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CHARACTERS OF THE INFINITE ALTERNATING GROUP $\mathfrak{A}_{\mathbb{N}}$ AND $\mathbb{N}$-GRADED QUOTIENT GRAPHS OVER INVOLUTION

Abstract. In 1964, German mathematician E. Thoma published the complete list of extreme characters of the infinite symmetric and alternating groups; the translation of this work and the commentary on it have been published in the current volume. Thoma has deduced the classification of extreme characters of the infinite alternating group $\mathfrak{A}_{\mathbb{N}}$ from the corresponding result for the symmetric group and general properties of countable groups that he has shown in another work. We suggest another, more direct proof of this result using different technique, – we consider the graph (Bratelli diagram), which may be viewed as a quotient of the Young graph by its natural involution. The branching graph of the infinite alternating group is not determined by the definition of a quotient graph over an involution. In particular, the branching graph of the infinite alternating group differs from the quotient of the branching graph of the infinite symmetric group. We are going to explore this connection later.

Effectively, we prove a general result, namely, given the set of ergodic measures on a graph with an involution, we explain how to describe the set of ergodic central measures on the quotient graph. The problems of how the traces (the characters) change after various changes of a graph, have not been sufficiently explored.

§1. $\mathbb{N}$-graded quotient graphs

In this paper, we assume that $\mathbb{N}$ is the set of positive integers. We will begin by recalling the definition of $\mathbb{N}$-graded graphs.

Definition 1.1. *Suppose that the set of vertices of an oriented graph* Γ *can be represented as a disjoint union*

$$\Gamma = \bigsqcup_{n\in\mathbb{N}} \Gamma_n \tag{1.1}$$

such that the following conditions hold:

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(1) *The set* Γ_1 *consists of a single vertex* e_1;
(2) *The set* Γ_n *is finite for every* $n \in \mathbb{N}$;
(3) *If the number* $m(v,u)$ *of edges going from a vertex* v *to a vertex* u *does not equal zero then* $v \in \Gamma_n$, $u \in \Gamma_{n+1}$ *for some* $n \in \mathbb{N}$,
(4) *for every vertex* $v \in \Gamma$ *there exists a path that goes from* e_1 *to* v,

then Γ *is called* a locally finite $\mathbb{N}$-graded graph. *Next, we are going to call them* $\mathbb{N}$*-graded graphs for short.*

Definition 1.2. *A map* $\omega : \Gamma \to \Gamma$ *is called* an involution (on the set of vertices) of the $\mathbb{N}$-graded graph Γ *if the following conditions hold:*

(1) $\omega(\Gamma_n) = \Gamma_n$, $\forall n \geqslant 1$;
(2) $\omega^2 = Id$;
(3) $m(\omega(v), \omega(u)) = m(v,u), \quad \forall v \in \Gamma_n, u \in \Gamma_{n+1}, n \geqslant 1.$

Definition 1.3. *For an* $\mathbb{N}$*-graded graph* Γ *with an involution* ω *we define* an $\mathbb{N}$-graded quotient graph mod the action of the involution Γ/ω *by the following rule. Its set of vertices of the* n*-th floor* $(\Gamma/\omega)_n$ *consists of two types:*
1) *A pair* $\{v, \omega(v)\}$ *if* $\omega(v) \neq v$, $v \in \Gamma_n$;
2) *A one-element set* $\{v\}$ *if* $\omega(v) = v$, $v \in \Gamma_n$.

The multiplicity of an edge of Γ/ω *depends on the types of vertices* $v \in \Gamma_n$ *and* $u \in \Gamma_{n+1}$ *it connects:*

(1) $m(\{v,\omega(v)\}, \{u,\omega(u)\}) := m(v,u) + m(v,\omega(u))$
$= m(v,u) + m(\omega(v),u)$ *for the case* $\omega(v) \neq v$, $\omega(u) \neq u$;
(2) $m(\{v,\omega(v)\},\{u\}) := m(v,u)$ *for the case* $\omega(v) \neq v$, $\omega(u) = u$;
(3) $m(\{v\},\{u,\omega(u)\}) := 2m(v,u)$ *for the case* $\omega(v) = v$, $\omega(u) \neq u$;
(4) $m(\{v\},\{u\}) := m(v,u)$ *for the case* $\omega(v) = v$, $\omega(u) = u$.

Next, we are going to call those $\mathbb{N}$*-graded graphs quotient graphs for short.*

As we see from the definition, a quotient graph is unlikely to have multiplicity-free edges even when the initial graph is multiplicity-free.

Remark 1.4. There are different ways to define the multiplicities of a quotient graph. For example, consider the Pascal graph, which has multiplicity-free edges, with the natural involution. Then on the quotient graph, we may define multiplicity-free edges, see [8], or (according to our definition) we may put that the multiplicities of some edges equal 2. Both quotient graphs have their applications but the sets of central measures on those graphs are different.

Definition 1.5. *Consider the canonical projection* proj : $\Gamma \to \Gamma/\omega$ *on the set of vertices:*

$$\operatorname{proj}(v) := \begin{cases} \{v, \omega(v)\} & \text{if } \omega(v) \neq v, \\ \{v\}, & \text{otherwise.} \end{cases} \tag{1.2}$$

Definition 1.6. *A map* $\varphi : \Gamma \to \mathbb{R}_+ \cup \{0\}$ *is called* a normed harmonic function *if it satisfies the equalities*

(1) $\varphi(e_1) = 1$;

(2)
$$\varphi(v) = \sum_{u \in \Gamma_{n+1}} m(v,u)\varphi(u), \ \forall v \in \Gamma_n. \tag{1.3}$$

Denote this set of normed harmonic functions on Γ *by* $\operatorname{Har}(\Gamma)$. *Next, we are going to consider only the harmonic functions that are normed. So we will call them harmonic functions for short.*

Proposition 1.7. *Suppose that* Γ *is an* $\mathbb{N}$*-graded graph with an involution* ω *and* φ *is a harmonic function on* Γ. *Define the function* $\operatorname{proj}(\varphi)$ *on* Γ/ω *by the following formula:*

$$\operatorname{proj}(\varphi)(\{v, \omega(v)\}) := \frac{\varphi(v) + \varphi(\omega(v))}{2} \text{ for the case } \omega(v) \neq v; \tag{1.4}$$

$$\operatorname{proj}(\varphi)(\{v\}) := \varphi(v) \text{ for the case } \omega(v) = v. \tag{1.5}$$

Then $\operatorname{proj}(\varphi)$ *is a harmonic function on* Γ/ω.

Proof. Let us show that the equality (1.3) holds for $\operatorname{proj}(\varphi)$ and an arbitrary vertex $y \in \Gamma/\omega$. Consider the two cases depending on the type of the vertex $y \in (\Gamma/\omega)_n$.

If $y = \{v\}$ then

$$m(v, \omega(u)) = m(v,u), \quad \forall u \in \Gamma_{n+1}. \tag{1.6}$$

Therefore,

$$\begin{aligned} (\operatorname{proj}(\varphi))(\{v\}) = \varphi(v) &= \sum_{u \in \Gamma_{n+1}} m(v,u)\varphi(u) \\ &= \sum_{\substack{\{u,\omega(u)\} \in (\Gamma/\omega)_{n+1} \\ \omega(u) \neq u}} 2m(v,u)\left(\frac{\varphi(u) + \varphi(\omega(u))}{2}\right) + \sum_{\substack{u \in \Gamma_{n+1} \\ \omega(u) = u}} m(v,u)\varphi(u) \\ &= \sum_{z \in (\Gamma/\omega)_{n+1}} m(y,z)(\operatorname{proj}(\varphi)(z)). \end{aligned} \tag{1.7}$$

If $y = \{v, \omega(v)\}$ then

$$
\begin{aligned}
(\mathrm{proj}(\varphi))(\{v,\omega(v)\}) &= \frac{\varphi(v)+\varphi(\omega(v))}{2} \\
&= \frac{1}{2}\sum_{u\in\Gamma_{n+1}} m(v,u)\varphi(u) + \frac{1}{2}\sum_{u\in\Gamma_{n+1}} m(\omega(v),u)\varphi(u) \\
&= \sum_{\substack{\{u,\omega(u)\}\in(\Gamma/\omega)_{n+1}\\ \omega(u)\neq u}} ((m(v,u)+m(\omega(v),u))\left(\frac{\varphi(u)+\varphi(\omega(u))}{2}\right) \\
&\quad + \sum_{\substack{u\in\Gamma_{n+1}\\ \omega(u)=u}} \frac{m(v,u)+m(\omega(v),u)}{2}\cdot\varphi(u) \\
&= \sum_{z\in(\Gamma/\omega)_{n+1}} m(y,z)(\mathrm{proj}(\varphi)(z)). \qquad (1.8)
\end{aligned}
$$

□

Proposition 1.8. *The map* proj : Har(Γ) → Har(Γ/ω) *is surjective.*

Proof. Suppose that $\psi \in \mathrm{Har}(\Gamma/\omega)$. We define a function $\overline{\psi}$ on the set of vertices of Γ by the formula

$$\overline{\psi}(v) := \begin{cases} \psi(\{v,\omega(v)\}) & \text{if } \ \omega(v)\neq v, \\ \psi(\{v\}), & \text{otherwise.} \end{cases} \qquad (1.9)$$

Clearly, $\mathrm{proj}(\overline{\psi}) = \psi$. Let us show that $\overline{\psi} \in \mathrm{Har}(\Gamma)$. Note that

$$\overline{\psi}(\omega(v)) = \overline{\psi}(v), \quad \forall v \in \Gamma. \qquad (1.10)$$

Consider two cases.

First, assume that $\omega(v) = v \in \Gamma_n$. Because $m(\omega(v),u) = m(v,u)$ we get

$$
\begin{aligned}
\overline{\psi}(v) = \psi(\{v\}) &= \sum_{\substack{\{u,\omega(u)\}\in(\Gamma/\omega)_{n+1}\\ \omega(u)\neq u}} 2m(v,u)\psi(\{u,\omega(u)\}) + \sum_{\substack{u\in\Gamma_{n+1}\\ \omega(u)=u}} m(v,u)\psi(\{u\}) \\
&= \sum_{\substack{\{u,\omega(u)\}\in(\Gamma/\omega)_{n+1}\\ \omega(u)\neq u}} 2m(v,u)\overline{\psi}(u) + \sum_{\substack{u\in\Gamma_{n+1}\\ \omega(u)=u}} m(v,u)\overline{\psi}(u) = \sum_{u\in\Gamma_{n+1}} m(v,u)\overline{\psi}(u).
\end{aligned}
\qquad (1.11)
$$

Now let us consider the case $\omega(v) \neq v$. Then we obtain a similar chain of equalities

$$\begin{aligned}
&\overline{\psi}(v) = \psi(\{v, \omega(v)\}) \\
&= \sum_{\substack{\{u,\omega(u)\}\in(\Gamma/\omega)_{n+1} \\ \omega(u)\neq u}} (m(v,u)+m(v,\omega(u)))\psi(\{u,\omega(u)\}) + \sum_{\substack{u\in\Gamma_{n+1} \\ \omega(u)=u}} m(v,u)\psi(\{u\}) \\
&\quad = \sum_{\substack{\{u,\omega(u)\}\in(\Gamma/\omega)_{n+1} \\ \omega(u)\neq u}} (m(v,u)+m(v,\omega(u)))\overline{\psi}(u) + \sum_{\substack{u\in\Gamma_{n+1} \\ \omega(u)=u}} m(v,u)\overline{\psi}(u) \\
&\qquad = \sum_{u\in\Gamma_{n+1}} m(v,u)\overline{\psi}(u). \qquad (1.12)
\end{aligned}$$

□

Definition 1.9. *A normed harmonic function φ on an $\mathbb{N}$-graded graph is called* extreme (undecomposable) *if it can't be represented in the form*

$$\varphi = a \cdot \varphi_1 + (1-a)\varphi_2, \tag{1.13}$$

where ψ_1 and ψ_2 are two distinct normed harmonic functions, and $0 < a < 1$.

Proposition 1.8 and the formula (1.9) imply the following result.

Proposition 1.10. *If a harmonic function φ on an $\mathbb{N}$-graded graph Γ is extreme then* $\mathrm{proj}(\varphi)$ *is also an extreme harmonic function on Γ/ω.*

Proof. It follows from the fact that the map defined by the equality (1.9) is linear. □

We can also formulate a statement close to the reverse of the Proposition 1.10.

Proposition 1.11. *Suppose that Γ is an $\mathbb{N}$-graded graph with an involution, and ψ is an extreme harmonic function on the quotient graph Γ/ω. Then there exists an extreme harmonic function φ on the $\mathbb{N}$-graded graph Γ such that* $\mathrm{proj}(\varphi) = \psi$.

Proof. By Proposition 1.8 the preimage $(\mathrm{proj})^{-1}(\psi)$ is not an empty set. On the other hand, as a preimage under a continuous map, it is a closed subset of a compact set of harmonic functions on Γ. Therefore, $(\mathrm{proj})^{-1}(\psi)$ is a non-empty convex compact set. By Krein–Milman Theorem it has an extreme point φ. Then Proposition 1.8 implies that φ is also undecomposable as a harmonic function on Γ. □

From Proposition 1.10 and Proposition 1.11 we obtain the following result.

Theorem 1.12. *The map* proj *surjectively maps the set of extreme harmonic function on Γ onto the set of extreme harmonic function on Γ/ω.*

Notice that Definitions 1.6 and 1.9 and subsequent statements hold if we do not require that the multiplicities $m(v,u)$ are integers. Now let us consider the case when all multiplicities $m(v,u)$ are non-negative integers.

Definition 1.13. *For an $\mathbb{N}$-graded graph Γ we define the topological space* $\mathrm{Paths}(\Gamma)$ *as the projective limit as $n \to \infty$ of discrete spaces of paths that go from e_1 to the n-th floor Γ_n. Note that depending on the multiplicities of the edges there may be multiple paths determined by the same sequence of vertices*

$$(e_1, t_2, t_3, \dots), \quad t_i \in \Gamma_i. \tag{1.14}$$

Denote by $\dim(v)$ *the number of paths that connect the vertices e_1 and v. Equivalently,*

$$\dim(v) = \sum_t \prod_{j=2}^{n} m(t_{j-1}, t_j), \quad \forall v \in \Gamma_n \tag{1.15}$$

where the sum is taken over all possible n-tuples such that $t_k \in \Gamma_k$, $\forall 2 \leqslant k \leqslant n$, and $t_1 := e_1$.

Remark 1.14. If Γ is an $\mathbb{N}$-graded graph with an involution ω then

$$\dim(\omega(v) = \dim(v), \quad \forall v \in \Gamma, \tag{1.16}$$

and

$$\dim(\mathrm{proj}(v)) = \begin{cases} \dim(v) + \dim(\omega(v)) & \text{if } \ \omega(v) \neq v, \\ \dim(v), & \text{otherwise}, \end{cases}, \quad \forall v \in \Gamma. \tag{1.17}$$

The proof is similar to the proof of Proposition 1.7. Note that the equality (1.17) determines the multiplicities of the quotient graph.

Definition 1.15. *We are going to consider Borel measures on the space* $\mathrm{Paths}(\Gamma)$ *such that* $M(\mathrm{Paths}(\Gamma) = 1$*. For a vertex $v \in \Gamma_n$ denote by* $\mathrm{Cyl}(n, v)$ *the subset of all infinite paths such that they go through the vertex v. Now let us fix a path t from the vertex e_1 to the vertex v and consider* *a subset of the set* $\mathrm{Cyl}(n, v)$ *consisting of the paths that repeat the first* $n-1$ *edges of the path* t. *A Borel measure on the space* $\mathrm{Paths}(\Gamma)$ *is called central if the measure of this subset does not depend on the choice of* t. *For a central measure* M *and* $v \in \Gamma_n$ *we put*

$$M(v) := M(\mathrm{Cyl}(n, v)). \tag{1.18}$$

Remark 1.16. For a central measure M on an $\mathbb{N}$-graded graph Γ the ratio

$$\varphi(v) := \frac{M(v)}{\dim(v)} \tag{1.19}$$

defines a harmonic function on Γ. This formula establishes a one-to-one correspondence between the set of extreme normed harmonic functions and the set of ergodic central measures, see, for example, [12, 3].

From Proposition 1.7 we obtain the analogous result for central measures.

Proposition 1.17. *Suppose that* Γ *is an* $\mathbb{N}$*-graded graph with an involution* ω *and* M *is a central measure on* Γ. *Define the measure* $\mathrm{proj}(M)$ *on* Γ/ω *by the following formula:*

$$\mathrm{proj}(M)(\{v, \omega(v)\}) := M(v) + M(\omega(v)) \text{ for the case } \omega(v) \neq v; \tag{1.20}$$

$$\mathrm{proj}(M)(\{v\}) := M(v) \text{ for the case } \omega(v) = v. \tag{1.21}$$

Then $\mathrm{proj}(M)$ *is indeed a central measure on* Γ/ω.

From Theorem 1.12 we get a similar result for central measures.

Theorem 1.18. *The map* proj *surjectively maps the set of ergodic central measures on* Γ *onto the set of ergodic central measures on* Γ/ω.

Remark 1.19. One may also consider semifinite measures and harmonic functions (see, for example, [12]) and define projections for those classes of measures and functions. Following the approach of this section, we can obtain results similar to those formulated above.

§2. The branching graph for the infinite alternating group $\mathfrak{A}_\mathbb{N}$

Definition 2.1. *When* λ *is a partition of the number* n *we will write* $\lambda \vdash n$. *For a partition* λ *let us denote by* λ' *its conjugate partition.*

Consider the Young graph $\mathbb{Y}$ *of partitions. It is easy to check that the map*

$$\lambda \mapsto \lambda' \tag{2.1}$$

is an involution of the $\mathbb{N}$*-graded graph* $\mathbb{Y}$*. Denote the corresponding quotient graph by* $\mathbb{Y}/(\,\cdot\,)'$.

From the classification of the ergodic measures on the Young graph (see [6, 11, 3]) and Theorem 1.18, we obtain the classification of ergodic central measures on the quotient graph.

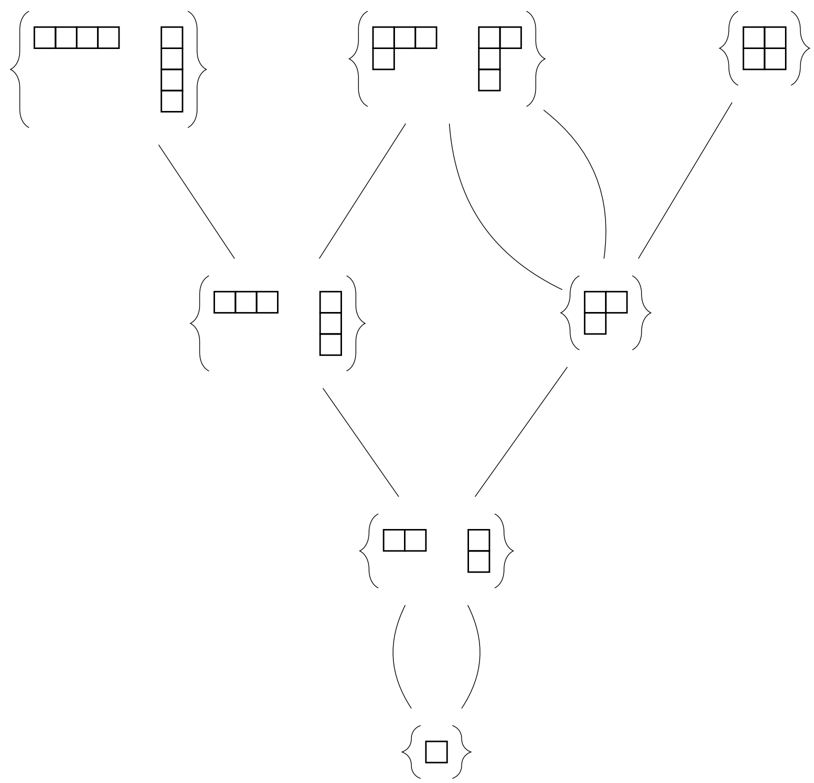

Figure 1. A part of the quotient graph $Y/(\,\cdot\,)$.

Theorem 2.2. *The central ergodic measures on the quotient graph* $\mathbb{Y}/(\cdot)'$ *are indexed by unordered pairs* (α, β) *such that*

$$\begin{aligned} &\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0), \quad \beta = (\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant 0), \\ &\text{and} \quad \sum_{j=1}^{\infty} (\alpha_j + \beta_j) \leqslant 1. \end{aligned} \tag{2.2}$$

The measure $M_{\alpha,\beta}$ *is defined the following way*

(1) *For* $\lambda \neq \lambda'$ *we get*

$$M_{\alpha,\beta}(\{\lambda, \lambda'\}) = \dim(\lambda)(\widehat{s_\lambda(\alpha, \beta)} + \widehat{s_{\lambda'}(\alpha, \beta)}); \tag{2.3}$$

(2) *For* $\lambda = \lambda'$

$$M_{\alpha,\beta}(\{\lambda\}) = \dim(\lambda)\widehat{s_\lambda(\alpha, \beta)}; \tag{2.4}$$

where $\dim(\lambda)$ *is the dimension on the Young graph (the number of standard tableaux of the form* λ*), and* $\widehat{s_\lambda(\alpha, \beta)}$ *is the Schur function* s_λ *corresponding to the specialization of Newton sums*

$$p_1 := 1, \quad p_k := \sum_{j=1}^{\infty} (\alpha_j^k + (-1)^{k+1} \beta_j^k). \tag{2.5}$$

Note that from Frobenius's formula that expresses Schur's functions via Newton's functions, we obtain that

$$\widehat{s_{\lambda'}(\alpha, \beta)} = \widehat{s_\lambda(\beta, \alpha)}. \tag{2.6}$$

Hence, $M_{\alpha,\beta} = M_{\beta,\alpha}$.

The Plansherel measure $M_{0,0}$ *is given by the formula*

$$\text{if } \lambda \neq \lambda' \text{ then } M_{0,0}(\{\lambda, \lambda'\}) = \frac{2(\dim(\lambda))^2}{|\lambda|!}; \tag{2.7}$$

$$\text{if } \lambda = \lambda' \text{ then } M_{0,0}(\{\lambda\}) = \frac{(\dim(\lambda))^2}{|\lambda|!}. \tag{2.8}$$

Now we are going to recall some well-known properties of alternating groups $\mathfrak{A}_n$ and the infinite alternating group $\mathfrak{A}_\mathbb{N} = \mathfrak{A}_2 \subset \mathfrak{A}_3 \subset \mathfrak{A}_4 \subset \ldots$.

Proposition 2.3. *The conjugacy class of the symmetric group* $\mathfrak{S}_n$ *corresponding to a parition* $\rho \vdash n$ *splits into two conjugacy classes in* $\mathfrak{A}_n$ *if and only if the parts of* ρ *are pairwise different and odd.*

This proposition has been shown in, for example, [2, Lemma 1.2.10]. The next two statements follow from [2, Theorem 2.5.7], see also [1].

Proposition 2.4. *Denote by $T_\lambda, \lambda \vdash n$ the irreducible complex representation of the group $\mathfrak{S}_n$ corresponding to a partition λ. Then:*

(1) *if $\lambda \neq \lambda'$ then the restriction of T_λ to the subgroup $\mathfrak{A}_n$ is irreducible. Denote this restriction by R_λ. We also have that*
$$R_\lambda \simeq R_{\lambda'}; \tag{2.9}$$
(2) *if $\lambda = \lambda'$ then the restriction of T_λ to the subgroup $\mathfrak{A}_n$ can be decomposed into the sum of two irreducible non-equivalent representations:*
$$T_\lambda|_{\mathfrak{A}_n} = R_\lambda^+ \oplus R_\lambda^- \tag{2.10}$$
The representations R_λ^+ and R_λ^- are conjugate, namely,
$$R_\lambda^+((1,2)x(1,2)) = R_\lambda^-(x), \quad \forall x \in \mathfrak{A}_n. \tag{2.11}$$

Proposition 2.5. *The restrictions of irreducible representations from $\mathfrak{A}_{n+1}$ to $\mathfrak{A}_n$ are multiplicity-free. Consider μ, $\mu \vdash n$ and λ, $\lambda \vdash (n+1)$. The branching rule breaks down into four cases:*

(1) *$\lambda \neq \lambda', \mu \neq \mu'$. Then R_μ enters in the decomposition of the restriction of R_λ iff either $\mu \subset \lambda$, or $\mu \subset \lambda'$;*
(2) *$\lambda \neq \lambda', \mu = \mu'$. Then R_μ^+ enters in the decomposition of the restriction of R_λ iff $\mu \subset \lambda$. Similarly, R_μ^- enters in the decomposition of the restriction of R_λ iff $\mu \subset \lambda$;*
(3) *$\lambda = \lambda', \mu \neq \mu'$. Then R_μ enters in the decomposition of the restriction of R_λ^+ iff $\mu \subset \lambda$. Similarly, R_μ enters in the decomposition of the restriction of R_λ^- iff $\mu \subset \lambda$;*
(4) *$\lambda = \lambda', \mu = \mu'$. Then R_μ^- enters in the decomposition of the restriction of R_λ^- iff $\mu \subset \lambda$. Similarly, R_μ^+ enters in the decomposition of the restriction of R_λ^+ iff $\mu \subset \lambda$. Also, R_μ^- does not enter in the decomposition of the restriction of R_λ^+, and R_μ^+ does not enter in the decomposition of the restriction of R_λ^-.*

Remark 2.6. The vertices of the branching graph $\Gamma(\mathfrak{A}_\mathbb{N})$ for $\mathfrak{A}_\mathbb{N}$ can be represented as a disjoint union
$$\{\{\lambda, \lambda'\} \mid \lambda \neq \lambda'\} \cup \{\lambda^+ \mid \lambda = \lambda'\} \cup \{\lambda^- \mid \lambda = \lambda'\}. \tag{2.12}$$
The graph should begin on the second floor corresponding to the group $\mathfrak{A}_2$, where there is a single vertex $\{(2), (1,1)\}$. To satisfy our definition of the $\mathbb{N}$-graded graph we will add a vertex e_1 to the first level and two edges

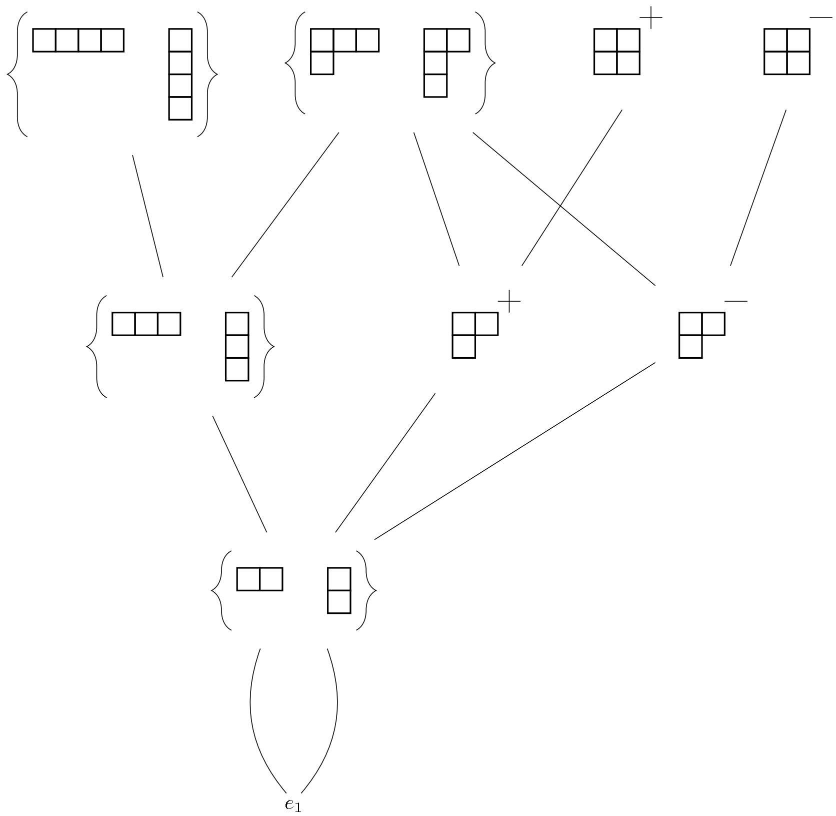

Figure 2. A part of the branching graph $\Gamma(\mathfrak{A}_{\mathbb{N}})$.

that go from e_1 to $\{(2),(1,1)\}$. That will not change the properties that we are going to consider.

From Proposition 2.5 we obtain the connection between the $\mathbb{N}$-graded graphs $\Gamma(\mathfrak{A}_{\mathbb{N}})$ and $\mathbb{Y}/(\,\cdot\,)'$.

Proposition 2.7. *If we replace a pair of representations R_λ^+ and R_λ^+ by their formal linear combination $\frac{1}{2}(R_\lambda^+ + R_\lambda^-)$ for each λ such that $\lambda' = \lambda$ then the new "branching graph" will coincide with the quotient Young graph.*

This result allows us to obtain the classification of the central ergodic measures on the branching graph of the group $\mathfrak{A}_{\mathbb{N}}$ from Thoma's Theorem.

Proposition 2.8. *To describe a central ergodic measures on the branching graph* $\Gamma(\mathfrak{A}_{\mathbb{N}})$ *of the group* $\mathfrak{A}_{\mathbb{N}}$ *we still need to pick an unordered pair* (α, β) *of sequences such that*

$$\begin{aligned} &\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0), \quad \beta = (\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant 0), \\ &\quad \textit{and} \quad \sum_{j=1}^{\infty} (\alpha_j + \beta_j) \leqslant 1. \end{aligned} \tag{2.13}$$

But to describe the value of the measure of the sets $\mathrm{Cyl}(\lambda^+)$ *and* $\mathrm{Cyl}(\lambda^-)$ *we will also need to pick a function*

$$\theta : \{\lambda \textit{ is a partition} \mid \lambda' = \lambda\} \to \{-, +\}, \tag{2.14}$$

where θ *takes the same values if one Young diagram differs from another by a single cell with equal coordinates* (j, j) *(for example, the partitions* $(2,1)$ *and* $(2,2)$*). The measure* $\widetilde{M}_{\alpha,\beta,\theta}$ *is defined the following way*

(1) *For* $\lambda \neq \lambda'$ *it does not depend on* θ

$$\widetilde{M}_{\alpha,\beta,\theta}(\{\lambda, \lambda'\}) = M_{\alpha,\beta}(\{\lambda, \lambda'\}); \tag{2.15}$$

(2) *For* $\lambda = \lambda'$

$$\widetilde{M}_{\alpha,\beta,\theta}(\lambda^+) = \begin{cases} M_{\alpha,\beta}(\{\lambda\}) & \textit{if } \theta(\lambda) = +; \\ 0, & \textit{otherwise}; \end{cases} \tag{2.16}$$

$$\widetilde{M}_{\alpha,\beta,\theta}(\lambda^-) = \begin{cases} M_{\alpha,\beta}(\{\lambda\}) & \textit{if } \theta(\lambda) = -; \\ 0, & \textit{otherwise}. \end{cases} \tag{2.17}$$

Note that

$$\widetilde{M}_{\alpha,\beta,\theta}(\lambda^+) + \widetilde{M}_{\alpha,\beta,\theta}(\lambda^-) = M_{\alpha,\beta}(\{\lambda\}). \tag{2.18}$$

Remark 2.9. The limit shape for Young diagrams distributed according to the Plansherel measure has been obtained in [9, 4]. The formula (2.7) shows that the question about the limit shape in the case of the group $\mathfrak{A}_{\mathbb{N}}$ trivially reduces to the classical result.

§3. Characters of the infinite alternating group $\mathfrak{A}_{\mathbb{N}}$

Next, we are going to describe the classification of extreme characters of the infinite alternating group $\mathfrak{A}_{\mathbb{N}}$ obtained by E. Thoma in [5], see also [6].

Definition 3.1. *For a partition ρ of some number n denote by $m_k(\rho)$ the number of its parts that are equal to k, $k \geqslant 1$.*

The description of conjugacy classes of the group $\mathfrak{A}_{\mathbb{N}}$ is well known, see, for example, [5].

Proposition 3.2. *The conjugacy classes in the group $\mathfrak{A}_{\mathbb{N}}$ are indexed by the partitions ρ such that*

$$m_1(\rho) = 0, \;\; m_{2k}(\rho) \;\; \text{is even for all} \;\; k \geqslant 1. \tag{3.1}$$

For an element $g \in \mathfrak{A}_{\mathbb{N}}$ denote by $\rho(g)$ its cycle type.

Now we will repeat the formulation of Thoma's classification of the extreme characters of the group $\mathfrak{A}_{\mathbb{N}}$ [5, Satz 6].

Theorem 3.3. *The extreme characters of the infinite alternating group $\mathfrak{A}_{\mathbb{N}}$ are indexed by the sequences such that*

$$\begin{gathered} \alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0), \quad \beta = (\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant 0), \\ \text{and} \;\; \sum_{j=1}^{\infty}(\alpha_j + \beta_j) \leqslant 1. \end{gathered} \tag{3.2}$$

The values of the extreme character $\chi_{\alpha,\beta}$ are given by the following formula

$$\chi_{\alpha,\beta}(g) = \prod_{k=2}^{\infty}\left(\sum_{j=1}^{\infty}(\alpha_j^k + (-1)^{k+1}\beta_j^k)\right)^{m_k(\rho(g))}, \quad \forall g \in \mathfrak{A}_{\mathbb{N}}. \tag{3.3}$$

At the same time the following equality holds

$$\chi_{\alpha,\beta} = \chi_{\beta,\alpha}. \tag{3.4}$$

In other words, every extreme character of $\mathfrak{A}_{\mathbb{N}}$ is the restriction of an extreme character of $\mathfrak{S}_{\mathbb{N}}$. And the restriction of an extreme character of $\mathfrak{S}_{\mathbb{N}}$ to $\mathfrak{A}_{\mathbb{N}}$ is always extreme.

Remark 3.4. The restriction of the character to the subgroup $\mathfrak{A}_{\mathbb{N}}$ corresponds to the canonical projection described by the equality (1.20).

Also, let us consider the equality (1.9) in our case. For a given character χ on the subgroup $\mathfrak{A}_{\mathbb{N}}$ we construct a character $\overline{\chi}$ on the group $\mathfrak{S}_{\mathbb{N}}$ by

$$\overline{\chi}(g) := \begin{cases} \chi(g) & \text{if} \;\; g \in \mathfrak{A}_{\mathbb{N}}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.5}$$

This allows us to make a shorter proof of Theorem 3.3 than the one we present below. However, the latter clarifies the connection between the ergodic measures on the quotient graph $\mathbb{Y}/(\cdot)'$ and the extreme characters of the infinite alternating group.

Proof of Theorem 3.3. The results, obtained in [11, 12, 3] imply that for every extreme characters χ of the group $\mathfrak{A}_\mathbb{N}$ there exists an ergodic measure $\widetilde{M}$ on the space of paths of the branching graph $\Gamma(\mathfrak{A}_\mathbb{N})$ of $\mathfrak{A}_\mathbb{N}$ such that

$$\chi(g) = \lim_{n\to\infty} \sum_{v\in(\Gamma(\mathfrak{A}_\mathbb{N}))_n} \widetilde{M}(v)\cdot\frac{\chi^v(g)}{\chi^v(e)}, \tag{3.6}$$

where by χ^v we denote the character of the irreducible complex representation corresponding to the label v of the branching graph $\Gamma(\mathfrak{A}_\mathbb{N})$. The sum is taken over the labels of irreducible complex representations of the group $\mathfrak{A}_n$, described in Proposition 2.4. It is sufficient to evaluate χ at representatives of conjugacy classes of the group $\mathfrak{A}_\mathbb{N}$. We may assume that each representative g leaves the elements 1 and 2 fixed. Then the equality (2.11) implies that the values of the characters of the representations R^+_λ and R^-_λ (for $\lambda=\lambda'$) at the element g are equal. Then from the equalities (2.18) and (3.6) we get that for some unordered pair (α,β)

$$\chi(g) = \lim_{n\to\infty} \sum_{\substack{\{\lambda,\lambda'\}\\ \lambda\vdash n,\,\lambda'\neq\lambda}} M_{\alpha,\beta}(\{\lambda,\lambda'\})\cdot\frac{\chi^\lambda(g)}{\chi^\lambda(e)} + \sum_{\substack{\lambda\\ \lambda\vdash n,\,\lambda'=\lambda}} M_{\alpha,\beta}(\{\lambda\})\cdot\frac{\chi^\lambda(g)}{\chi^\lambda(e)}, \tag{3.7}$$

where, as usual, χ^λ stands for the corresponding irreducible character of the symmetric group $\mathfrak{S}_n$, and $M_{\alpha,\beta}$ is an ergodic measure on the quotient graph $\mathbb{Y}/(\,\cdot\,)'$. As we know,

$$\chi^{\lambda'}(g) = \chi^\lambda(g), \quad \forall g\in\mathfrak{A}_n. \tag{3.8}$$

Hence, from Theorem 2.2 we see that the RHS of the equality (3.7) coincides with the sum that we obtain if evaluate the corresponding expression for the infinite symmetric group $\mathfrak{S}_\mathbb{N}$. Therefore, the LHS is given by the equality (3.3). □

Remark 3.5. The character defined by the equalty (3.6) does not depend on the parameter θ in the classification of Proposition 2.8. Here we see an example when different ergodic measures on the branching graph may define the same extreme character.

Remark 3.6. One may construct factor representations of the type II_1, corresponding to the characters of $\mathfrak{A}_\mathbb{N}$ similar to the consruction for $\mathfrak{S}_\mathbb{N}$, see [10] and also [7]. On the other hand, for every character the restriction of the factor representation of $\mathfrak{S}_{\mathbb{N}}$ onto the subgroup $\mathfrak{A}_{\mathbb{N}}$ is exactly the factor representation with that character.

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