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A. Sakharov

ANALYTIC SEQUENT CALCULI WITH NONLOGICAL AXIOMS

Abstract. This paper investigates a variety of sequent calculi including substructural ones and calculi with equality that can be used for characterizing AI systems. These calculi have introduction inference rules for logical connectives and contain nonlogical axioms. Nonlogical axioms represent domain knowledge. Derivations in these calculi can be restricted to a normal form and to an ordered form. Inference rules are constrained in these forms. It is proved that these calculi are analytic. Infinite branching can be avoided in inference procedures for these calculi.

§1. Introduction

The core of the AI systems known as knowledge bases is domain knowledge [28]. This is knowledge about a concrete set of object domains. Domain knowledge is exhibited via properties of relevant many-sorted functions and predicates. The sort of every argument of such predicate or function as well as the sort of the function range is one of object domains from the set.

In AI systems, domain knowledge is represented as logical programs or as rules and facts of various forms. The most common form of the rules is *if* $A_1, \ldots, A_n$ *then* B, and facts are usually either atoms or other syntactically constrained logical expressions. The rules hold for any values of variables occurring in them. A great deal of research has been devoted to logical characterizations of these AI systems. See [12,26] for example. These characterizations give formal descriptions of otherwise obscure systems and make their results explainable. The results are obtained by recursive application of inference rules to axioms in the respective logical calculi.

The value of logical characterizations is determined by their properties. The efficiency of inference methods is dependent on these properties. Due to the diversity of AI systems, they are regularly categorized by various non-standard calculi. Domain knowledge is mapped to nonlogical axioms

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that are incorporated into the calculi. Models may not be available for these calculi. In this situation, the task of investigating properties of a particular non-standard logic with a variable set of nonlogical axioms is often a complicated research project with an uncertain outcome.

Arguably, sequents are the most common and refined notation in the specification of proof theories. Sequent calculi have been used in formalizations of a variety of logics including non-standard ones. One advantage of sequent calculi is the explicit definition of every logical connective. Sequent calculi are well-suited for analysis and comparison of logics. The majority of the sequent calculi described in the literature possess desirable properties. Nonetheless, these calculi do not facilitate inference methods because of the variety of rule choices at any derivation step and because their axioms and some of their inference rules constitute infinite branching points in derivation search. i.e., an infinite number of choices exists at one step.

We suggest a family of sequent calculi as a framework for representing standard and non-standard logics having AI utility. Logical rules in these calculi are introduction rules for logical connectives. Substructural calculi, i.e., calculi without some structural inference rules, are included in this family. Calculi with equality are included as well. Domain knowledge is expressed by nonlogical axioms in the form of sequents in these calculi. A standard sequent notation of these calculi helps simplify the task of characterizing AI systems.

Our goal is the specification of such framework that certain properties hold for all calculi in the framework. Derivations in these calculi can be restricted to a normal form for which a variant of the subformula property holds, i.e., these calculi are analytic. Inference rules are constrained in the normal form. Furthermore, inference rules are ordered in this form. Contraction and weakening are merged with other inference rules. Infinite branching can be avoided in inference procedures for calculi of this framework.

§2. Sequent Calculi

Terms are expressions built from object constants, object variables, and function constants. Atoms are expressions $p(t_1, \ldots, t_k)$ where p is a predicate constant and $t_1, \ldots, t_k$ are terms. Atoms $p(x_1, \ldots, x_k)$ where $x_1, \ldots, x_k$ are distinct object variables are called basic. The infix form can be used for some standard binary functions and predicates. We consider many-sorted object constants, object variables, functions, and predicates. Logical formulas are built recursively from atoms and logical connectives.

We limit logical connectives to unary and binary. The majority of logics do not have connectives of higher arity. We consider quantifier-free languages because typical AI languages such as logic programming languages and knowledge base languages exclude quantifiers [28]. Languages without quantifiers are comprehensible for a broader audience. Skolem functions [5] serve as an alternative to quantifiers. Skolemization methods have been developed for a range of logics [9]. Quantifiers are problematic for some non-standard logics.

The syntax of the languages of particular calculi could be more restricted. For example, some connective could be applied only to logical expressions of a certain form. A number of calculi related to AI are propositional - they do not include variables. Database languages such as Datalog [28] do not include functions.

A sequent is an expression $\Gamma \vdash \Pi$ where Γ and Π are multisets of formulas, Γ is an antecedent and Π is a succedent. Sequent calculi have axioms and inference rules. Inference rules have one or more premises and one conclusion, each of them is a sequent. In concert with connectives, we limit the number of premises to two. Axioms are basically inference rules with zero premises. Upper-case Latin letters are metavariables denoting formulas in inference rules and logical axioms. Upper-case Greek letters are metavariables denoting formula multisets. The multiset metavariables occur in inference rules only. Sequent calculi include logical axiom $A \vdash A$ or a similar one where A is a formula metavariable.

It is known that standard sequent calculi do not necessarily have an adequate expressiveness for some intricate logics. Sequent calculus extensions such as hypersequents [37] have been developed to address these unusual cases. These various extensions are not covered in this paper. Allowing additional logical axioms also makes it possible to express complicated logics as ordinary sequent calculi. Sequent calculi with multiple logical axioms mix Gentzen-style and Hilbert-style proof systems. Additional logical axioms may compromise important properties of sequent calculi.

A substitution is a finite mapping of object variables to terms. Let $x_1, \ldots, x_k$ be object variables and $t_1, \ldots, t_k$ be terms. The result of applying substitution $\theta = \{x_1/t_1, \ldots, x_k/t_k\}$ to formula A is the expression $A\theta$ obtained from A by simultaneously replacing every occurrence of x_i

by term t_i for $i = 1 \dots k$. $A\theta$ is called an instance of A. The notion of substitution can be extended onto formula multisets.

Nonlogical axioms are sequents containing formulas, no multiset or formula metavariables occur in them. Any nonlogical axiom with variables represents infinitely many sequents. Each of these sequents is an instance of the axiom. A nonlogical axiom is called repetitive if it has an instance with two or more identical formulas in the antecedent or in the succedent.

Usually, the outcome of inference is sequents $\vdash G$ where formula G is called a goal. Unlike goals for theorem provers, goals for AI systems as well as formulas in nonlogical axioms are shallow formulas. A calculus is called consistent if sequent $\vdash$ is not derivable. Inconsistent calculi without nonlogical axioms are not worth investigating but it is acceptable for nonlogical axioms to be the source of inconsistency. Argumentation deals with inconsistent sets of nonlogical axioms [1].

A sequent calculus is called analytic if for any sequent $\int$ derivable from a set of sequents $\mathcal{S}$, this sequent can be derived using the syntactic material available in $\mathcal{S} \cup \int$ [17]. Adapting this informal definition to our setting, calculus analyticity means that whenever sequent $\vdash G$ is derivable, the set of formulas used in its derivation could be restricted to G, its subformulas, and instances of formulas from nonlogical axioms and their subformulas.

Inference rules in sequent calculi are split into structural and logical. The structural rules are essentially universal for all of the calculi whereas logical rules vary. Given the multiset view of antecedents and succedents, the structural rules are weakening, contraction, and cut. Some of these structural rules may be missing in some calculi. We do not consider calculi without cut. The cut rule plays the role of Modus Pones. Withot Modus Pones, nonlogical axioms are useless. In some calculi, the maximum number of formulas in succedents is restricted by one. We do not consider calculi with other constraints on sequents.

Weakening: $$\frac{\Gamma \vdash \Pi}{A, \Gamma \vdash \Pi} LW \qquad \frac{\Gamma \vdash \Pi}{\Gamma \vdash A, \Pi} RW$$

Contraction: $$\frac{A, A, \Gamma \vdash \Pi}{A, \Gamma \vdash \Pi} LC \qquad \frac{\Gamma \vdash A, A, \Pi}{\Gamma \vdash A, \Pi} RC$$

Cut: $$\frac{\Gamma \vdash A, \Delta \quad A, \Pi \vdash \Sigma}{\Gamma, \Pi \vdash \Delta, \Sigma} cut$$

Every formula from the conclusion that is not identical to a formula from a premise is called principal. Every formula from premises that is not identical to a formula from the conclusion is called active. All other formulas are called contexts. Let $\diamond$ denote a unary connective, $\circ$ denote a binary connective. Let $\diamond\Pi$ denote $\{\diamond A | A \in \Pi\}$.

Nonlogical axioms. Nonlogical axioms express properties of concrete predicates and functions that are relevant to a given set of domains. Nonlogical axioms may also serve as predicate definitions including inductive ones. Some nonlogical axioms could be user-defined and are not a part of core domain knowledge. Here are some examples of nonlogical axioms:

$$\begin{array}{rl} t(x,y), t(y,z) \vdash t(x,z) & 0 < x, x < y \vdash g(x) < g(y) \\ \neg(s(x) \prec x) \vdash m(x) & y \prec x \vdash \neg m(x) \\ p(x), x = y \cdot z \vdash y = 1 \vee z = 1 & \vdash x \star (y \star z) = (x \star y) \star z \end{array}$$

where x, y, z are object variables.

All formulas in these examples either are atoms or contain one connective. Succedents containing one formula is a typical format for logic programs and knowledge base rules, but there is no such constraint for nonlogical axioms. The first axiom expresses the transitivity of predicate t. The second axiom states that function g is strictly increasing for positive values. The third and fourth axioms define predicate m. If s is a Skolem function, then m holds for minimal elements of a partially ordered domain. Every Skolem function occurs in one axiom only. The fifth axiom gives a property of prime numbers. The sixth axiom states associativity of binary function $\star$.

In mathematical logic, functions and predicates are uninterpreted symbols. But in AI systems, some functions and predicates could be evaluable [22]. For any evaluable function, there is an algorithm that calculates the value of this function for any given constant arguments. And for any evaluable predicate, there is an algorithm that calculates the truth value of this predicate for constant arguments. Evaluable functions and predicates give rise to implicit nonlogical axioms. For any atom A of an evaluable predicate with arguments that are terms comprised of evaluable functions and object constants, either $\vdash A$ or $A \vdash$ is an implicit nonlogical axiom depending on the truth value of A.

Properties of predicates defined by nonlogical axioms can be inferred without their explicit specification. Suppose predicate t is transitive closure of commutative predicate e.

$$t(x,x) \qquad e(x,y), t(y,z) \vdash t(x,z) \qquad e(x,y) \vdash e(y,x)$$

If $t(a, b)$ is derived by using these nonlogical axioms for some constants a and b, then $t(b, a)$ is derivable too. Cut is the only inference rule needed in these derivations.

Nonlogical axioms can be used to define functions. There is no need to combine the logic language with another one [18] for this purpose. If equality is an evaluable predicate, then the nonlogical axioms below define the factorial function.

$$f(1) = 1 \qquad\qquad \neg(n = 1) \vdash f(n) = f(n-1) \cdot n$$

§3. $\boldsymbol{L_A}$ CALCULI

Definition 1. *A logical inference rule is called an introduction rule if it has one of the following forms and does not have any additional applicability provisos.*

$$\frac{A, \Gamma \vdash \Pi}{\diamond A, \Gamma \vdash \Pi}L1 \quad \frac{A \vdash \diamond\Pi}{\diamond A \vdash \diamond\Pi}LP \quad \frac{\Gamma \vdash A, \Pi}{\Gamma \vdash \diamond A, \Pi}R1 \quad \frac{\diamond\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}RP$$

$$\frac{A, \Gamma \vdash \Pi}{\Gamma \vdash \diamond A, \Pi}F1 \qquad \frac{\Gamma \vdash A, \Pi}{\diamond A, \Gamma \vdash \Pi}B1$$

$$\frac{\Gamma \vdash}{\diamond\Gamma \vdash}LO \quad \frac{\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}RL \quad \frac{\vdash \Pi}{\vdash \diamond\Pi}RO \quad \frac{A \vdash \Pi}{\diamond A \vdash \diamond\Pi}LR$$

$$\frac{A, B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}L2 \quad \frac{\Gamma \vdash A, B, \Pi}{\Gamma \vdash A \circ B, \Pi}R2 \quad \frac{A, \Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}F2 \quad \frac{A, \Gamma \vdash B, \Pi}{A \circ B, \Gamma \vdash \Pi}B2$$

$$\frac{A, \Gamma \vdash \Pi \quad B, \Delta \vdash \Sigma}{A \circ B, \Gamma, \Delta \vdash \Pi, \Sigma}LM \qquad \frac{\Gamma \vdash A, \Pi \quad \Delta \vdash B, \Sigma}{\Gamma, \Delta \vdash A \circ B, \Pi, \Sigma}RM$$

$$\frac{B, \Gamma \vdash \Pi \quad \Delta \vdash A, \Sigma}{\Gamma, \Delta \vdash A \circ B, \Pi, \Sigma}FM \qquad \frac{B, \Gamma \vdash \Pi \quad \Delta \vdash A, \Sigma}{A \circ B, \Gamma, \Delta \vdash \Pi, \Sigma}BM$$

The idea of introduction rules is that every formula from a premise either has a copy in the conclusion or is a subformula of some formula from the conclusion. The choice of rule forms is dictated by interest in the subformula property. No surprise that these forms correspond to the calculi that enjoy cut admissibility in the absence of nonlogical axioms.

This list of introduction inference rules could be expanded. We included only the rules utilized in known logics and not adverse to cut admissability. Besides, we excluded additive rules, i.e., double-premise rules whose premises share the same context. Additive rules are interchangeable with their respective multiplicative counterparts in calculi with both weakening and contraction [23]. Therefore, additive rules can be abandoned in such calculi. Augmenting the list of introduction rules with additive ones would have added plenty of technicalities.

Rules LP and RP impose syntactic restrictions on the context in either the antecedent or the succedent. Rules LO, RL, RO, LR include provisos on the number of formulas in one of the two sides of sequents. It is difficult to guarantee cut admissibility in the presence of rules with context dependencies or with multiple principal formulas in both the antecedent and succedent.

Definition 2. *A sequent calculus is called a L_A calculus if it has one logical axiom $A \vdash A$ and possibly nonlogical axioms, the cut rule, possibly the two weakening rules, possibly the two contraction rules, some introduction logical rules, and*

- *for every unary connective $\diamond$, the rules with this connective are limited to one R1 rule and one of L1, LP rules, one RP rule and one L1 rule, one F1 rule and one B1 rule, one RL rule and one of L1, LO rules, or one LR rule and one of R1, RO rules,*
- *for every binary connective $\circ$, the rules with this connective are limited to one R2 rule and one LM rule, one RM rule and one L2 rule, one F2 and one BM rule, or one FM and one B2 rule.*

The object of this investigation is sets (families) of L_A calcili in which structural and logical inference rules are fixed and every calculus in the set has its own set of nonlogical axioms. Any calculus in a set corresponds to a finite set of domains, predicates and functions are determined by the domains. Basically, such calculus set corresponds to a logic for a variety of domains.

The quantifier-free fragments of classical and intuitionistic first-order logics are L_A calculi. The size of succedents is limited by one in the latter calculus. There are some non-standard logics that cannot be specified by L_A calculi. One example of those is temporal logics. Their sequent calculi include logical rules which are not introduction rules [24]. Single-succedent calculi may include the $R2^+$ and $R2^*$ rules instead of R2 (see section Weakening-Free Form below). For brevity, these other rules are not included in L_A calculi. They are treated similarly to $R2$, and they are considered later.

Examples of $\boldsymbol{L_A}$ calculi. 1. The set of connectives of multiplicative linear logic is comprised of linear negation $^\perp$, multiplicative conjunction $\otimes$, and multiplicative disjunction $\oplus$ [4]. Multiplicative linear logic does not have the weakening and contraction rules. It is a propositional logic. The two $^{\perp}$ rules are similar to the classical negation rules [32]. The conjunction and disjunction rules are

$$\frac{A, B, \Gamma \vdash \Pi}{A \otimes B, \Gamma \vdash \Pi} L\otimes \qquad \frac{\Gamma \vdash A, \Pi \;\; \Delta \vdash B, \Sigma}{\Gamma, \Delta \vdash A \otimes B, \Pi, \Sigma} R\otimes$$
$$\frac{B, \Gamma \vdash \Pi \;\; B, \Delta \vdash \Sigma}{L \oplus B, \Gamma, \Delta \vdash \Pi, \Sigma} L\oplus \qquad \frac{\Gamma \vdash A, B, \Pi}{\Gamma \vdash A \oplus B, \Pi} R\oplus$$

2. Classical negation $\neg$ is the only connective of the calculi LK_{-c} of evaluable non-Horn knowledge bases [29]. The language of LK_{-c} is limited to literals, that is, atoms and their negations. LK_{-c} do not include the contraction rule. The classical negation rules are the only logical rules of LK_{-c}.

3. Modal logic $S4$ is an extension of classical propositional logic. The modal connectives are $\Box$ and $\Diamond$. Symbol $\Diamond$ is an abbreviation for $\neg\Box\neg$. $S4$ includes inference rules of classical propositional logic. Additional inference rules are [37]:

$$\frac{A, \Gamma \vdash \Pi}{\Box A, \Gamma \vdash \Pi} L\Box \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A} R\Box$$

Modal logic $S5$ can be modeled by $S4$. It is known that A is provable in $S5$ if and only if $\Box\Diamond\Box A$ is provable in $S4$ [11].

4. Standard deontic logic is also an extension of classical propositional logic. It has one additional connective: $\mathcal{O}$ (obligation). Permission connective $\mathcal{P}$ is defined as $\mathcal{P} = \neg\mathcal{O}\neg$. Standard deontic logic has two additional inference rules [1].

$$\frac{\Gamma \vdash A}{\mathcal{O}\Gamma \vdash \mathcal{O}A} D\mathcal{O} \qquad \frac{\Gamma \vdash}{\mathcal{O}\Gamma \vdash} L\mathcal{O}$$

The $D\mathcal{O}$ rule is identical to the KR rule from [1] but the $L\mathcal{O}$ rule is different from the DR rule from that paper because the DR rule does not fit the definition of introduction rules. It is easy to prove that our two rules can be derived from the KR and DR rules and vice versa.

§4. Contraction-free form

Let $[\Gamma]$ denote the result of applying zero or more possible contractions to multiset Γ. If a calculus does not include contraction, then $[\Gamma] = \Gamma$. If a calculus includes both weakening and contraction, then the $[\ ]$ operation eliminates all duplicate formulas. If a calculus includes contraction and does not include weakening, then this operation is non-deterministic, i.e., none, some, or all contractions are applied.

Let us modify the conclusion of cut and all logical rules by applying [ ] to both the antecedent and the succedent. For instance, cut and BM become

$$\frac{\Gamma \vdash A, \Delta \quad A, \Pi \vdash \Sigma}{[\Gamma, \Pi] \vdash [\Delta, \Sigma]} cut \qquad \frac{A, \Gamma \vdash \Pi \quad \Delta \vdash B, \Sigma}{[A \circ B, \Gamma, \Delta] \vdash [\Pi, \Sigma]} BM$$

Definition 3. *The calculi obtained from L_A by applying [ ] to both the antecedent and the succedent of the conclusion of cut and all logical inference rules are called L'_A.*

Proposition 1. *For any L_A calculus and its L'_A counterpart, any L_A derivation can be transformed into a L'_A derivation with the same endsequent and vice versa.*

Proof. If the two calculi in question do not have contraction, then their inference rules are identical. Otherwise, any L'_A derivation can be turned into a L_A derivation because the rules with [ ] in the conclusions correspond to the respective original rules combined with contractions. The logical rules and cut in L_A are modeled by the respective L'_A rules followed by weakenings when necessary. If the L_A calculus does not have the weakening rules, then cut and logical rules in L_A are modeled by the variants of the respective L'_A rules that do not apply contractions. □

Proposition 2. *The contraction rules are admissible in L'_A derivations for calculi whose nonlogical axioms are not repetitive.*

Proof. Consider a L'_A derivation and a topmost contraction in this derivation. It cannot follow a nonlogical axiom because these axioms are non-repetitive. If the L'_A calculus does not include weakenings, then this contraction can be merged into the preceding cut or logical rule.

If the L'_A calculus includes weakenings, then the premise of this contraction can be neither cut nor a logical rule because the [ ] operation removes all duplicate formulas. If the premise of this contraction is the conclusion of a weakening rule and the contraction formula is the principal formula of the weakening rule, then these two rules can be dropped. If the contraction formula is not the principal formula of the preceding weakening rule, then either the following permutation is done or one of similar permutations for the three other cases.

$$\dfrac{\dfrac{A, A, \Delta \vdash B, \Pi}{A, A, \Delta \vdash B, C, \Pi}\,RW}{A, \Gamma \vdash B, C, \Pi}\,LC \quad \rightarrow \quad \dfrac{\dfrac{A, A, \Delta \vdash B, \Pi}{A, \Gamma \vdash B, \Pi}\,LC}{A, \Gamma \vdash B, C, \Pi}\,RW$$

Each of these permutations reduces the size of the subderivation ending in this contraction, i.e., the number of rules in the subderivation. This permutation is repeated until its premise is the conclusion of weakening whose principal formula is the contraction formula. By induction on the size of the subderivation, the topmost contraction is eliminated. By induction on the number of contractions, all of them can be eliminated. □

§5. Normal form

Theorem 1. *(normal form for L_A) For a consistent L'_A calculus with nonrepetitive nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into such derivation with the same endsequent and without contractions that the following holds:*

1) (*subformula property*) *Every formula in the derivation is G, its subformula, or an instance of a formula from a nonlogical axiom or its subformula.*
2) *Every cut formula is an instance of a formula from a nonlogical axiom.*
3) *The conclusion of every weakening is the premise of L2, R2, F2, B2 rule in which the weakening formula is active.*

Proof. Consider a derivation of $\vdash G$ without contractions. First, we move weakening rules down so that the condition (3) is satisfied. The proof is by induction on the total number of cuts and logical rules in the derivation. Look at a bottommost weakening rule not satisfying this condition. If this rule is followed by another weakening, then the two rules can be permuted. If this weakening rule is followed by a logical rule in which the weakening formula is not active, then the weakening rule can be permuted with the logical rule. If this weakening rule is followed by cut and the weakening formula is not the cut formula, then the weakening rule is permuted with the cut. Below, we cover two other cases.

1. The conclusion of this weakening is cut and the weakening formula is the cut formula. Suppose $\Gamma \vdash \Pi$ is the premise of this weakening. If the calculus does not have contraction, then the two rules can be replaced by a series of weakenings. If the calculus has contraction, then duplicate formulas in Γ or Π could only be created by weakenings above this one

and below the closest preceding cut or logical rule. Let us remove these weakenings creating duplicates. The premise of this weakening becomes $[\Gamma] \vdash [\Pi]$. After that, the cut rule is replaced by weakenings.

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma \vdash \Pi}}{\Gamma \vdash A, \Pi}\,RW \qquad A, \Delta \vdash \Sigma}{[\Gamma, \Delta] \vdash [\Pi, \Sigma]}\,cut \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\cdots}{[\Gamma] \vdash [\Pi]}}{\cdots}}{[\Gamma, \Delta] \vdash [\Pi, \Sigma]}$$

The symmetrical case is similar.

2. The weakening formula is an active formula in the following logical rule except L2, R2, F2, B2.

$$\dfrac{\dfrac{\Gamma \vdash}{\Gamma \vdash A}\,LW}{[\diamond\Gamma] \vdash \diamond A}\,RL \quad \rightarrow \quad \dfrac{\dfrac{\Gamma \vdash}{[\diamond\Gamma] \vdash}\,LO}{[\diamond\Gamma] \vdash \diamond A}\,LW$$

$$\dfrac{\dfrac{\Gamma \vdash B}{A, \Gamma \vdash B}\,LW}{[\diamond A, \diamond\Gamma] \vdash \diamond B}\,RL \quad \rightarrow \quad \dfrac{\dfrac{\Gamma \vdash B}{[\diamond\Gamma] \vdash \diamond B}\,RL}{[\diamond A, \diamond\Gamma] \vdash \diamond B}\,LW$$

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma \vdash \Pi}}{\Gamma \vdash A, \Pi}\,RW}{[\Gamma] \vdash [\diamond A, \Pi]}\,R1 \quad \rightarrow \quad \dfrac{\dfrac{\cdots}{[\Gamma] \vdash [\Pi]}}{[\Gamma] \vdash [\diamond A, \Pi]}\,RW$$

$$\dfrac{\Gamma \vdash A, \Pi \qquad \dfrac{\dfrac{\cdots}{\Delta \vdash \Sigma}}{\Delta \vdash B, \Sigma}\,RW}{[\Gamma, \Delta] \vdash [A \circ B, \Pi, \Sigma]}\,RM \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\cdots}{[\Delta] \vdash [\Sigma]}}{\cdots}}{[\Gamma, \Delta] \vdash [A \circ B, \Pi, \Sigma]}$$

If LW from the second transformation creates a duplicate formula in a calculus with contraction, then this LW rule is dropped. The last two transformations are applicable to calculi with contraction. See an explanation in case 1. If the calculus does not have contraction, then the pair of rules RW, R1 is replaced by one weakening rule and the pair of rules RW, RM is replaced by a series of weakenings. All the remaining cases from the same category are similar. They are left to the reader.

The outcome of each of the aforementioned transformations is one of the following: the weakening rule in question is permuted with the following rule; the weakening rule is dropped; the weakening rule, possibly preceding weakening rules, and the following cut or logical rule are replaced by one or more weakening rules. No new cuts or logical rules are created by these transformations. If a cut rule or logical rule is eliminated, then the derivation can be transformed to one satisfying the condition (3) by the induction assumption.

Now suppose no cut or logical rule is eliminated. If this weakening was permuted with another weakening, then the latter was followed by a L2, R2, F2, or B2 rule and the other weakening formula is active in the logical rule. Hence, the weakening formula is not active in that logical rule and the two rules are permuted as well. The condition (3) is restored for the other weakening rule after that. The distance between the weakening rule and the endsequent is reduced by two. If this weakening was permuted with a cut or logical rule, then the distance between the weakening rule and the endsequent is reduced by one. In all these cases, no new weakening rules are generated, the condition (3) for other weakening rules is not affected.

A weakening rule cannot be the last rule in the derivation of $\vdash G$ because sequent $\vdash$ is not derivable in consistent calculi. By induction on the distance between the weakening rule and the endsequent, either this weakening will be dropped or the conclusion of this weakening will become the premise of such L2, R2, F2, B2 rule that the weakening formula is active in it. The number of weakening rules not satisfying the condition (3) is reduced by one. By induction on the number of such weakenings, all of them can be eliminated provided that the total number of cuts and logical rules does not change.

Next, look at a topmost cut rule not satisfying the condition (2) of this theorem. If a nonlogical axiom is a premise of the cut, then the condition (2) is satisfied. If one premise is the logical axiom $A \vdash A$, then the application of cut has no effect and can be dropped. Since the condition (3) is satisfied, a weakening rule cannot be a premise of this cut. Below, we cover all other cases.

1. One premise is the conclusions of a logical rule whose principal formula is not the cut formula. The cut can be permuted with the logical rule. Such permutations for rules L2, R2, F2, BM are presented in [32]. The use of cut and the [ ] operation instead of the mix rule actually simplifies the permutations. Permutations for rules L1, R1, B2, LM, RM, FM are examined below. The cases of the right premise being the conclusion of RP and the left premise being the conclusion of LP are analyzed later. All formulas are principal in the conclusion of RL, LO, LR, RO rules. To fit the permutations on a page, we leave out the cut rule name and the [ ] notation.

$$\dfrac{\Gamma \vdash C, \Pi \quad \dfrac{A, C, \Delta \vdash \Sigma}{\diamond A, C, \Delta \vdash \Sigma}\, L1}{\Gamma, \diamond A, \Delta \vdash \Pi, \Sigma} \quad \to \quad \dfrac{\dfrac{\Gamma \vdash C, \Pi \quad A, C, \Delta \vdash \Sigma}{\Gamma, A, \Delta \vdash \Pi, \Sigma}}{\Gamma, \diamond A, \Delta \vdash \Pi, \Sigma}\, L1$$

The symmetrical case for $L1$ is similar and so are the two cases for $R1$.

$$\dfrac{\Gamma \vdash C, \Pi \quad \dfrac{A, C, \Delta \vdash B, \Sigma}{A \circ B, C, \Delta \vdash \Sigma}\,B2}{\Gamma, A \circ B, \Delta \vdash \Pi, \Sigma} \quad \to \quad \dfrac{\dfrac{\Gamma \vdash C, \Pi \quad A, C, \Delta \vdash B, \Sigma}{\Gamma, A, \Delta \vdash \Pi, B, \Sigma}}{\Gamma, A \circ B, \Delta \vdash \Pi, \Sigma}\,B2$$

If the B2 rule is preceeded by a weakening rule, then the following transformation is done.

$$\dfrac{\Gamma \vdash C, \Pi \quad \dfrac{\dfrac{A, C, \Delta \vdash \Sigma}{A, C, \Delta \vdash B, \Sigma}\,RW}{A \circ B, C, \Delta \vdash \Sigma}\,B2}{\Gamma, A \circ B, \Delta \vdash \Pi, \Sigma} \quad \to \quad \dfrac{\dfrac{\dfrac{\Gamma \vdash C, \Pi \quad A, C, \Delta \vdash \Sigma}{\Gamma, A, \Delta \vdash \Pi, \Sigma}}{\Gamma, A, \Delta \vdash \Pi, B, \Sigma}\,RW}{\Gamma, A \circ B, \Delta \vdash \Pi, \Sigma}\,B2$$

The symmetrical cases for the two above permutations are similar. Permutations for L2, R2, F2 rules and possibly preceeding weakenings are similar to the above permutations for B2.

$$\dfrac{\dfrac{\Gamma \vdash A, C, \Pi \quad \Delta \vdash B, \Sigma}{\Gamma, \Delta \vdash A \circ B, C, \Pi, \Sigma}\,RM \quad C, \Psi \vdash \Omega}{\Gamma, \Delta, \Psi \vdash A \circ B, \Pi, \Sigma, \Omega} \quad \to \quad \dfrac{\dfrac{\Gamma \vdash A, C, \Pi \quad C, \Psi \vdash \Omega}{\Gamma, \Psi \vdash A, \Pi, \Omega} \quad \Delta \vdash B, \Sigma}{\Gamma, \Psi, \Delta \vdash A \circ B, \Pi, \Omega, \Sigma}\,RM$$

$$\dfrac{\dfrac{\Gamma \vdash A, C, \Pi \quad \Delta \vdash B, C, \Sigma}{\Gamma, \Delta \vdash A \circ B, C, \Pi, \Sigma}\,RM \quad C, \Psi \vdash \Omega}{\Gamma, \Delta, \Psi \vdash A \circ B, \Pi, \Sigma, \Omega} \quad \to \quad \dfrac{\dfrac{\Gamma \vdash A, C, \Pi \quad C, \Psi \vdash \Omega}{\Gamma, \Psi \vdash A, \Pi, \Omega} \quad \dfrac{\Delta \vdash B, C, \Sigma \quad C, \Psi \vdash \Omega}{\Delta, \Psi \vdash B, \Sigma, \Omega}}{\Gamma, \Psi, \Delta \vdash A \circ B, \Pi, \Omega, \Sigma}\,RM$$

The last permutation applies to calculi with contraction. In calculi without contraction, the first of the two permutations is done. The symmetrical case for RM is similar. Permutations for LM and FM are left to the reader because they are similar to the above.

2. Both premises are the conclusions of logical rules and the cut formula is principal in both of the logical rules. Again, the cut rule can be permuted with the logical rules. Below, we present only the cases that are different from the standard ones [23, 32].

$$\dfrac{\dfrac{\Gamma \vdash A, \Pi}{\Gamma \vdash \diamond A, \Pi}\,R1 \quad \dfrac{A, \Delta \vdash \Sigma}{\diamond A, \Delta \vdash \Sigma}\,L1}{\Gamma, \Delta \vdash \Pi, \Sigma} \quad \to \quad \dfrac{\Gamma \vdash A, \Pi \quad A, \Delta \vdash \Sigma}{\Gamma, \Delta \vdash \Pi, \Sigma}$$

$$\dfrac{\dfrac{\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\,RL \quad \dfrac{A, \Delta \vdash \Sigma}{\diamond A, \Delta \vdash \Sigma}\,L1}{\diamond\Gamma, \Delta \vdash \Sigma} \quad \to \quad \dfrac{\dfrac{\dfrac{\Gamma \vdash A \quad A, \Delta \vdash \Sigma}{\Gamma, \Delta \vdash \Sigma}}{\cdots}}{\diamond\Gamma, \Delta \vdash \Sigma}\,L1$$

$$\dfrac{\dfrac{\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\,RL \quad \dfrac{A, \Delta \vdash B}{\diamond A, \diamond\Delta \vdash \diamond B}\,RL}{\diamond\Gamma, \diamond\Delta \vdash \diamond B} \quad \to \quad \dfrac{\dfrac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}}{\diamond\Gamma, \diamond\Delta \vdash \diamond B}\,RL$$

$$\dfrac{\dfrac{\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\,RL \qquad \dfrac{A, \Delta \vdash}{\diamond A, \diamond\Delta \vdash}\,LO}{\diamond\Gamma, \diamond\Delta \vdash} \quad \rightarrow \quad \dfrac{\dfrac{\Gamma \vdash A \quad A, \Delta \vdash}{\Gamma, \Delta \vdash}}{\diamond\Gamma, \diamond\Delta \vdash}\,LO$$

The cases for the pairs $R1/LR$, LR/LR, RO/LR are symmetrical to the three last permutations.

$$\dfrac{\dfrac{\diamond\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\,RP \qquad \dfrac{A, \Delta \vdash \Sigma}{\diamond A, \Delta \vdash \Sigma}\,L1}{\diamond\Gamma, \Delta \vdash \Sigma} \quad \rightarrow \quad \dfrac{\diamond\Gamma \vdash A \quad A, \Delta \vdash \Sigma}{\diamond\Gamma, \Delta \vdash \Sigma}$$

The case for the pair $R1/LP$ is symmetrical to the above.

$$\dfrac{\dfrac{\Delta \vdash A, \Sigma \quad \Psi \vdash B, \Omega}{\Delta, \Psi \vdash A \circ B, \Sigma, \Omega}\,RM \qquad \dfrac{A, B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\,L2}{\Gamma, \Delta, \Psi \vdash \Pi, \Sigma, \Omega} \quad \rightarrow \quad \dfrac{\Psi \vdash B, \Omega \qquad \dfrac{\Delta \vdash A, \Sigma \quad A, B, \Gamma \vdash \Pi}{B, \Gamma, \Delta \vdash \Pi, \Sigma}}{\Gamma, \Delta, \Psi \vdash \Pi, \Sigma, \Omega}$$

If the L2 rule is preceeded by a weakening rule, then the following transformation is done.

$$\dfrac{\dfrac{\Delta \vdash A, \Sigma \quad \Psi \vdash B, \Omega}{\Delta, \Psi \vdash A \circ B, \Sigma, \Omega}\,RM \qquad \dfrac{\dfrac{A, \Gamma \vdash \Pi}{A, B, \Gamma \vdash \Pi}\,LW}{A \circ B, \Gamma \vdash \Pi}\,L2}{\Gamma, \Delta, \Psi \vdash \Pi, \Sigma, \Omega} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\Delta \vdash A, \Sigma \quad A, \Gamma \vdash \Pi}{\Gamma, \Delta \vdash \Pi, \Sigma}}{\cdots}}{\Gamma, \Delta, \Psi \vdash \Pi, \Sigma, \Omega}$$

The case that A is the weakening formula is similar. The case for the pair R2/LM is similar to the above. The case for the pair B2/FM is similar to one for the pair F2/BM [32]. If R2, F2, B2 from one of these pairs is preceded by a weakening rule, then the pair is permuted along with the weakening rule like it is done for the pair RM/L2.

3. One premise is the conclusions of another cut. Note that the upper cut formula is an instance of a formula from a nonlogical axiom. We can permute these two cuts.

$$\dfrac{\dfrac{\Gamma \vdash \Pi, A \quad A, \Delta \vdash B, \Sigma}{\Gamma, \Delta \vdash B, \Pi, \Sigma} \qquad B, \Psi \vdash \Omega}{\Gamma, \Delta, \Psi \vdash \Pi, \Sigma, \Omega} \quad \rightarrow \quad \dfrac{\Gamma \vdash \Pi, A \qquad \dfrac{A, \Delta \vdash B, \Sigma \quad B, \Psi \vdash \Omega}{A, \Delta, \Psi \vdash \Sigma, \Omega}}{\Gamma, \Delta, \Psi \vdash \Pi, \Sigma, \Omega}$$

$$\dfrac{\Gamma \vdash B, \Pi \qquad \dfrac{B, \Delta \vdash A, \Sigma \quad B, A, \Psi \vdash \Omega}{B, \Delta, \Psi \vdash \Sigma, \Omega}}{\Gamma, \Delta, \Psi \vdash \Pi, \Sigma, \Omega} \quad \rightarrow \quad \dfrac{\dfrac{\Gamma \vdash B, \Pi \quad B, \Delta \vdash A, \Sigma}{\Gamma, \Delta \vdash A, \Pi, \Sigma} \qquad \dfrac{\Gamma \vdash B, \Pi \quad B, A, \Psi \vdash \Omega}{A, \Gamma, \Psi \vdash \Pi, \Omega}}{\Gamma, \Psi, \Delta \vdash \Pi, \Omega, \Sigma}$$

The second transformation is applicable to calculi with contraction. Otherwise, only one upper cut is needed like in the first of the two above permutations. The other symmetrical cases are similar. After these permutations, the condition (2) still holds for the new lower cut.

4. The right premise is the conclusion of RP. If the left premise is the conclusion of another cut, these two cuts are permuted. The left premise

cannot be the conclusion of the LP rule by the definition of L_A. If the left premise is the conclusion of any other logical rule and the cut formula is not principal in the logical rule, then the two rules are permuted. If the cut formula is principal, then the other rule must be RP.

$$\dfrac{\dfrac{\diamond\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\,RP \quad \dfrac{\diamond A, \diamond\Delta \vdash B}{\diamond A, \diamond\Delta \vdash \diamond B}\,RP}{\diamond\Gamma, \diamond\Delta \vdash \diamond B} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\diamond\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\,RP \quad \diamond A, \diamond\Delta \vdash B}{\diamond\Gamma, \diamond\Delta \vdash B}}{\diamond\Gamma, \diamond\Delta \vdash \diamond B}\,RP$$

The case of the left premise of this cut being the conclusion of LP is similar.

In all cases, one of the following is achieved by each of these transformations: the cut rule is eliminated, the size of the cut formula is reduced, the size of the subderivation ending in this cut is reduced. If two cuts are created by a permutation, then the size of both cut formulas is reduced or the size of both subderivations is reduced. The condition (3) is still satisfied for the subderivation after these transformations. By double induction on the size of the cut formula and the size of the subderivation, this cut can be eliminated or replaced by one or more cut rules having a nonlogical axiom as a premise.

These transformations do not affect the condition (2) for the other cut rules including the ones involved in permutations in case 3. Hence, the number of cuts violating the condition (2) is reduced by one after the completion of the transformations related to the cut rule in question. The condition (3) could be violated because of series of weakening rules created during transformations in case 2. The procedure for attaining the condition (3) is applied to the modified derivation. This procedure does not generate new cut rules and it does not change cut formulas. Hence, this procedure does not increase the number of cuts violating the condition (2).

All the above transformations related to the condition (2) can be repeated for any topmost cut not satisfying the condition (2). By induction on the number of the cut rules violating the condition (2) in the derivation of $\vdash G$, all of them can be eliminated, and the condition (3) is restored. The condition (1) is a corollary of the condition (2). For the calculi in which the length of succedents is limited by one, none of the transformations considered in the proof produces succedents that are longer than one. $\square$

Theorem 1 establishes that L'_A calculi are analytic.

§6. Equality

Equality plays an important role in nearly all knowledge-based systems. In sequent calculi, the following axioms specify equality [23].

$$x = x \qquad\qquad y = z, A\{x/y\} \vdash A\{x/z\}$$

The first axiom is a nonlogical axiom. The second equality axiom specifies a property of a concrete predicate but this property applies to all formulas. This axiom should be qualified as logical since it contains a formula metavariable. In the remainder of the paper, the equality axiom will refer to the second axiom.

The equality axioms make it possible to infer properties of functions defined by nonlogical axioms. For instance, if h is defined as composition of two strictly decreasing functions f and g, then $h(a) < h(b)$ is derivable for any such constants a and b that $a < b$ provided that equality is evaluable.

$$x < y \vdash f(x) > f(y) \qquad x < y \vdash g(x) > g(y) \qquad \vdash h(x) = f(g(x))$$

Definition 4. *The L_A (L'_A) calculi with weakening and contraction, without LP, RP, LO, RL, RO, LR rules, and extended with the two equality axioms will be called $L^=_A$ ($L'^=_A$) calculi.*

Clearly, Proposition 1 and Proposition 2 hold for $L'^=_A$ calculi too.

Proposition 3. *The equality axiom is admissible for non-atomic formulas in $L^=_A$ ($L'^=_A$) calculi.*

Proof. The proof for standard sequent calculi is well-known. It is by induction on the structure of non-atomic formulas. The cases for F1, B1, L2, R2, F2, LM, RM, BM rules are covered in the literature. The cases for other $L^=_A$ rules are similar. The respective proofs apply to $L'^=_A$ too. $\square$

Theorem 2. *(normal form for $L^=_A$) For a consistent $L'^=_A$ calculus $\mathcal{C}$ with non-repetitive nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into such derivation with the same endsequent and without contractions that the following conditions hold:*

1) (*weak subformula property*) *Every formula in the derivation is G, its subformula, an instance of a basic atom, or an instance of a formula from a nonlogical axiom or its subformula.*
2) *Every cut formula is an instance of a basic atom or an instance of a formula from a nonlogical axiom.*
3) *The conclusion of every weakening is the premise of L2, R2, F2, B2 rule in which the weakening formula is active.*

Proof. Consider a derivation of $\vdash G$ without contractions. Given that there is a finite number of predicates and constants in the nonlogical axioms of $\mathcal{C}$, the equality axiom is replaced by a finite set of the following axioms

$$x_1 = y_1, \ldots, x_k = y_k, p(x_1, \ldots, x_k) \vdash p(y_1, \ldots, y_k) \tag{1}$$

$$x_1 = y_1, \ldots, x_k = y_k \vdash f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k) \tag{2}$$

for every predicate p and every function f occurring in the nonlogical axioms. These axioms have the form of nonlogical axioms. This replacement does not change the set of derivable formulas. First, the axiom (1) is derived by k applications of cut to the equality axiom for atom $p(x_1, \ldots, x_k)$. At step i, the following instance of the equality axiom is used:

$$x_i = y_i, p(y_1, \ldots, x_i, \ldots, x_k)\{x_i/x_i\} \vdash p(y_1, \ldots, x_i, \ldots, x_k)\{x_i/y_i\}$$

The axiom (2) is also derived by k applications of cut to the equality axiom. At step i, the following instance of the equality axiom is used:

$$x_i = y_i, f(x_1, \ldots, x_i, \ldots, z_k) = f(y_1, \ldots, z_i, \ldots, z_k)\{z_i/x_i\} \vdash \\ f(x_1, \ldots, x_i, \ldots, z_k) = f(y_1, \ldots, z_i, \ldots, z_k)\{z_i/y_i\}$$

Second, the proof that the equality axiom for any atom $A(t_1, \ldots, t_k)$ is derivable from the axioms (1) and (2) is done by induction on the depth of $A(t_1, \ldots, t_k)$. Clearly, the equality axiom is derivable from the axioms (1) and (2) if all $t_1, \ldots, t_k$ are variables or constants. Suppose the equality axiom holds for atoms of depth n. Let the depth of atom $A(t_1, \ldots, t_k)$ be $n+1$. Look at any term $t_i = f(s_1, \ldots, s_m)$. By the induction assumption, $x = y \vdash s_j\{z/x\} = s_j\{z/y\}$ for $j = 1, \ldots, m$. By the axiom (2)

$$s_1\{z/x\} = s_1\{z/y\}, \ldots, s_m\{z/x\} = s_m\{z/y\} \vdash t_i\{z/x\} = t_i\{z/y\}$$

Clearly, $x = y \vdash t_i\{z/x\} = t_i\{z/y\}$ holds for every t_i which is a variable or constant. By the axiom (1)

$$t_1\{z/x\} = t_1\{z/y\}, \ldots, t_k\{z/x\} \\ = t_k\{z/y\}, A(t_1, \ldots, t_k)\{z/x\} \vdash A(t_1, \ldots, t_k)\{z/y\}$$

Hence, the equality axiom holds for $A(t_1, \ldots, t_k)$.

Since the axioms (1) and (2) are no different than nonlogical axioms, this derivation is transformed to a derivation satisfying all the conditions of Theorem 1. Now, we replace the axioms (1) and (2) with their derivations from the equality axiom for basic atoms. This replacement produces a derivation satisfying all the conditions of this theorem. □

Theorem 2 establishes weak anlyticity of $L'^{=}_{A}$ calculi. Basic atoms may occur in normal-form derivations along with syntactic material from the goal and nonlogical axioms. In general, logical axioms are not reducible to nonlogical axioms. If logical axioms are included in a calculus, then analyticity is lost in cut rules with these axioms as premises.

§7. Weakening-Free Form

Consider the following rules:

$$\frac{A, \Gamma \vdash \Pi}{[A \circ B, \Gamma] \vdash [\Pi]}\ L2^{+} \qquad \frac{B, \Gamma \vdash \Pi}{[A \circ B, \Gamma] \vdash [\Pi]}\ L2^{*} \qquad \frac{\Gamma \vdash A, \Pi}{[\Gamma] \vdash [A \circ B, \Pi]}\ R2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{[\Gamma] \vdash [A \circ B, \Pi]}\ R2^{*}$$

$$\frac{A, \Gamma \vdash \Pi}{[\Gamma] \vdash [A \circ B, \Pi]}\ F2^{+} \qquad \frac{\Gamma \vdash B, \Pi}{[A \circ B, \Gamma] \vdash [\Pi]}\ B2^{+}$$

Definition 5. *The calculi obtained from L'_A ($L'^{=}_A$) calculi with weakening by adding the $L2^+, R2^+, L2^*, R2^*, F2^+, B2^+$ rules for the respective L'_A ($L'^{=}_A$) rules are called L''_A ($L''^{=}_A$). The L'_A ($L'^{=}_A$) calculi without weakening have identical L''_A ($L''^{=}_A$) counterparts.*

Proposition 4. *For any L'_A ($L'^{=}_A$) calculus and its L''_A ($L''^{=}_A$) counterpart, any L'_A ($L'^{=}_A$) derivation can be transformed into a L''_A ($L''^{=}_A$) derivation with the same endsequent and vice versa.*

Proof. Any L'_A ($L'^{=}_A$) derivation is also a L''_A ($L''^{=}_A$) derivation. The $L2^+$, $R2^+$, $L2^*$, $R2^*, F2^+, B2^+$ rules are admissible in calculi with weakening. Any of them can be modeled by a combination of a weakening rule and the respective L2, R2, F2, B2 rule. □

Proposition 5. *For a consistent L''_A ($L''^{=}_A$) calculus with non-repetitive nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into such derivation with the same endsequent and without the weakening rules that the first two conditions of the normal-form theorem are satisfied.*

Proof. First, we transform a given L''_A ($L''^{=}_A$) derivation into a L'_A ($L'^{=}_A$) derivation. The latter derivation is transformed into the normal form. Second, we replace every L2, R2, F2, B2 rule preceded by a weakening rule with the respective $L2^+, R2^+, L2^*, R2^*, F2^+, B2^+$ rule. The outcome of these transformations is a L''_A ($L''^{=}_A$) derivation without weakenings. The first two conditions of the normal-form theorem hold in this derivation. □

§8. Ordered form

Definition 6. *Order relation $\succ$ on formulas is called a simplification order* [14] *if it is:*

– well-founded: there is no infinite sequence of formulas $T_0 \succ T_1 \succ \dots$

– *monotone: if R is a subformula of L and $L \neq R$, then $L \succ R$*
– *stable: if $L \succ R$, then $L\theta \succ R\theta$ for any substitution θ*

Definition 7. *Formula A is maximal (minimal) with respect to the set of formulas Σ if $B \succ A$ ($A \succ B$) does not hold for any other formula $B \in \Gamma$.*

Theorem 3. (*ordered form*) *For any consistent L''_A calculus without LP, RP rules (or $L''^{=}_A$ calculus) and for any simplification order $\succ$, every derivation of $\vdash G$ can be transformed into such derivation with the same endsequent and without the contraction and weakening rules that the first two conditions of the normal-form theorem are satisfied and*

– *every cut formula is maximal with respect to such formulas from its succedent/antecedent that are not G, its subformulas, or instances of proper subformulas of nonlogical-axiom formulas*
– *for any two consecutive rules from the set $\{L1, R1, F1, B1, L2, R2, F2, B2, L2^+, R2^+, L2^*, R2^*, F2^+, B2^+\}$ or any two consecutive rules from the set $\{LM, RM, FM, BM\}$, the principal formula of the lower rule is maximal with respect to the principal formula of the upper rule*
– *if the calculus includes weakening, contraction and does not have LO, RL, RO, LR rules, then for any two consecutive logical rules, the principal formula of the lower rule is maximal with respect to the principal formula of the upper rule*

Proof. Consider a L''_A derivation of $\vdash G$, convert it into the normal form and then eliminate the weakening rules. The case of $L''^{=}_A$ is similar. Let $\mathcal{M}$ be the set of all cut formulas from this derivation and $\mathcal{N}$ be the empty set. All formulas that are not G, its subformulas, or instances of proper subformulas of nonlogical-axiom formulas belong to $\mathcal{M}$. These formulas do not occur in $\vdash G$ and they cannot be active formulas in logical rules, and thus, they must be cut formulas. $\mathcal{M}$ may contain other formulas as well.

Let us pick up a formula A that is maximal in $\mathcal{M}$. Any bottommost occurrence of A is in the two premises of a cut rule as the cut formula. If this cut follows a logical rule in which A is not principal, then this cut is permuted with the preceding logical rule. If one premise is another cut with C as the other cut formula, then the two rules are permuted if $C \notin \mathcal{N}$. Relevant permutations were shown in the proof of Theorem 1. Additional cases involve rules $L2^+, R2^+, L2^*, R2^*, F2^+, B2^+$, for example:

$$\dfrac{\dfrac{B, \Gamma \vdash \Pi, C}{[A \circ B, \Gamma] \vdash [\Pi, C]}\,L2^* \qquad C, \Delta \vdash \Sigma}{[A \circ B, \Gamma, \Delta] \vdash [\Pi, \Sigma]}\,cut \quad \to \quad \dfrac{\dfrac{B, \Gamma \vdash \Pi, C \qquad C, \Delta \vdash \Sigma}{[B, \Gamma, \Delta] \vdash [\Pi, \Sigma]}\,cut}{[A \circ B, \Gamma, \Delta] \vdash [\Pi, \Sigma]}\,L2^*$$

None of these permutations violates the first two conditions of the normal-form theorem.

These cut permutations continue until each premise of the cut of A is a nonlogical axiom or a logical rule in which A is principal. Each of the permutations reduces the size of the subderivation ending with the cut eliminating A. Therefore, this process will terminate. The same transformation is repeated for every other bottommost occurrence of A. After the completion of all these transformations, only immediate predecessors of cuts of A may contain A. All cuts of A satisfy the first condition of this theorem because the set of formulas that are not G, its subformulas, or instances of proper subformulas of nonlogical-axiom formulas is a subset of $\mathcal{M}$.

Now we move A from $\mathcal{M}$ to $\mathcal{N}$ and repeat the same procedure for a maximal element B of $\mathcal{M}$. After this procedure is done for B, any occurrence of such cut formula F that $F \succ B$ is in $\mathcal{N}$, and hence all F occurrences are above the cuts of B. Consequently, all cuts where B is the cut formula satisfy the condition of this theorem after the completion of the procedure for B. This procedure for a maximal element of $\mathcal{M}$ does not violate the first condition of this theorem for elements of $\mathcal{N}$ because their cuts are not transformed by the procedure for B. By induction on the number of elements of $\mathcal{M}$, the transformed derivation satisfies the first condition of this theorem when $\mathcal{M} = \emptyset$.

Look at any longest chain of rules from the set $\{L1, R1, F1, B1, L2, R2, F2, B2, L2^+, R2^+, L2^*, R2^*, F2^+, B2^+\}$. Let us pick up a rule whose principal formula P is maximal among principal formulas in the chain. Suppose this rule is not the lowermost rule in the chain. P cannot be active in the following rule with principal formula Q because $Q \succ P$ in this case by the definition of simplification orders. Otherwise, this rule can be permuted with the following rule.

Below are shown three permutations of this kind. The cases for other pairs of logical rules are similar.

$$\dfrac{\dfrac{A, \Gamma \vdash B, C, \Pi}{[\Gamma] \vdash [A \circ B, C, \Pi]}\, F2}{[\diamond C, \Gamma] \vdash [A \circ B, \Pi]}\, B1 \;\rightarrow\; \dfrac{\dfrac{A, \Gamma \vdash B, C, \Pi}{[\diamond C, A, \Gamma] \vdash [B, \Pi]}\, B1}{[\diamond C, \Gamma] \vdash [A \circ B, \Pi]}\, F2$$

$$\dfrac{\dfrac{A, B, \Gamma \vdash C, D, \Pi}{[A \circ B, \Gamma] \vdash [C, D, \Pi]}\, L2}{[A \circ B, \Gamma] \vdash [C \cdot D, \Pi]}\, R2 \;\rightarrow\; \dfrac{\dfrac{A, B, \Gamma \vdash C, D, \Pi}{[A, B, \Gamma] \vdash [C \cdot D, \Pi]}\, R2}{[A \circ B, \Gamma] \vdash [C \cdot D, \Pi]}\, L2$$

$$\dfrac{\dfrac{D, \Gamma \vdash B, \Pi}{[A \circ B, D, \Gamma] \vdash [\Pi]}\,B2^{+}}{[A \circ B, C \cdot D, \Gamma] \vdash [\Pi]}\,L2^{*} \;\to\; \dfrac{\dfrac{D, \Gamma \vdash B, \Pi}{[C \cdot D, \Gamma] \vdash [B, \Pi]}\,L2^{*}}{[A \circ B, C \cdot D, \Gamma] \vdash [\Pi]}\,B2^{+}$$

These permutations of the two logical rules reduce the distance from the rule in question to the end of the chain. By induction on this distance, this rule can be moved to the lowermost position in the chain. After this transformation, the second condition of this theorem will hold for at least the last two rules in the modified chain. This transformation does not violate any condition of the normal-form theorem.

Now we pick up a rule whose principal formula is maximal in the remaining part of the chain, i.e., excluding the previously selected rule, and repeat the same procedure for this rule until it is followed by the previously selected rule. This adds one more pair of rules at the bottom of the chain for which the second condition holds. By induction on the length of the chain, it can be transformed so that all pairs of logical rules in the chain satisfy the second condition of this theorem. The other chains and other logical rules are not modified by this transformation. By induction on the number of such chains, the derivation of $\vdash G$ can be transformed into another derivation satisfying the second condition for single-premise rules. The first condition is not affected by permutations related to these chains.

Similarly to the case of single-premise rules, look at any largest derivation subtree comprised of rules from the set $\{LM, RM, FM, BM\}$. Let us pick up a rule whose principal formula P is maximal among principal formulas in the subtree. Suppose this rule is not the lowermost in the subtree. Like in the case of single-premise rules, P cannot be active in the following rule. Otherwise, this rule can be permuted with the following rule from the subtree.

Some of these permutations are shown below. The cases for other rule pairs are similar. Again, the [ ] notation is left out in order to fit the permutations on a page.

$$\dfrac{\dfrac{\Gamma \vdash A, \Pi \quad \Delta \vdash B, C, \Sigma}{\Gamma, \Delta \vdash A \circ B, C, \Pi, \Sigma}\,RM \qquad \Psi \vdash D, \Omega}{\Gamma, \Delta, \Psi \vdash A \circ B, C \cdot D, \Pi, \Sigma, \Omega}\,RM \;\to\; \dfrac{\Gamma \vdash A, \Pi \qquad \dfrac{\Delta \vdash B, C, \Sigma \quad \Psi \vdash D, \Omega}{\Delta, \Psi \vdash B, C \cdot D, \Sigma, \Omega}\,RM}{\Gamma, \Delta, \Psi \vdash A \circ B, C \cdot D, \Pi, \Sigma, \Omega}\,RM$$

$$\dfrac{C, \Gamma \vdash \Pi \qquad \dfrac{A, \Delta \vdash D, \Sigma \quad B, \Psi \vdash D, \Omega}{A \circ B, \Delta, \Psi \vdash D, D, \Sigma, \Omega}\,LM}{A \circ B, \Gamma, \Delta, \Psi \vdash C \cdot D, \Pi, \Sigma, \Omega}\,FM \;\to\; \dfrac{\dfrac{C, \Gamma \vdash \Pi \quad A, \Delta \vdash D, \Sigma}{A, \Gamma, \Delta \vdash C \cdot D, \Pi, \Sigma}\,FM \qquad \dfrac{C, \Gamma \vdash \Pi \quad B, \Psi \vdash D, \Omega}{B, \Gamma, \Psi \vdash C \cdot D, \Pi, \Omega}\,FM}{A \circ B, \Gamma, \Delta, \Psi \vdash C \cdot D, \Pi, \Sigma, \Omega}\,LM$$

$$\dfrac{C, \Gamma \vdash \Pi \qquad \dfrac{A, \Delta \vdash D, \Sigma \quad B, \Psi \vdash D, \Omega}{A \circ B, \Delta, \Psi \vdash D, D, \Sigma, \Omega}\,LM}{A \circ B, \Gamma, \Delta, \Psi \vdash C \cdot D, D, \Pi, \Sigma, \Omega}\,FM \;\to\; \dfrac{\dfrac{C, \Gamma \vdash \Pi \quad A, \Delta \vdash D, \Sigma}{A, \Gamma, \Delta \vdash C \cdot D, \Pi, \Sigma}\,FM \qquad B, \Psi \vdash D, \Omega}{A \circ B, \Gamma, \Delta, \Psi \vdash C \cdot D, D, \Pi, \Sigma, \Omega}\,LM$$

The second permutation applies to calculi with contraction, and the third one applies to calculi without contraction.

These permutations of the two logical rules reduce the distance from the rule in question to the root of the subtree. By induction on this distance, this rule can be moved to the lowermost position in the subtree. After this transformation, the second condition of this theorem will hold for at least the two pairs of rules at the bottom of the modified subtree.

Now we pick up a rule whose principal formula is maximal in the remaining part of the subtree, i.e. excluding the previously selected rule, and repeat the same procedure for this rule until it is followed by the previously selected rule. This adds two more pairs of rules at the bottom of the subtree for which the second condition holds. By induction on the size of the subtree, it can be transformed so that all pairs of logical rules in the subtree satisfy the second condition of this theorem. The other subtrees and the chains of single-premise logical rules are not modified by this transformation. By induction on the number of such subtrees, the derivation of $\vdash G$ can be transformed into another derivation satisfying the second condition. The first condition of this theorem and the conditions of the normal-form theorem are not affected by permutations related to double-premise logical rules.

Suppose the calculus includes weakening, contraction and does not have LO, RL, RO, LR rules. Look at any largest derivation subtree comprised of logical rules. Let us pick up a rule whose principal formula P is maximal among principal formulas in the subtree. Suppose this rule is not the lowermost in the subtree. Given that P cannot be active in the following rule, this rule can be permuted with the following rule from the subtree. We need to consider only the pairs combining single-succedent and double succedent rules.

$$\dfrac{\dfrac{\Gamma \vdash \Pi, A, B, C}{\Gamma \vdash \Pi, A \circ B, C}\,R2 \quad \Delta \vdash \Sigma, D}{\Gamma, \Delta \vdash \Pi, \Sigma, A \circ B, C \cdot D}\,RM \quad \to \quad \dfrac{\dfrac{\Gamma \vdash \Pi, A, B, C \quad \Delta \vdash \Sigma, D}{\Gamma, \Delta \vdash \Pi, \Sigma, A, B, C \cdot D}\,RM}{\Gamma, \Delta \vdash \Pi, \Sigma, A \circ B, C \cdot D}\,R2$$

$$\dfrac{\dfrac{B, C, \Gamma \vdash \Pi \quad D, \Delta \vdash \Sigma, A}{C, D, \Gamma, \Delta \vdash \Pi, \Sigma, A \circ B}\,FM}{C \cdot D, \Gamma, \Delta \vdash \Pi, \Sigma, A \circ B}\,L2 \quad \to \quad \dfrac{\dfrac{B, C, \Gamma \vdash \Pi}{B, C \cdot D, \Gamma \vdash \Pi}\,L2^{+} \quad \dfrac{D, \Delta \vdash \Sigma, A}{C \cdot D, \Delta \vdash \Sigma, A}\,L2^{*}}{C \cdot D, \Gamma, \Delta \vdash \Pi, \Sigma, A \circ B}\,FM$$

Permutations of other pairs mixing L2, R2, F2, B2 on one side and LM, RM, FM, BM on the other are similar. Permutations for pairs including L1, R1, F1, B1 are even simpler. All of them are left to the reader. The rest of the proof is similar to the proof for subtrees comprised of LM, RM, FM, BM rules. □

The last condition of Theorem 3 is not applicable to calculi with LO, RL, RO, LR rules because these rules have multiple active and principal formulas.

§9. Inference

We discuss effects of the properties of the normal and other forms on inference in general. As a consequence of incorporating domain knowledge into sequent calculi, unification is adapted as well. The design of particular inference algorithms for L_A and $L_A^=$ calculi including adaptation of existing algorithms is beyond the scope of this paper.

The derivation forms specified earlier give significant constraints for the application of inference rules. The most important are the constraints on formulas in the conclusions of logical inference rules and in the premises of cut. Embedding weakening into logical rules further reduces choices for inference steps. Hence, the derivation search space is smaller for the respective derivations.

Theorem 3 adapts ordered resolution [14] to sequent derivations. It states additional constraints for the cut rule and for certain sequences of logical inference rules. Ordered resolution is considered one of the fastest inference methods for classical first-order logic. For calculi with numerous nonlogical axioms, cut is expected to be the most frequently used inference rule. Having shallow goals and shallow formulas in nonlogical axioms along with the subformula property warrants short chains of consecutive logical rules in derivations. Constraining cuts to ordered ones is expected to have a similar effect on inference in L_A and $L_A^=$ calculi as the effect of ordered resolution steps on first-order logic inference.

Infinite branching can be avoided in normal-form derivations. The instantiation of nonlogical axioms is a potential source of infinite branching. Fortunately, the problem of instantiating nonlogical axioms and the equality axiom for basic atoms is solved by using these axioms 'as is' and by employing unification. Unique object variables are generated for every copy of a nonlogical axiom used in inference.

Since functions and predicates are sorted in the calculi under consideration, sorted unification is utilized. If the sorts form a semilattice, then the most general unifier is unique [36]. In particular, this is the case if the set of sorts (domains) is flat, i.e., all distinct domains are disjoint. Unification algorithms are applicable to quantifier-free first-order formulas because these formulas can be treated as terms whose signature is extended with predicates and logical connectives.

Unification is embedded in cut and in the [ ] operation for calculi with contraction.

$$\frac{\Gamma \vdash A, \Delta \quad B, \Pi \vdash \Sigma}{\Gamma\theta, \Pi\theta \vdash \Delta\theta, \Sigma\theta}\ cut$$

Substitution θ is the most general unifier of A and B [28].

Proposition 6. *Inference with unification embedded in cut and in the* [ ] *operation is sound and complete for any* L''_A $(L''^{=}_A)$ *calculus.*

Proof. It is easy to prove by induction on the depth of derivations that if $\Gamma \vdash \Pi$ is derivable in a L''_A $(L''^{=}_A)$ calculus, then $\Gamma\theta \vdash \Pi\theta$ is derivable as well for any substitution θ. Consider a derivation with embedded unification and a topmost inference rule employing unification. The subderivation whose endsequent is this rule can be modified by applying the respective substitution, which eliminates unification from the rule. By induction on the number of rules with unification in this derivation, it can be transformed into a derivation without unification.

It is well-known that the lifting lemma [5] can be adapted to cut and contraction. Consider a derivation without unification. By inductively applying the lifting lemma to this derivation, it can be transformed into a derivation with embedded unification. □

The choice of formulas in the logical axiom and in the rules which embed weakening is another potential source of infinite branching in inference procedures for sequent calculi. Due to the subformula property of normal-form derivations in L''_A calculi and due to the use of unification, this choice of formulas can be limited in these calculi to the goal formula, its subformulas, formulas from nonlogical axioms and their subformulas. In $L^{=}_A$ calculi, basic atoms are added to this set of formulas. Unique object variables are generated for every copy of a formula or subformula from a nonlogical axiom. Consequently, infinite branching is reduced to finite branching for normal-form derivations for both L''_A and $L''^{=}_A$. Given that the majority of formulas in nonlogical axioms are shallow, the respective finite sets of formulas are rather narrow.

It is well-known that the logical axiom $A \vdash A$ is admissible for nonatomic formulas in standard first-order logics. The same is true for L''_A and

$L_A''^{=}$ calculi without LP, RP, LR, RL rules. Hence, the choice of formulas in this logical axiom can be restricted to atomic formulas in such calculi.

§10. Related Work

Local cut permutations are used in the proof of the normal-form and ordered-form theorems. These permutations were introduced in the seminal work of Gentzen [32]. Since then, they are widely used in cut elimination proofs for a variety of calculi. Other styles of cut elimination emerged as well [3]. The presence of axioms prevents cut elimination. Local permutations of logical rules were investigated by Kleene [13].

The subformula property is a desirable property for any calculus. This property is a corollary and a primary reason for cut elimination. Investigation of cut admissibility is a central part of the majority of research devoted to sequent calculi. Development of sequent calculus extensions is often driven by the desire to have cut-free calculi. Examples of such extensions are hypersequent calculi [2], labeled calculi [25], etc.

As far as sequent calculi that are not cut-free are concerned, analyticity of numerous such calculi have been investigated. The majority of analyticity proofs are semantic. These proofs show the completeness of analytic derivations with respect to calculus models. For instance, the proof of analyticity of pure propositional sequent calculi is semantic [17]. Analyticity of several modal logics is proved in [10]. An analytic sequent calculus for bi-intuitionistic logic is presented in [15]. Takano's result [33] is an exception. This paper gives a syntactic proof of the analyticity of modal logic S5. Our presumption about calculi with nonlogical axioms is that models are not available.

The set of formulas occurring in derivations could be restricted without sacrificing completeness for some sequent calculi. They are called bounded sequent calculi. Bounded sequent calculi for conditional logics are presented in [19]. Bounded sequent calculi for sub-classical logics are presented in [16].

Paper [6] investigates the logics that have been formalized as both sequent calculi extended with additional logical axioms and analytic hypersequent calculi. The proof of boundedness of the former is done by transforming analytic hypersequent derivations into sequent derivations in which substitutions for metavariables from the additional logical axioms are restricted. How to transform derivations in sequent calculi extended with logical axioms into bounded derivations without taking a detour through another formalism remains an open question. Paper [7] extends the results of [6] by proving stronger boundedness.

The logical axiom $A, \Gamma \vdash A, \Pi$ is used in numerous sequent calculi [23] because it subsumes weakening. The version of this axiom for single-succedent calculi does not include Π. Sequent calculi with this logical axiom are not a good basis for inference procedures. This axiom creates an infinite number of infinite branching points in every derivation step where it is used. In contrast, the logical axiom $A \vdash A$ creates just one infinite branching point.

In [23], axioms extending sequent calculus G3c for classical first-order logic are converted to inference rules. Cut is admissible in the extended calculus but they are not analytic because of these additional inference rules. The same is possible for axioms of a certain form extending sequent calculus G3im for intuitionistic first-order logic. Both of these calculi employ the axiom $A, \Gamma \vdash A, \Pi$. Paper [8] specifies several classes of axioms that can be converted to analytic hypersequent inference rules.

In some papers [17,27,38], sequent calculus properties are proved for any calculus with inference rules satisfying certain constraints. In this work, rules are given explicitly. Specifying inference rule constraints has a theoretical advantage over concrete inference rules. In practice, it is easier to match logical connectives in newly created characterizations of AI systems against concrete inference rules. Besides, L_A inference rules are heterogeneous: LP, RP are context-dependent; LO, RO, RL, LR have multiple principal formulas. Each of these rule groups would require its own constraints.

To the best of author's knowledge, sequent calculi with axioms not containing formula metavariables have not been investigated in the literature. Unlike most of research, both calculi with a complete set of structural rules and substructural logics are covered here. This work represents a departure from the typical approach in proof theory where structural enrichments of standard sequent calculi, and even new formalisms, are acceptable. Our research aims at logical characterizations of a variety of AI systems. Hence, the formalism should be as simple as possible, and so, we deliberately limit ourselves to standard sequent calculi without cumbersome inference rules. These calculi are arguably the simplest logical formalism of all.

The use of unification in cut and the [ ] operation is similar to its use in inference rules of a number of sequent calculi in [34]. Numerous inference methods have been designed for sequent calculi. Among these methods that are applicable to more than one logic, some of them are based on Maslov's inverse method [34, 35] and some others are based on focusing [20, 21]. Focused extensions of standard sequent calculi serve to advance inference. Preliminary results of this research are outlined in [30].

§11. Concluding Remarks

Knowledge bases normally have tons of facts and rules which are represented by nonlogical axioms in the sequent calculi characterizing these AI systems. These axioms are crucial for inference. The sets of nonlogical axioms are fluid, some axioms may be devised for particular derivation goals. Evaluable predicates and functions incorporate computing into these logical systems. In these calculi, nonlogical axioms are more frequently used in derivations than logical rules because inference goals are usually shallow formulas with predicates and functions that are specific for the relevant domains.

Quantifiers could be added to the language of logical formulas considered here. It is known that the cut rule can be permuted with standard quantifier introduction rules [32]. Therefore, L_A calculi extended with quantifiers remain analytic. Unification can be done for quantified formulas [31]. Investigation of substructural calculi with additive logical inference rules is a topic for future research.

One contribution of this paper is adaptation of syntactic cut elimination techniques to calculi in which nonlogical axioms play a major role in inference, and thus, cut is not admissible. Discovery of new cut elimination methods was not a goal of this research. Another contribution of this paper is adaptation of ordered resolution to sequent calculi. It is expected that the effect of ordered inference is most significant in sequent calculi with numerous nonlogical axioms because of the heavy use of cuts in the respective derivations. At the same time, the benefits of ordered inference can be experienced for L''_A calculi without nonlogical axioms. Cut is admissible in them. In such L''_A calculi, entire derivations can be ordered because contraction and weakening are merged with logical rules.

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Synstretch
Framingham, Massachusetts, USA
E-mail: mail@sakharov.net

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