degenerate stratum with isotropy class $[\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))]$ is the locus of density matrices with the degeneracies $\boldsymbol{k} = (2,1)$ and $\boldsymbol{k} = (1,2)$,

$$\mathfrak{P}_{[\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))]}: \quad \mathfrak{P}_{1,2} \bigcup \mathfrak{P}_{2,1}\,,$$

  with components

$$\mathfrak{P}_{1,2}: \quad \{\varrho \in \mathfrak{P}_3 \,|\, \mathrm{spec}(\varrho) := (r_1, r_2, r_3)\,, 1 > r_1 \neq r_2 = r_3 > 0\},$$
$$\mathfrak{P}_{2,1}: \quad \{\varrho \in \mathfrak{P}_3 \,|\, \mathrm{spec}(\varrho) := (r_1, r_2, r_3)\,, 1 > r_1 = r_2 \neq r_3 > 0\};$$

- the 0-dimensional stratum, $\mathfrak{P}_{[SU(3)]}$, the mixed state with the triple degeneracy $\boldsymbol{k} = (3)$, $r_1 = r_2 = r_3 = 1/3$.

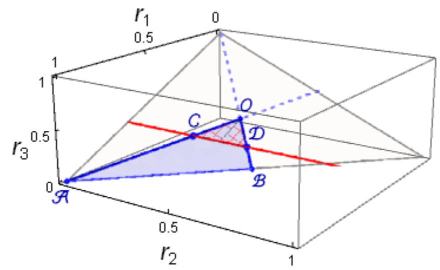


Figure 1. Triangle $\triangle AOB$ as the ordered 2-simplex of qutrit eigenvalues, and the hatched triangle $\triangle COD$ corresponds to the classical states.

Global indicator $\mathcal{Q}_{[\mathrm{T}^3]}$. The regular stratum $\mathfrak{P}_{[T^3]}$ consists of density matrices with a simple spectrum: $1 > r_1 > r_2 > r_3 \geqslant 0\,,\quad \sum\limits_{i=1}^{3} r_i = 1$. The plane separating classical and quantum states of qutrits,

$$H_3: \quad \pi_1 r_3 + \pi_2 r_2 + \pi_3 r_1 = 0 \tag{49}$$

intersects the partially ordered simplex of qutrit eigenvalues by the straight line passing through the points

$$C = \frac{1}{3\pi_3 - 1}(\pi_3 - 1, \pi_3, \pi_3)\,, \qquad D = \frac{1}{3\pi_1 - 1}(\pi_1, \pi_1, \pi_1 - 1)\,. \tag{50}$$

Hence, the eigenvalues of qutrit classical states belong to WF positivity triangle $\triangle COD$ with the vertices (50) and the vertex of maximally mixed state $\boldsymbol{O} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with triple degeneracy. Note that

$$\begin{aligned}
\min_{\boldsymbol{\pi}} |OD| &= \frac{1}{2\sqrt{6}} \text{ at } \boldsymbol{\pi} = \frac{1}{3}(5, -1, -1),\\
\max_{\boldsymbol{\pi}} |OD| &= \frac{1}{\sqrt{6}} \text{ at } \boldsymbol{\pi} = (1, 1, -1),\\
\min_{\boldsymbol{\pi}} |OC| &= \frac{1}{2\sqrt{6}} \text{ at } \boldsymbol{\pi} = \frac{1}{3}(5, -1, -1),\\
\max_{\boldsymbol{\pi}} |OC| &= \frac{1}{2\sqrt{2}} \text{ at } \boldsymbol{\pi} = (1, 1, -1),
\end{aligned}$$

and the line H_3 is tangent to the disc of the "absolutely" classical states: $r \leqslant \frac{1}{4}$. Here we use the relation $r_1^2 + r_2^2 + r_3^2 = 1/3 + 2/3r^2$ between eigenvalues of a qutrit and its Bloch radius r.

Following the suggested generic scheme, the evaluation of volume of classical states of qutrit reduces to the integration over the WF positivity triangle $\triangle COD$. The integrand of equivalent canonical K_2-simplex is given by a sextic homogeneous polynomial of the following form:

$$I(\boldsymbol{\pi}) \propto \frac{1}{(3\pi_1 - 1)^3(1-3\pi_3)^3} \int\limits_{K_2} dudv\, u^2 v^2 \left(\frac{u}{3\pi_1 - 1} + \frac{v}{1-3\pi_3}\right)^2 . \quad (51)$$

Based on the 2nd Lasserre method (33), we evaluate the associated Bombieri polynomial at point $\boldsymbol{s}_6 = (\frac{2}{8!})^{1/6}\,(1,1)$ and arrive at the following exact expression for the indicator $\mathcal{Q}_3 = I(\boldsymbol{\pi})/I(1,0,0)$:

$$\mathcal{Q}_{[T^3]} = \frac{1}{(3\pi_1 - 1)^3(1-3\pi_3)^3} \times \left[\frac{4}{(3\pi_1-1)^2} + \frac{4}{(1-3\pi_3)^2} + \frac{6}{(3\pi_1-1)(1-3\pi_3)}\right] \quad (52)$$

for all possible SW kernels of qutrit states from the principle stratum.

$\mathcal{Q}_3$-indicator of qutrits from degenerate stratum. The stratum $\mathfrak{P}_{[S(U(2)\times U(1))]}$ has two pieces, associated to density matrices with the degenerate eigenvalues $r_1 = r_2 \neq r_3$ and $r_1 \neq r_2 = r_3$, respectively. Hence, the $\mathcal{Q}_3$-indicator for the degenerate stratum of a qutrit reads:

$$\mathcal{Q}_{[S(U(2)\times U(1))]} = \frac{\mathrm{vol}_{\mathrm{HS}}(\mathfrak{P}^{\mathrm{Cl}}_{1,2} \bigcup \mathfrak{P}^{\mathrm{Cl}}_{2,1})}{\mathrm{vol}_{\mathrm{HS}}\,(\mathfrak{P}_{1,2} \bigcup \mathfrak{P}_{2,1})} . \quad (53)$$

Exploiting the suggested techniques of integration for (53), we obtain:

$$\mathcal{Q}_{[S(U(2)\times U(1))]} = \frac{2^5}{1+2^5}\left(\frac{1}{(3\pi_1-1)^5} + \frac{1}{(1-3\pi_3)^5}\right) . \quad (54)$$

Order relations between indicators. Now we are in position to compare the classicality indicators for different strata. Introducing the ζ-angle parameterization ($\zeta \in [0, \pi/3]$) for the SW kernel eigenvalues

$$\pi_1 = \frac{1}{3} + \frac{2}{\sqrt{3}}\sin\zeta + \frac{2}{3}\cos\zeta\,,\ \pi_2 = \frac{1}{3} - \frac{2}{\sqrt{3}}\sin\zeta + \frac{2}{3}\cos\zeta\,,\ \pi_3 = \frac{1}{3}1 - \frac{4}{3}\cos\zeta\,, \quad (55)$$

one can easily verify the inequalities for the classicality indicators of qutrit:

$$0 < \mathcal{Q}_{[\mathrm{T}^3]} < \mathcal{Q}_{[S(U(2)\times U(1))]} < 1\,. \tag{56}$$

The Fig. 2 demonstrates how the partial order of the corresponding isotropy groups $[\mathrm{T}^3] < [S(U(2) \times U(1))]$ is reproduced at the level of their classicality indicators.

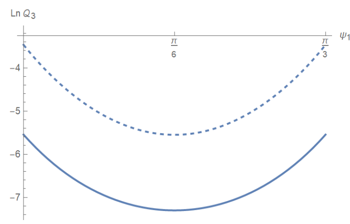


Figure 2. Comparing indicators of qutrit regular (solid curve) and degenerate strata (dashed curve).

4.3. Quatrit.

$\mathcal{Q}_4$-indicator for quatrit regular stratum. The orbit space of quatrit represents tetrahedron and the stratum $\mathfrak{P}_{[\mathrm{T}^4]}$ is given by the density matrices with the regular spectrum, $1 > r_1 > r_2 > r_3 > r_4 \geqslant 0$, $\sum\limits_{i=1}^{4} r_i = 1$. In order to describe the subset of classical states of quatrit, we analyse intersections of 3-simplex with the supporting plane

$$H_4: \ \{\boldsymbol{r}\in\mathcal{C}_3, \boldsymbol{\pi}\in\mathcal{P}_3 \,|\, \pi_1=(\pi_1-\pi_4)r_1+(\pi_1-\pi_3)r_2+(\pi_1-\pi_2)r_3\}, \tag{57}$$

where

$$\mathcal{P}_3: \qquad \sum_{i=1}^{3}\pi_i = 1\,, \quad \sum_{i=1}^{3}\pi_i^2 = 4\,. \tag{58}$$

The possible cross-sections of the plane (57) with the tetrahedron are either a triangular, or a quadrilateral depending on the moduli space $\mathcal{P}_3$. Indeed, one can see that the maximally mixed state $\boldsymbol{r}_* = (1/4, 1/4, 1/4, 1/4)$ has positive WF and the rays emanating from $\boldsymbol{r}_*$ along the edges of tetrahedron intersect the plane H_4 at three points. Then there are two possibilities and hence, only two types of admissible cross-sections:

(A) triangles, if the intersection points belong to edges of the tetrahedron emanating from vertex of maximally mixed states;
(B) quadrilaterals, if an intersection point lies outside the edge of the tetrahedron.

An explicit form of intersection points, taking into account the eigenvalues order $\pi_1 \geqslant \pi_2 \geqslant \pi_3 \geqslant \pi_4$, are:

(1) Intersection with edge OC at point with symmetry $S(U(3)\times U(1))$:

$$\boldsymbol{P}_{OC} = \frac{1}{4\pi_1 - 1}\begin{pmatrix} \pi_1 \\ \pi_1 \\ \pi_1 \\ \pi_1 - 1 \end{pmatrix}, \quad \text{if} \quad \pi_1 \geqslant 1; \tag{59}$$

(2) There is no intersection with the edge AB. The plane H_4 intersects the ray passing through the edge AB:

$$\boldsymbol{P}_{AB} = \frac{1}{\pi_3 - \pi_4}\begin{pmatrix} \pi_3 \\ -\pi_4 \\ 0 \\ 0 \end{pmatrix}; \tag{60}$$

and this point belongs to the edge AB if

$$\pi_3 > \pi_4 \,\&\, \pi_4 < 0 \,\&\, \pi_3 + \pi_4 > 0 \,\&\, \pi_3 > 0 \,,$$

but these conditions never hold.

(3) Intersection with the edge AC at point with symmetry $U(1) \times SU(2) \times U(1)$:

$$\boldsymbol{P}_{AC} = \frac{1}{1 - \pi_1 - 3\pi_4}\begin{pmatrix} 1 - (\pi_1 + \pi_4) \\ -\pi_4 \\ -\pi_4 \\ 0 \end{pmatrix}, \quad \text{if} \quad \pi_1 \leqslant 1 \,\&\, \pi_4 \leqslant 0; \tag{61}$$

(4) Intersection with edge AO at point with symmetry $S(U(1)\times U(3))$:

$$\boldsymbol{P}_{OA} = \frac{1}{1 - 4\pi_4}\begin{pmatrix} 1 - \pi_4 \\ -\pi_4 \\ -\pi_4 \\ -\pi_4 \end{pmatrix}, \quad \text{if} \quad \pi_4 \leqslant 0; \tag{62}$$

(5) Intersection with the edge OB at point with symmetry $S(U(2)\times U(2))$:

$$\boldsymbol{P}_{OB}=\frac{1}{2((\pi_1+\pi_2)-(\pi_3+\pi_4))}\begin{pmatrix}(\pi_1+\pi_2)\\(\pi_1+\pi_2)\\-(\pi_3+\pi_4)\\-(\pi_3+\pi_4)\end{pmatrix},\ \text{if } \pi_3+\pi_4\leqslant 0; \tag{63}$$

(6) Intersection with the edge BC at point $S(U(2)\times U(1)\times U(1))$:

$$\boldsymbol{P}_{BC}=\frac{1}{\pi_1+3\pi_2-1}\begin{pmatrix}\pi_2\\\pi_2\\(\pi_1+\pi_2)-1\\0\end{pmatrix},\ \text{if } \frac{1}{2}\leqslant\pi_2\leqslant 1\ \&\ \pi_2\leqslant\pi_1\leqslant 1. \tag{64}$$

As we will see below, the **A**-type configurations have either the maximal symmetry groups, $SU(4)$, or sub-maximal, $S(U(1)\times U(3)), S(U(3)\times U(1)$ and $S(U(2)\times U(2))$ respectively, while for the **B**-type configurations, when the cross-section of separating plane with the simplex of quatrit eigenvalues represents a quadrilateral, the isotropy groups are $S(U(1)\times U(2)\times U(1)), S(U(2)\times U(1)^2)$.

WF positivity polytope of A-type. For this class of SW kernels $\pi_1\geqslant 1$ the cross-section WF positivity polytope is a 3-simplex (see Fig. 3). Following the suggested method, in order to compute the H-S volume of classical states, we map the WF positivity simplex – the conv $(\boldsymbol{O},\boldsymbol{P_{OA}},\boldsymbol{P_{OB}},\boldsymbol{P_{OC}})$ – to a canonical 3-simplex K_3. As a result, we arrive at calculation of the following integral:

$$\begin{aligned}I(\boldsymbol{\pi})\propto{}&\frac{1}{32(4\pi_1-1)^3(1-4\pi_4)^3(\pi_1+\pi_2-\pi_3-\pi_4)^3}\\&\times\int\limits_{K_3}du\,dv\,dt\,u^2v^2t^2\left(\frac{u}{4\pi_1-1}+\frac{t}{1-4\pi_4}+\frac{v}{2\,(\pi_1+\pi_2-\pi_3-\pi_4)}\right)^2\\&\times\left(\frac{u}{4\pi_1-1}+\frac{v}{2(\pi_1+\pi_2-\pi_3-\pi_4)}\right)^2\left(\frac{t}{1-4\pi_4}+\frac{v}{2(\pi_1+\pi_2-\pi_3-\pi_4)}\right)^2.\end{aligned}\tag{65}$$

Evaluating then the associated Bombieri polynomial at

$$\boldsymbol{s}_{12}=\left(\frac{6}{15!}\right)^{1/12}(1,1,1)$$

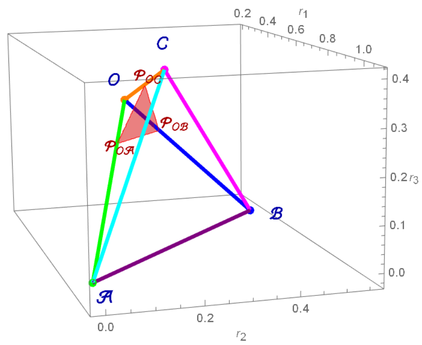


Figure 3. An ordered quatrit 3-simplex with the vertices, $\boldsymbol{O} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \boldsymbol{C} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right), \boldsymbol{B} = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$, and $\boldsymbol{A} = (1, 0, 0, 0)$ and WF positivity simplex $\operatorname{conv}\left(\boldsymbol{O}, \boldsymbol{P_{OA}}, \boldsymbol{P_{OB}}, \boldsymbol{P_{OC}}\right)$.

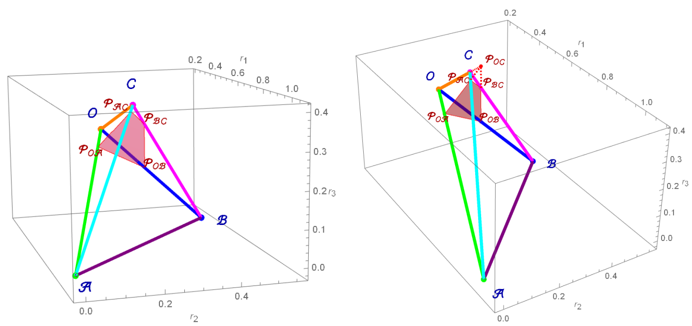
and using the Lasserre formula (33), we arrive at the following expression for (65):

$$
\begin{aligned}
I(\boldsymbol{\pi}) \propto\ & \frac{1}{(4\pi_1 - 1)^3 (1 - 4\pi_4)^3 (\pi_1 + \pi_2 - \pi_3 - \pi_4)^3} \\
\times \Bigg[& \frac{480}{(4\pi_1 - 1)^2 (1 - 4\pi_4)^4} + \frac{480}{(4\pi_1 - 1)^4 (1 - 4\pi_4)^2} + \frac{35}{(\pi_1 + \pi_2 - \pi_3 - \pi_4)^6} \\
& + \frac{105}{(4\pi_1 - 1)(\pi_1 + \pi_2 - \pi_3 - \pi_4)^5} + \frac{105}{(1 - 4\pi_4)(\pi_1 + \pi_2 - \pi_3 - \pi_4)^5} \\
& + \frac{180}{(4\pi_1 - 1)^2 (\pi_1 + \pi_2 - \pi_3 - \pi_4)^4} + \frac{180}{(1 - 4\pi_4)^2 (\pi_1 + \pi_2 - \pi_3 - \pi_4)^4} \\
& + \frac{200}{(4\pi_1 - 1)^3 (\pi_1 + \pi_2 - \pi_3 - \pi_4)^3} \\
& + \frac{540}{(4\pi_1 - 1)(1 - 4\pi_4)^2 (\pi_1 + \pi_2 - \pi_3 - \pi_4)^3} \\
& + \frac{540}{(4\pi_1 - 1)^2 (1 - 4\pi_4)(\pi_1 + \pi_2 - \pi_3 - \pi_4)^3} \\
& + \frac{120}{(4\pi_1 - 1)^4 (\pi_1 + \pi_2 - \pi_3 - \pi_4)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{120}{(1-4\pi_4)^{4}\,(\pi_1+\pi_2-\pi_3-\pi_4)^{2}}\\
&+\frac{912}{(4\pi_1-1)^{2}\,(1-4\pi_4)^{2}\,(\pi_1+\pi_2-\pi_3-\pi_4)^{2}}\\
&+\frac{360}{(4\pi_1-1)\,(1-4\pi_4)^{4}\,(\pi_1+\pi_2-\pi_3-\pi_4)}\\
&+\frac{960}{(4\pi_1-1)^{2}\,(1-4\pi_4)^{3}\,(\pi_1+\pi_2-\pi_3-\pi_4)}\\
&+\frac{960}{(4\pi_1-1)^{3}\,(1-4\pi_4)^{2}\,(\pi_1+\pi_2-\pi_3-\pi_4)}\\
&+\frac{800}{(4\pi_1-1)^{3}\,(1-4\pi_4)^{3}}\\
&+\frac{600}{(4\pi_1-1)\,(\pi_1+\pi_2-\pi_3-\pi_4)^{2}\,(1-4\pi_4)^{3}}\\
&+\frac{200}{(\pi_1+\pi_2-\pi_3-\pi_4)^{3}\,(1-4\pi_4)^{3}}\\
&+\frac{315}{(4\pi_1-1)\,(\pi_1+\pi_2-\pi_3-\pi_4)^{4}\,(1-4\pi_4)}\\
&+\frac{600}{(4\pi_1-1)^{3}\,(\pi_1+\pi_2-\pi_3-\pi_4)^{2}\,(1-4\pi_4)}\\
&\left.+\frac{360}{(4\pi_1-1)^{4}\,(\pi_1+\pi_2-\pi_3-\pi_4)\,(1-4\pi_4)}\right]. \qquad (66)
\end{aligned}
$$

WF positivity polytope of B-type. For the class of SW kernels with $\frac{1}{4} \leqslant \pi_1 < 1$ the cross-section of the separating hyperplane of quatrit with the ordered 3-simplex of eigenvalues represents the quadrilateral which is the base of the WF positivity cone with vertex at maximally mixed state $\boldsymbol{O}$ depicted in Fig. 4. For computation of the H-S volume of WF positivity polytope one can use either its decomposition into simplicies or signed simplices. An example illustrating the signed simplices decomposition is shown in Fig. 4b,

$$
\begin{aligned}
&\mathrm{Vol}\big[\boldsymbol{O}\boldsymbol{C}\boldsymbol{P}_{AC}\boldsymbol{P}_{BC}\boldsymbol{P}_{OB}\boldsymbol{P}_{OA}\big]\\
&\qquad = \mathrm{Vol}\big[\boldsymbol{O}\boldsymbol{P}_{OC}\boldsymbol{P}_{OA}\boldsymbol{P}_{OB}\big] - \mathrm{Vol}\big[\boldsymbol{C}\boldsymbol{P}_{OC}\boldsymbol{P}_{AC}\boldsymbol{P}_{BC}\big]\,. \qquad (67)
\end{aligned}
$$



(a) (b)

Figure 4. The WF positivity 3-polytope formed by a cross-section $(\boldsymbol{P}_{AC}\boldsymbol{P}_{BC}\boldsymbol{P}_{OB}\boldsymbol{P}_{OA})$ of the quatrit simplex and the maximally mixed state.

Using the LA method of computation, we obtain the representation for the classicality indicators in the form of the piecewise rational functions of the SW kernels eigenvalues. Due to the combinatorial complexity, the corresponding expressions are too cumbersome to be written explicitly in the text. However, being interested in comparing the classicality indicators $\mathcal{Q}_4$ for different strata in relations to their symmetry type, we can effectively use these expressions. In the next section we briefly summarize the relevant observations.

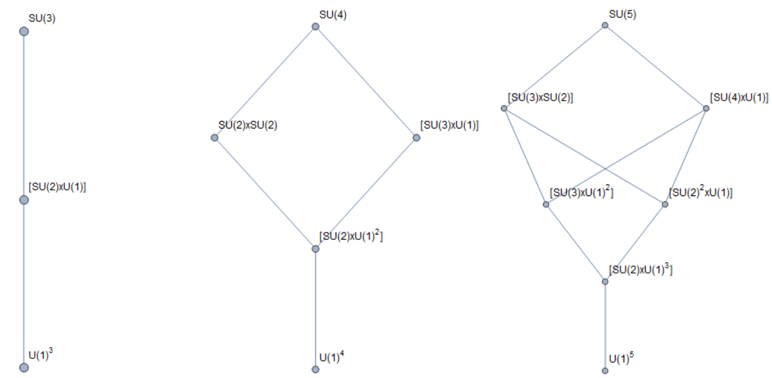
§5. Summary

Our calculations reveal interrelation between hierarchy of quantum states symmetry and their classicality/quantumness which in our opinion deserve a certain attention.

We found that the classicality indicators of qutrit and quatrit for the regular stratum and degenerate strata respect the order of the corresponding isotropy groups in agreement with their Hasse diagram for partially ordered subgroups of unitary groups (Fig. 5).

The curves for qutrit $\mathcal{Q}_3-$indicators in Fig. 2 and the surfaces in Fig. 6 describing quatrit $\mathcal{Q}_4$-indicators for all possible strata as function of a quatrit moduli parameters ψ_1 and ψ_2 [6] illustrate the mentioned hierarchical structure of classicality in relation with the symmetry properties of states.

Making the corresponding slices of $\mathcal{Q}_4$-indicators for the fixed values of the moduli parameter $\psi_2 = \{0, \pi/6, \pi/3\}$ in Fig. 7, we distinctly see that for the groups at the same "level" in Hasse diagram the values of Q-indicators are of the same order (even equal for certain WF representations), otherwise their magnitudes significantly vary.



$N = 3$ $N = 4$ $N = 5$

Figure 5. Hasse diagram for SU(N) group, $N = 3, 4, 5$.

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[6] These quatrit moduli parameters are angles of the Möbius spherical triangle $(2, 3, 3)$ on a unit sphere (cf. [2]).

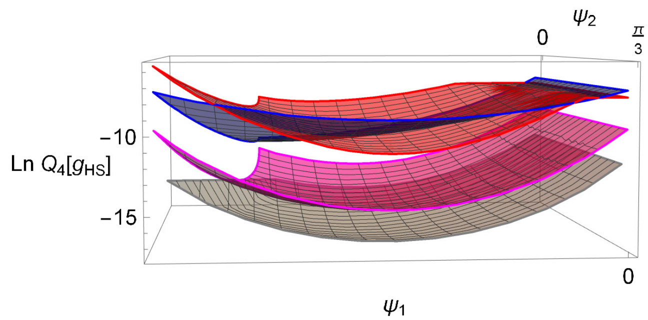

Figure 6. $\mathcal{Q}_4$-indicators for strata of different isotropy types: $\mathcal{Q}_{[S(U(3)\times U(1))]}$ (red surface); $\mathcal{Q}_{[U(1)\times SU(2)\times U(1)]}$ (blue surface); $\mathcal{Q}_{[SU(2)\times U(1)^2]}$ (magenta surface); and regular $\mathcal{Q}_{[T^4]}$-indicator (gray surface).

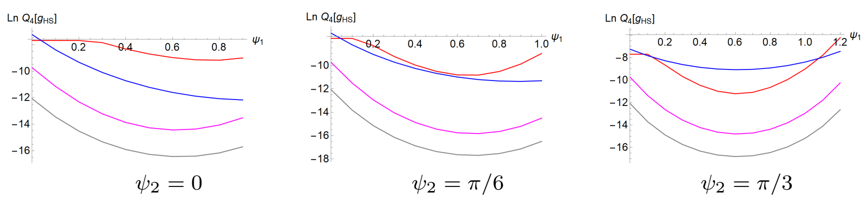

Figure 7. A quatrit classicality indicators for different strata as a function of WF moduli parameter ψ_1 for the fixed values $\psi_2 = \{0, \frac{\pi}{6}, \frac{\pi}{3}\}$.

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