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ON JORDAN STRUCTURE OF NILPOTENT MATRICES FROM LIE ALGEBRA $\mathfrak{so}(N,\mathbb{C})$

Abstract. The Jordan structure of matrices of the Lie algebra of a complex orthogonal group, nilpotent case, is considered. These matrices have an arbitrarily complicated Jordan structure, under the known condition that the number of Jordan blocks of even size is even. A normal form for such matrices is proposed. Gram matrices of Jordan chains are described.

§1. Introduction

Our ground field is $\mathbb{C}$. We investigate complex groups and algebras and use the following notations:

$$\mathrm{GL} := \mathrm{GL}(N,\mathbb{C}), \quad \mathrm{SO} := \mathrm{SO}(N,\mathbb{C}), \quad \mathfrak{gl} := \mathfrak{gl}(N,\mathbb{C}), \quad \mathfrak{so} := \mathfrak{so}(N,\mathbb{C})$$

We treat elements of the group and its Lie algebra as linear operators in the auxiliary $\mathsf{V} \simeq \mathbb{C}^N$.

Let us equip V with a non-degenerated symmetric scalar product $\langle \dots, \dots \rangle$. The elements of SO and $\mathfrak{so}$ can be considered as linear transformations of V satisfying some additional conditions.

Matrix Φ, belongs to SO if it preserves the scalar product

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle \Leftrightarrow \Phi^\top g \Phi = g,$$

where g is the metric tensor.

If $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle$,

$$0 = \langle d\Phi X, \Phi Y \rangle + \langle \Phi X, d\Phi Y \rangle = \langle d\Phi \Phi^{-1} \Phi X, \Phi Y \rangle + \langle \Phi X, d\Phi \Phi^{-1} \Phi Y \rangle.$$

It takes place for all X, Y and $\Phi X, \Phi Y$. Let us denote $A = d\Phi\Phi^{-1}$. We get

$$A \in \mathfrak{sl} \Leftrightarrow \langle AX, Y \rangle + \langle X, AY \rangle = 0.$$

We can see that an element A belongs to the algebra $\mathfrak{so}$ iff the corresponding operator is antiself-adjoint: $A = -A^* \subset \mathrm{End}\ \mathsf{V}$.

Key words and phrases: Lie algebra of complex orthogonal group, Jordan normal form, cyclic chains of vectors.

We need it in the coordinate form. Let $\mathbf{e} = (e_1, \dots, e_N)$ be a basis of V, and g be its metric tensor: $\langle e_i, e_j \rangle = \mathrm{g}_{ij}$, $X = e_i x^i$, $Y = e_i y^i$:

$$\langle X, Y \rangle == \operatorname{tr} x^\top \mathrm{g} y = x^i \mathrm{g}_{ij} y^j .$$

The matrix of the adjoint operator is similar to the transposed matrix:

$$\langle AX, Y \rangle = \operatorname{tr}\ (Ax)^\top \mathrm{g} y = \operatorname{tr} x^\top A^\top \mathrm{g} y, \quad \langle X, A^* Y \rangle = \operatorname{tr}\ x^\top \mathrm{g} A^* y,$$

consequently $A^\top \mathrm{g} = \mathrm{g} A^*$, or $A^* = \mathrm{g}^{-1} A^\top \mathrm{g}$:

$$A \in \mathfrak{so} \Leftrightarrow \mathrm{g} A + A^\top \mathrm{g} = 0.$$

We need one more transposing, namely the transposing with respect to the antidiagonal. We call it a $\vdash$-transposing:

$$A^\vdash := \tau A^\top \tau, \quad \text{or} \quad (A^\vdash)^i_j = A^{-j}_{-i}\ ,$$

where τ is the matrix of the inversion (the units on the antidiagonal).

We treat the change of the index sign as the counting of the coordinates from the opposite side:

$$\{a, b, c, d, e, \dots, v, w, x, y, z\}$$

– the elements a and z, b and y, c and x etc. have the opposite indexes.

The $\vdash$-transposing interchange raws and columns, but states them in the special order. For example:

$$x^\vdash = x^\top \tau = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^\vdash = (c, b, a) = (a, b, c)\tau,$$

or

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}^\vdash = \begin{pmatrix} f & e & d \\ c & b & a \end{pmatrix} = \tau \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \tau$$

Let us denote $\tau \mathrm{g} =: \varrho$. The adjoint-operation using $\vdash$-transposing can be written as follows

$$\varrho A^* = A^\vdash \varrho, \quad A^* = \varrho^{-1} A^\vdash \varrho,$$

consequently $A^* = -A$ is equivalent to $A^\vdash \varrho + \varrho A = 0$ or $A^\vdash = -\varrho A \varrho^{-1}$.

We get two versions of the formulas, for example:

$$\langle X, Y \rangle = \operatorname{tr} x^\top \mathrm{g} y = \operatorname{tr} x^\vdash \varrho y,$$

where x and y are the columns of the coordinates of the vectors X and Y.

We will use different bases on V. Let us consider the operation of the change of the basis. Let index $\langle e\rangle$ corresponds to the values in the basis $\mathbf{e}$ and the index $\langle f\rangle$ corresponds to the values in the basis $\mathbf{f}$, the bases are connected by matrix ϑ:

$$X = \mathbf{e}x_{\langle e\rangle} = \mathbf{f}x_{\langle f\rangle} = \mathbf{e}\vartheta(\vartheta^{-1}x_{\langle e\rangle}),$$

consequently

$$\mathbf{f} = \mathbf{e}\vartheta \Rightarrow x_{\langle f\rangle} = \vartheta^{-1}x_{\langle e\rangle}.$$

Let us consider the metric tensor:

$$\begin{aligned}\langle X, Y\rangle &= x^{\top}_{\langle e\rangle}\mathrm{g}_{\langle e\rangle}y_{\langle e\rangle} = x^{\vdash}_{\langle e\rangle}\varrho_{\langle e\rangle}y_{\langle e\rangle}\\ &= x^{\top}_{\langle f\rangle}\vartheta^{\top}\mathrm{g}_{\langle e\rangle}\vartheta y_{\langle f\rangle} = x^{\vdash}_{\langle f\rangle}\tau\vartheta^{\top}\tau\tau\mathrm{g}_{\langle e\rangle}\vartheta y_{\langle e\rangle} = x^{\vdash}_{\langle f\rangle}\vartheta^{\vdash}\varrho_{\langle e\rangle}\vartheta y_{\langle f\rangle},\end{aligned}$$

consequently

$$\mathbf{f} = \mathbf{e}\vartheta \Rightarrow \mathrm{g}_{\langle f\rangle} = \vartheta^{\top}\mathrm{g}_{\langle e\rangle}\vartheta, \quad \varrho_{\langle f\rangle} = \vartheta^{\vdash}\varrho_{\langle e\rangle}\vartheta, \quad A_{\langle f\rangle} = \vartheta^{-1}A_{\langle e\rangle}\vartheta.$$

Let us fix so called *hyperbolic basis* in V. Its Gram-matrix (the metric tensor) is the inversion matrix τ. It means that $\langle e_i, e_j\rangle = \delta_{i,-j}$.

The connection between the hyperbolic and the orthonormal bases can be chosen in different ways. We fix one of them:

$$\left(e^{\langle h\rangle}_{-k}, e^{\langle h\rangle}_{+k}\right)\begin{pmatrix} 1 & i \\ 1 & 1/i \end{pmatrix}/\sqrt{2} = \left(e^{\langle o\rangle}_{2k}, e^{\langle o\rangle}_{2k+1}\right),$$

where $e^{\langle h\rangle}_{-k}, e^{\langle h\rangle}_{+k}$ is a couple of the conjugated vectors ($\langle e^{\langle h\rangle}_{-k}, e^{\langle h\rangle}_{+k}\rangle = 1$) from the hyperbolic basis, and $e^{\langle o\rangle}_{2k}, e^{\langle o\rangle}_{2k+1}$ are two orthogonal unit vectors.

§2. Normal form in hyperbolic basis.

The base of our construction is the classical theorem that states: *"Two similar skew-symmetrical matrices are orthogonally similar"* (see [1]). It means that the parameters of the conjugation classes in the case of the orthogonal group are the same as for the general linear group, namely the number of the Jordan (cyclic) chains of vectors and the lengths of the chains. A length is the number of the segments of the chain. There is only one special restriction, that is *the number of the Jordan chains with the even lengths must be even* for each length.

Let us choose the "normal form" of the $\vdash$-skew-symmetrical matrix as follows

$$
J^{\langle h\rangle}=\left(\begin{array}{ccc|c|ccccccc}
\ddots & \ddots & \ddots & & & & & & & & \\
 & 0 & 0 & -\mathrm{P}_r^{\vdash} & & & & & & & \\
 & & 0 & 0 & \sigma & & & & & & \\
\hline
 & & & 0 & 0 & \mathrm{P}_r & & & & & \\
\hline
 & & & & 0 & 0 & \lvert\varkappa\rvert & & & & \\
 & & & & & 0 & 0 & \varkappa\rvert & & & \\
 & & & & & & 0 & 0 & \lvert\varkappa\rvert & & \\
 & & & & & & & 0 & 0 & \varkappa\rvert & \\
 & & & & & & & & 0 & 0 & \ddots \\
 & & & & & & & & & 0 & \ddots \\
 & & & & & & & & & & \ddots
\end{array}\right), \tag{1}
$$

where we use the notations $\sigma := \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$, $\varkappa\rvert := \begin{pmatrix} I \\ 0 \end{pmatrix}$, $\lvert\varkappa\rvert := \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix}$. Matrix P_r we define later. The sizes of the blocks have different values depending on their positions. Let us explain the construction carefully.

The matrix $J^{\langle h\rangle}$ has the block structure. The blocks correspond to the splitting of the basis on the subsets. We numerate the subsets by the index "k": $k \in \{-M, 1-M, \ldots, M-1, M\}$. So we have $2M+1$ block-rows and block-columns, numerated from $-M$ to $+M$. The central block-line and block-column are marked by the index zero. Zero row and column are placed between the couples of lines.

We will use the tensor notations. The first index corresponds to this block structure, i.e. $(J^{\langle h\rangle})^{k_1;}_{k_2;}$ is the block in the row k_1 and column k_2. For example $(J^{\langle h\rangle})^{0;}_{2;} = \mathrm{P}_r$, $(J^{\langle h\rangle})^{2;}_{4;} = \varkappa\rvert$ etc.. The next indexes like $(J^{\langle h\rangle})^{k_1;j_1;i_1}_{k_2;j_2;i_2}$ will correspond to a finer structure. The block $(J^{\langle h\rangle})^{k_1;j_1;}_{k_2;j_2;}$ is the block in the j_1-s row and j_2-s column of the partition of the block $(J^{\langle h \rangle})^{k_1;}_{k_2;}$. These partitions will be described later.

Let us denote the sizes of the partition that we are discussing now by m_k. It means that the size of $(J^{\langle h \rangle})^{\pm k_1;}_{\pm k_2;}$ is $m_{k_1} \times m_{k_2}$. If some $m_k = 0$, then the block-rows and block-columns with numbers $\pm k$ are absent.

The number m_k is defined in the following way. The difference $m_k - m_{k+2}$ is the number of the Jordan chains with $k+1$ units. The number m_0 is the total number of the chains with the odd lengths, the number m_1 is the total number of the chains with the even lengths. I remind that *all the values m_k with the odd k are even.*

The numbers of the Jordan chains and their lengths can be calculated using the ranks of the powers of the matrix, see formulas (4),(5) and the speculations after them.

From the definition of m_k follows, particularly, that the dimension of $\ker J^{\langle h \rangle}$ is equal to $m_0 + m_1$ and the dimension $N = \dim \mathsf{V}$ of the matrices from our Lie algebra $\mathfrak{so}$ is $m_0 + 2\sum_{k=1}^{M} m_k$.

The sequence of pairs $(\not{\nu}|, |\not{\nu}|)$ in the matrix (1) repeats periodically up to the right border of the matrix, their sizes depend on their positions in $J^{\langle h \rangle}$.

The elements over anti-diagonal are $\vdash$-antisymmetric to the elements that we have already described, matrix $J^{\langle h \rangle}$ is $\vdash$-antisymmetric: $(J^{\langle h \rangle})^{\vdash} = -J^{\langle h \rangle}$.

All we need to define now is $m_0 \times m_2$ matrix P_r. Let us define square $m_0 \times m_0$ matrix P first:

$$\mathrm{P} := \frac{1}{\sqrt{2}} \begin{pmatrix} i\mathrm{I} & \tau \\ \tau/i & \mathrm{I} \end{pmatrix}, m_0 \text{ is even;} \quad \mathrm{P} := \frac{1}{\sqrt{2}} \begin{pmatrix} i\mathrm{I} & 0 & \tau \\ 0 & \sqrt{2} & 0 \\ \tau/i & 0 & \mathrm{I} \end{pmatrix}, m_0 \text{ is odd.} \tag{2}$$

Matrix P_r, consists of the first m_2 columns of the matrix P.

From the property

$$\mathrm{P}^{\vdash}\mathrm{P} = \tau \tag{3}$$

follows[1] that $\mathrm{P}_r^{\vdash}\mathrm{P}_r = \tau$, and we see that the rank of the products of the blocks

$$(J^{\langle h \rangle})^{s+2;}_{s;} (J^{\langle h \rangle})^{s+4;}_{s+2;} \dots (J^{\langle h \rangle})^{s+2(n-1);}_{s+2(n-2);} (J^{\langle h \rangle})^{s+2n;}_{s+2(n-1);}$$

[1]The sizes of the matrices I, τ and 0 depend on the context.

is equal to the minimal size of these block-factors. The products are the only non-zero blocks of $(J^{\langle h\rangle})^n$. All these blocks stay in different rows and columns, consequently the rank of $(J^{\langle h\rangle})^n$ is equal to the sum of their ranks:

$$m_n + 2\sum_{k=n+1}^{M} m_k = \operatorname{rank}(J^{\langle h\rangle})^n. \qquad (4)$$

All values m_n can be calculated from this function because the second difference of $\operatorname{rank}(J^{\langle h\rangle})^n$ is

$$\operatorname{rank}(J^{\langle h\rangle})^{n-1} + \operatorname{rank}(J^{\langle h\rangle})^{n+1} - 2\operatorname{rank}(J^{\langle h\rangle})^n = m_{n-1} - m_{n+1}, \qquad (5)$$

and $m_n = 0$ if $n > M$.

From the other hand the difference $\operatorname{rank}(J^{\langle h\rangle})^n - \operatorname{rank}(J^{\langle h\rangle})^{n+1}$ is the number of the Jordan chains of the lengths $n+1$ and longer. The number of the chains of the length n is equal to the second difference of the ranks of $(J^{\langle h\rangle})^n$ that is $m_{n-1} - m_{n+1}$. We can make these values arbitrary, choosing the sizes m_k of the blocks.

It proves that any matrix from $\mathfrak{so}$ can be brought into the form (1) in the hyperbolic basis.

Note that we used *only ranks* of the blocks and their products. Consequently the following *"stability property"* takes place:

Any $\vdash$-antisymmetric matrix A from the algebraically open set belongs to one conjugation class if $(i+2 > j \Rightarrow A^{i;}_{j;} = 0)$, $m_{n-1} \geqslant m_{n+1}$, $m_{2k+1} = 0$ mod 2. *This class contains $J^{\langle h\rangle}$.*

§3. Jordan normal form of matrix from $\mathfrak{so}$.

It follows from the structure of $J^{\langle h\rangle}$, that its action is very similar to the action of a matrix in the Jordan normal form that either interchanges the basic vectors or annihilates them. Matrix $J^{\langle h\rangle}$ acts in the same way for all subsets of the basic vectors (up to a sign) except the basic vectors marked indexes "2;" and "0;".

Let us call the subsets of the basic vectors with the same first index *sectors*. How $J^{\langle h\rangle}$ acts in the sectors "2;" and "0;"? It follows from the equality $\mathrm{P}_r^{\vdash}\mathrm{P}_r = \tau$, that $J^{\langle h\rangle}$ sends a basic vector from the sector "2;" to some non-basic vector (it is a column of P_r), and sends *just this vector-columns of* P_r back to the basic set.

Note that $\mathrm{P}^{\vdash}\mathrm{P} = \tau$ can be treated as the Gram matrix of the set of vector-columns of P. Consequently *the images of the basic vectors from the sector "2;" form the orthonormal set of vectors.* Moreover, we can see that the last $m_0 - m_2$ vector-columns of P that complete P_r to P are an orthonormal set of vectors from the kernel of $J^{\langle h \rangle}$, and these $m_0 - m_2$ vectors do not belong to the image of $J^{\langle h \rangle}$.

Let us choose the columns of P as *a new basis* of the subspace enveloping the basic vectors from the "0;"-sector. We can see that the total space $\mathsf{V} \simeq \mathbb{C}^N$ is split on the direct sum of two orthogonal subspaces the dimensions of which are $N - m_0$ and m_0. The first subset is equipped with the hyperbolic basis and the second one is equipped with the orthonormal basis. The action of $J^{\langle h \rangle}$ on this basis is "Jordan-like", that means that the action is cyclic up to a sign. Let us formulate it more carefully as a Theorem.

Theorem 1. *The space V where matrix $A \in \mathfrak{so}$ acts as a linear transformation, can be split on the mutually orthogonal subspaces equipped with the non-degenerated scalar product induced from the ambient V. Such subspaces "reduce" the transformation A, i.e. the transformation can be contracted on the subspaces.*

The hyperbolic bases can be chosen on these subspaces in such a way that

- *If the subspace has odd dimension, there is a hyperbolic basis e_k that is cyclic up to the sign: $Ae_k = \pm e_{k-1}, Ae_{-k_{max}} = 0$, the dimension of the subspace is equal to $2k_{max} + 1$.*
- *If the subspace has even dimension, it is enveloping a pair of cyclic chains of the same even length. These cyclic (up to a sign) vectors form a hyperbolic basis of the subspace. The conjugated vectors of this hyperbolic basis belong to the different chains.*

We can see that the hyperbolic basis is very similar to the Jordan basis. The only problem is that the "central vector" of each lattice with the odd length is not isotropic, consequently there are several non-isotropic vectors in the Jordan basis. Their number is the number of the lattices with the odd lengths, we denoted this number by m_0.

Let us collect all these vectors to one coordinate subspace labelled index "0;". This subspace is equipped with the orthonormal basis formed by the columns of P. The orthogonal complement to the subspace is equipped with the hyperbolic basis.

The transformation of the initial hyperbolic basis where matrix from $\mathfrak{so}$ has form (1) to the Jordan basis, collected from the cyclic vectors of the matrix can be made by matrix φ:

$$\mathbf{e}^{\langle j\rangle} := \mathbf{e}^{\langle h\rangle} \left(\begin{array}{cccccc|c|ccccc}
\ddots &&&&&&&&&&& \\
& \sigma &&&&&&&&&& \\
&& \tau &&&&&&&&& \\
&&& -\sigma &&&&&&&& \\
&&&& -\tau &&&&&&& \\
&&&&& \sigma &&&&&& \\
\hline
&&&&&& \mathrm{P} &&&&& \\
\hline
&&&&&&& \mathrm{I} &&&& \\
&&&&&&&& \mathrm{I} &&& \\
&&&&&&&&& \mathrm{I} && \\
&&&&&&&&&& \mathrm{I} & \\
&&&&&&&&&&& \ddots
\end{array}\right) =: \mathbf{e}^{\langle h\rangle}\varphi. \tag{6}$$

We call basis $\mathbf{e}^{\langle j\rangle}$ *"a Jordan basis"* for the element of $\mathfrak{so}$. Matrix $J^{\langle h\rangle}$ becomes $\mathrm{J} = \varphi^{-1} J^{\langle h\rangle} \varphi$:

$$\mathrm{J} := \left(\begin{array}{ccccc|c|ccccc}
\ddots & \ddots & \ddots &&&&&&&& \\
& 0 & 0 & \underline{\varkappa} &&&&&&& \\
&& 0 & 0 & \overline{\underline{\varkappa}} &&&&&& \\
&&& 0 & 0 & \underline{\varkappa} &&&&& \\
&&&& 0 & 0 & \varkappa &&&& \\
\hline
&&&&& 0 & 0 & \varkappa| &&& \\
\hline
&&&&&& 0 & 0 & |\varkappa| && \\
&&&&&&& 0 & 0 & \varkappa| & \\
&&&&&&&& 0 & 0 & \ddots \\
&&&&&&&&& 0 & \ddots \\
&&&&&&&&&& \ddots
\end{array}\right) \tag{7}$$

We remind of the notations: $\varkappa| = \begin{pmatrix} \mathrm{I} \\ 0 \end{pmatrix}$, $|\varkappa| = \begin{pmatrix} 0 \\ \mathrm{I} \\ 0 \end{pmatrix}$, $\underline{\varkappa} = (\mathrm{I}|0)$, $\overline{\underline{\varkappa}} = (0|\mathrm{I}|0)$, and $\varkappa$ is the unit matrix.

The metric tensor $\varrho_{\langle j\rangle} := \varphi^{\vdash} \varrho_{\langle h\rangle} \varphi$ in this basis looks more complicated of course:

$$\varrho_{\langle j\rangle} := \left(\begin{array}{cccccc|c|cccccc}
\ddots &&&&&&&&&&&& \\
& \sigma &&&&&&&&&&& \\
&& \tau &&&&&&&&&& \\
&&& -\sigma &&&&&&&&& \\
&&&& -\tau &&&&&&&& \\
&&&&& \sigma &&&&&&& \\
\hline
&&&&&& \tau &&&&&& \\
\hline
&&&&&&& -\sigma &&&&& \\
&&&&&&&& -\tau &&&& \\
&&&&&&&&& \sigma &&& \\
&&&&&&&&&& \tau && \\
&&&&&&&&&&& -\sigma & \\
&&&&&&&&&&&& \ddots
\end{array}\right) \tag{8}$$

The quartet $(-\sigma, -\tau, \sigma, \tau)$ repeats periodically from the central "τ" to the right-lower corner. The quartet $(\sigma, -\tau, -\sigma, \tau) = (-\sigma, -\tau, \sigma, \tau)^{\vdash}$ repeats periodically from the central "τ" to the left-upper corner. Let us consider the fine structure of basis $\mathbf{e}^{\langle j\rangle}$ for the separation and describing the Jordan chains.

The non-trivial blocks of J are $\mathrm{J}^{i;}_{i+2;}$, consequently the iterations of $\mathrm{J} \in \mathrm{End}\,\mathsf{V}$ preserve the parity of the indexes and we can consider the subsets of the basis $\mathbf{e}^{\langle j\rangle}$ with the even and the odd indexes separately. Let us start from the even indexes that correspond to the lattices of odd lengths.

Consider the biggest even value of the index that is $k'' = 0 \mod 2$. The subset of the basic vectors $\mathbf{e}^{\langle j\rangle}_{k'';}$ will be moved by J to the subset $\mathbf{e}^{\langle j\rangle}_{k''-2;}$ by $m_{k''-2} \times m_{k''}$ matrix $\varkappa| = \begin{pmatrix} I \\ 0 \end{pmatrix}$. It means that the images of all $m_{k''}$ basic vectors from $\mathbf{e}^{\langle j\rangle}_{k'';}$ form the first part of the set $\mathbf{e}^{\langle j\rangle}_{k''-2;}$ preserving their order. We can see from the structure of J that the vectors $\mathbf{e}^{\langle j\rangle}_{k'';}$ do not belong to the image of J. So all the set $\mathbf{e}^{\langle j\rangle}_{k'';}$ consists of the starting vectors of the Jordan lattices. We will see later that they are the longest lattices with the odd lengths.

Consider the next pair of the sectors $\mathbf{e}^{\langle j\rangle}_{k''-2;}$ and $\mathbf{e}^{\langle j\rangle}_{k''-4;}$. Matrix $\varkappa|$ the size of which is $m_{k''-4}\times m_{k''-2}$ moves all the vectors of $\mathbf{e}^{\langle j\rangle}_{k''-2;}$ to the first part of $\mathbf{e}^{\langle j\rangle}_{k''-2;}$ preserving their order.

We see that there are three kinds of vectors in this sector. Those that are the images of the images of $\mathbf{e}^{\langle j\rangle}_{k'';}$, the images of the vectors from $\mathbf{e}^{\langle j\rangle}_{k''-2;}$ that do not belong to the image of J and the vectors that are not in the image of J.

We can continue the process. As a result the sector $\mathbf{e}^{\langle j\rangle}_{0;}$, the largest one, will be split on $k''/2+1$ parts that we call *districts*. The next sectors will be smaller and smaller if their numbers decrease from "0" to "$-k''$". This decreasing is made by matrix $\underline{\varkappa}=(\mathrm{I}|0)$. This matrix sends to zero the rightest district of the sector "$-k$" and shifts basic vectors from other districts to the sector "$-k-2$".

We can see that the step from $\mathbf{e}^{\langle j\rangle}_{0;}$ to $\mathbf{e}^{\langle j\rangle}_{-2;}$ annihilates the vectors from the "most fresh" district of $\mathbf{e}^{\langle j\rangle}_{0;}$, these are such vectors from the $\ker \mathrm{J}$ that do not belong to the image of J, the vectors of the Jordan lattices of the unit length. The next step annihilates the vectors that appear at the sector $\mathbf{e}^{\langle j\rangle}_{2;}$, these are the lattices of the length three and so on.

The splitting on the districts is the next, fine level of the splitting of the basis $\mathbf{e}^{\langle j\rangle}$. We numerate the districts by the positive even numbers in such a way that the iteration of J does not change the number of the district, it decreases the number of the sector on two units.

The vectors in $\mathbf{e}^{\langle j\rangle}_{k'';}$ and their J-images belong to the district number k''. The sector $\mathbf{e}^{\langle j\rangle}_{k''-2;}$ consists of vectors of two districts, namely the district k'' that is the image of $\mathbf{e}^{\langle j\rangle}_{k'';}$ and the district $k''-2$ that consists of the starting vectors of the lattices of the lengths $k''-1$.

The number of the district has the constant value for all the chains of the same length in all sectors. It is equal to the number of the generalized(!) *eigenvectors in the lattice.*

The minimum value of the district number in the sector coincides with the absolute value of the number of sector.

Consider the sectors with the odd indexes. The sectors contain the vectors of the lattices with even lengths. The action of the matrices $|\varkappa| = \begin{pmatrix} 0 \\ \mathrm{I} \\ 0 \end{pmatrix}, \underline{\overline{\varkappa}} = (0|\mathrm{I}|0)$ is very similar to the previous ones. The difference between this case and the previous ones is that now the increasing and decreasing of sectors occur simultaneously on both sides. Let us numerate the districts in the same manner:

The number of the district does not change during the J*-iterations and up to a sign coincides with the number of the generalized eigenvectors in the chain.*

For example. Let the number k' be the biggest odd number of the sectors. Then the last three sectors (we wrote them in brackets) consist of the following districts:

$$(\mathbf{e}^{\langle j\rangle}_{k'-4;-k'+4}\mathbf{e}^{\langle j\rangle}_{k'-4;-k'+2}\mathbf{e}^{\langle j\rangle}_{k'-4;-k'}\mathbf{e}^{\langle j\rangle}_{k'-4;k'}\mathbf{e}^{\langle j\rangle}_{k'-4;k'-2}\mathbf{e}^{\langle j\rangle}_{k'-4;k'-4})$$
$$(\mathbf{e}^{\langle j\rangle}_{k'-2;-k'+2}\mathbf{e}^{\langle j\rangle}_{k'-2;-k'}\mathbf{e}^{\langle j\rangle}_{k'-2;k'}\mathbf{e}^{\langle j\rangle}_{k'-2;k'-2})(\mathbf{e}^{\langle j\rangle}_{k';-k'}\mathbf{e}^{\langle j\rangle}_{k';k'}).$$

The sectors $\mathbf{e}^{\langle j\rangle}_{+1;}$ and $\mathbf{e}^{\langle j\rangle}_{-1;}$ have the same (the maximal) number of districts numbered as follows:

$$-1, -3, -5, \ldots, 2-k', -k', k', k'-2, \ldots, +5, +3, +1.$$

The action of J decreases the number of sector on two units and does not change the other indexes such as the numeration of the districts and the number inside a district that numbers the chains of the same lengths (we did not write that index). Let us describe the Gram matrix (8) of this basis.

§4. Scalar products of vectors of Jordan basis.

Consider the chains with the odd units first. The vectors that belong to the different chains are orthogonal. The non-vanishing products belong to the sectors with the opposite numbers and the same numbers of districts. The non-vanishing scalar products of the basic vectors are:

$$\langle \mathbf{e}^{\langle j\rangle}_{4k;s}, \quad \mathbf{e}^{\langle j\rangle}_{-4k;s}\rangle = +1, \quad \langle \mathbf{e}^{\langle j\rangle}_{4k-2;s}, \quad \mathbf{e}^{\langle j\rangle}_{2-4k;s}\rangle = -1.$$

We have the orthonormal basis in the zero sector, particularly. We do not write the index counting the chains of the same lengths.

Consider the chains with the even number of units. The vectors that belong to the different couples of chains are orthogonal. Consider the coupled chains. The non-vanishing products belong to the sectors with the opposite numbers and the opposite numbers of districts. Sometimes the non-vanishing products of the basic vectors are equal to "+1", sometimes they are equal to "−1", namely

$$\langle \mathbf{e}^{\langle j\rangle}_{4k-1;+s}, \mathbf{e}^{\langle j\rangle}_{1-4k;-s}\rangle = +1, \langle \mathbf{e}^{\langle j\rangle}_{4k-1;-s}, \mathbf{e}^{\langle j\rangle}_{2-4k;+s}\rangle = -1, \quad s > 0$$

$$\langle \mathbf{e}^{\langle j\rangle}_{4k-3;+s}, \mathbf{e}^{\langle j\rangle}_{3-4k;-s}\rangle = -1, \quad \langle \mathbf{e}^{\langle j\rangle}_{4k-3;-s}, \mathbf{e}^{\langle j\rangle}_{3-4k;+s}\rangle = +1, \quad s > 0.$$

We do not write the index counting couples of the chains of the same lengths again.

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