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AN ACTION OF THE KLEIN 4-GROUP ON THE ANGULAR VELOCITY

Abstract. Expressing the angular velocity via Euler angles is a key step, linking kinematics with rigid body dynamics. Once the components of angular velocity are found in a rotating frame, they are (simultaneously) found in an inertial (non-rotating) frame. And once the components are found for successive intrinsic rotations, they are just as readily found for successive extrinsic rotations. The action of the Klein 4-group on the angular velocity, which we describe in this paper, provides further insight into the kinematic relations of rigid body motion, including the critical motion of Dzhanibekov flipping wingnut.

Motivation

Consider the (rotational) motion of a rigid body about a fixed point O. Denote by $\boldsymbol{\omega}$ the pseudovector of its angular velocity. We shall, from now on, distinguish taking the "total" (time) derivative (d/dt), in an "absolute" (inertial) frame, from taking the "partial" (time) derivative ($\partial/\partial t$) in a rotating (body-fixed) frame. And we shall use the adjectives "absolute" and "rotating" in order to distinguish two corresponding coordinate systems, which origins are assumed to coincide with the point O.

In accordance with such notation, "Newton's second law for rotational motion" would be written as

$$\boldsymbol{\tau} = d\boldsymbol{m}/dt = \partial\boldsymbol{m}/\partial t + \boldsymbol{\omega} \times \boldsymbol{m},$$

where $\boldsymbol{\tau}$ and $\boldsymbol{m}$ are, respectively, the pseudovectors of torque (that is, the moment of external forces) and angular momentum, both measured about the point O, with the binary operation "$\times$" denoting the cross product.

Key words and phrases: angular velocity, angular momentum, Dzhanibekov effect, Euler angles, Galois axis, Klein 4-group, moving frame, orthonormal basis, principal axes of inertia, pseudovector, transition matrix.

The transition matrix

Let T denote the "transition matrix" from an (ordered) basis $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3$ of the absolute coordinate system to an (ordered) basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of the rotating coordinate system. The matrix T determines a linear operator, acting on the Euclidean space $\mathbb{E}^3$. Such an action might be represented via a (formal) multiplication by T (on the right):

$$\begin{pmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \boldsymbol{f}_3\end{pmatrix} T = \begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\end{pmatrix}.$$

The columns of the matrix T are the coordinates of the rotating basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, relative to the absolute basis $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3$, whereas, upon assuming orthonormality of both bases,[1] the rows of the same matrix T are coordinates of the absolute basis $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3$, relative to the rotating basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$.

Adopting Einstein summation convention,[2] we might express a given vector $\boldsymbol{u}$ as

$$\boldsymbol{u} = \alpha^i \boldsymbol{f}_i = \beta^i \boldsymbol{e}_i,$$

where the absolute coordinate α^i is the dot product $\boldsymbol{u} \cdot \boldsymbol{f}_i$, whereas the rotating coordinate β^i is the dot product $\boldsymbol{u} \cdot \boldsymbol{e}_i$, $i \in \{1, 2, 3\}$.
We point out that the same matrix T transforms, via a multiplication (on the left) the rotating coordinates of the vector $\boldsymbol{u}$ into its absolute coordinates, whereas the inverse S of the matrix T, transforms the absolute coordinates of the vector $\boldsymbol{u}$ into its rotating coordinates, that is,

$$T \begin{pmatrix}\beta^1 \\ \beta^2 \\ \beta^3\end{pmatrix} = \begin{pmatrix}\alpha^1 \\ \alpha^2 \\ \alpha^3\end{pmatrix}, \; S \begin{pmatrix}\alpha^1 \\ \alpha^2 \\ \alpha^3\end{pmatrix} = \begin{pmatrix}\beta^1 \\ \beta^2 \\ \beta^3\end{pmatrix}.$$

Thus, denoting the (linear) action of T on the vector $\boldsymbol{u}$ by $T \cdot \boldsymbol{u}$, we have

$$T \cdot \boldsymbol{u} = \begin{pmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \boldsymbol{f}_3\end{pmatrix} T \begin{pmatrix}\alpha^1 \\ \alpha^2 \\ \alpha^3\end{pmatrix} = \begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\end{pmatrix} \begin{pmatrix}\alpha^1 \\ \alpha^2 \\ \alpha^3\end{pmatrix} = \begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\end{pmatrix} T \begin{pmatrix}\beta^1 \\ \beta^2 \\ \beta^3\end{pmatrix}.$$

The inverse of the action of T is the action of S:

$$S \cdot \boldsymbol{u} = \begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\end{pmatrix} S \begin{pmatrix}\beta^1 \\ \beta^2 \\ \beta^3\end{pmatrix} = \begin{pmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \boldsymbol{f}_3\end{pmatrix} \begin{pmatrix}\beta^1 \\ \beta^2 \\ \beta^3\end{pmatrix} = \begin{pmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \boldsymbol{f}_3\end{pmatrix} S \begin{pmatrix}\alpha^1 \\ \alpha^2 \\ \alpha^3\end{pmatrix}.$$

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[1]The orthonormality of the bases implies the orthonormality of the matrix T, that is, the inverse of T coincides with its transpose.

[2]According to which the summation sign over a repeated index is omitted.

The rotating coordinates of $T \cdot \boldsymbol{u}$ coincide with the absolute coordinates of the vector $\boldsymbol{u}$, whereas the absolute coordinates of $S \cdot \boldsymbol{u}$ coincide with the rotating coordinates of the vector $\boldsymbol{u}$.

The coordinates of the angular velocity

Assume that the vector $\boldsymbol{u}$ is fixed in the absolute frame and put $\boldsymbol{v} := T \cdot \boldsymbol{u} = \alpha^i \boldsymbol{e}_i$. The (rotating) vector $\boldsymbol{v}$ is fixed in the rotating frame. Its velocity vector is

$$\dot{\boldsymbol{v}} = \begin{pmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \boldsymbol{f}_3\end{pmatrix} \dot{T} \begin{pmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\end{pmatrix} = \begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\end{pmatrix} S\dot{T} \begin{pmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\end{pmatrix} = \boldsymbol{\omega} \times \boldsymbol{v},^{3}$$

and we emphasize, that the indicated angular velocity $\boldsymbol{\omega}$ is precisely the angular velocity of the rotating frame. Thus, denoting the rotating coordinates of $\boldsymbol{\omega}$ by ω^i, $i \in \{1,2,3\}$, we must have

$$W := S\dot{T} = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}. \tag{1}$$

In a rotating coordinate system, the coordinates of the (fixed) vector $\boldsymbol{u}$ do not, of course, remain constant. In accordance with our notation, the following identity

$$\dot{\boldsymbol{u}} = \partial \boldsymbol{u}/\partial t + \boldsymbol{\omega} \times \boldsymbol{u} = \boldsymbol{0}$$

holds and might be be rewritten as

$$\begin{aligned}\dot{\boldsymbol{u}} = \begin{pmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \boldsymbol{f}_3\end{pmatrix} (T\dot{S} + \dot{T}S) \begin{pmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\end{pmatrix} \\ = \begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\end{pmatrix} \dot{S}T \begin{pmatrix}\beta^1\\ \beta^2\\ \beta^3\end{pmatrix} + \begin{pmatrix}\boldsymbol{f}_1 & \boldsymbol{f}_2 & \boldsymbol{f}_3\end{pmatrix} \dot{T}S \begin{pmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\end{pmatrix}.\end{aligned}$$

Note that the matrix $\dot{T}S$ (from which we might "extract" the absolute coordinates of $\boldsymbol{\omega}$) is orthogonally similar to the matrix W: $\dot{T}S = TWS$.

[3]The dot (above) denotes time derivative.

Euler angles kinematics

We introduce the Euler angles ψ (precession), θ (nutation) and ϕ (spin) via three successive "intrinsic" rotations, bringing the basis $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3$ to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. The first rotation, about $\boldsymbol{f}_3$ by the angle ψ, brings $\boldsymbol{f}_1$ to the vector $\boldsymbol{n}$, which lies along the so-called "line of nodes". The second rotation, about $\boldsymbol{n}$ by the angle θ, brings $\boldsymbol{f}_3$ to $\boldsymbol{e}_3$. The third rotation, about $\boldsymbol{e}_3$ by the angle ϕ, brings $\boldsymbol{n}$ to $\boldsymbol{e}_1$. Thereby, these three (elementary) rotations (eventually) bring $\boldsymbol{f}_2$ to $\boldsymbol{e}_2$. Note that, alternatively, we might have arrived at the same basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ via three "extrinsic" rotations, successively about $\boldsymbol{f}_3$ by the angle ϕ, then about $\boldsymbol{f}_1$ by the angle θ and, lastly, about $\boldsymbol{f}_3$ by the angle ψ.[4]

Put

$$\begin{aligned}
&W(\chi^1,\chi^2,\chi^3) := R(-\chi^3,-\chi^2,-\chi^1)\,\dot{R}(\chi^1,\chi^2,\chi^3),\\
&R(\chi^1,\chi^2,\chi^3) := R_3(\chi^1)R_1(\chi^2)R_3(\chi^3),\\
&R_3(\chi) := \begin{pmatrix}\cos\chi & -\sin\chi & 0\\ \sin\chi & \cos\chi & 0\\ 0 & 0 & 1\end{pmatrix}, \quad R_1(\chi) := \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\chi & -\sin\chi\\ 0 & \sin\chi & \cos\chi\end{pmatrix},
\end{aligned}$$

where the variables χ^1, χ^2 and χ^3 are presumed to be time-dependent.

As before, we denote the transition matrix from the absolute basis to the rotating basis by T, and denote its inverse, which is the transition matrix from the rotating basis to the absolute basis, by S. Thus,

$$\begin{aligned}
T &= R(\psi,\theta,\phi)\\
&= \begin{pmatrix} cc(\phi,\psi) - ss(\phi,\psi)\cos\theta & -cs(\psi,\phi) - cs(\phi,\psi)\cos\theta & ss(\psi,\theta)\\ cs(\phi,\psi) + cs(\psi,\phi)\cos\theta & -ss(\phi,\psi) + cc(\phi,\psi)\cos\theta & -cs(\psi,\theta)\\ ss(\phi,\theta) & cs(\phi,\theta) & \cos\theta \end{pmatrix},
\end{aligned}$$

where $cc(\phi,\psi) := \cos\phi\cos\psi$, $cs(\chi^1,\chi^2) := \cos\chi^1\sin\chi^2$, $ss(\chi^1,\chi^2) := \sin\chi^1\sin\chi^2$, and

$$S = R(-\phi,-\theta,-\psi).^5$$

[4] Hence, the Euler angles ψ, θ, ϕ reappear in a reversed order.

[5] Note that the function $cs(\cdot,\cdot)$ is an odd function, unlike the functions $cc(\cdot,\cdot)$ and $ss(\cdot,\cdot)$ which are even.

We might also calculate the rotating coordinates ω^i, $i \in \{1,2,3\}$, of the angular velocity $\boldsymbol{\omega}$ since

$$W(\psi,\theta,\phi) = S\dot{T} = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}$$

and so

$$\boldsymbol{\omega} = \omega^i\, \boldsymbol{e}_i = (\dot{\psi}\, ss(\phi,\theta) + \dot{\theta}\cos\phi)\boldsymbol{e}_1 + (\dot{\psi}\, cs(\phi,\theta) - \dot{\theta}\sin\phi)\boldsymbol{e}_2 + (\dot{\psi}\cos\theta + \dot{\phi})\boldsymbol{e}_3.$$

An action of the Klein 4-group

The Klein 4-group acts on the rotating coordinates ω^i, $i \in \{1,2,3\}$, viewed as functions of the Euler angles ψ,θ,ϕ. The two generating inversions are

$$\sigma_{01}: \boldsymbol{\omega} \leftrightarrow -\boldsymbol{\omega},^{6} \quad \sigma_{10}: (\psi,\theta,\phi) \leftrightarrow (-\phi,-\theta,-\psi).$$

Note that the action of σ_{10} on Euler angles induces the transposition $T \leftrightarrow S$ and further induces the transformation $W(\psi,\theta,\phi) = S\dot{T} \leftrightarrow T\dot{S} = W(-\phi,-\theta,-\psi)$. Thus, σ_{10} transforms the rotating coordinates of the pseudovector $\boldsymbol{\omega}$ to the absolute coordinates of the pseudovector $-\boldsymbol{\omega}$. The action of σ_{01} is induced by the transpositions $W(\psi,\theta,\phi) = S\dot{T} \leftrightarrow \dot{S}T = -W(\psi,\theta,\phi)$ and $-W(-\phi,-\theta,-\psi) = \dot{T}S \leftrightarrow T\dot{S} = W(-\phi,-\theta,-\psi)$.[7]

Denote by σ_{11} the (commutative) product of σ_{01} and σ_{10}. The action of σ_{11} upon the rotating coordinates of the angular velocity $\boldsymbol{\omega}$ is induced by the transformation $W(\psi,\theta,\phi) = S\dot{T} \leftrightarrow \dot{T}S = -W(-\phi,-\theta,-\psi)$, so it sends them to the absolute coordinates of the (same) pseudovector $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = (\dot{\phi}\, ss(\psi,\theta) + \dot{\theta}\cos\psi)\boldsymbol{f}_1 - (\dot{\phi}\, cs(\psi,\theta) - \dot{\theta}\sin\psi)\boldsymbol{f}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\boldsymbol{f}_3.$$

Discussion and conclusion

The group $\mathbb{Z}_2^3$ acts naturally on the (directed) principal axes of inertia. Its factor group, acting on two principal axes, corresponding to the two extreme moments of inertia is a Klein 4-group.[8] Two of the (non-trivial)

[6] Note that the instantaneous axis of rotation does not depend upon the sign of $\boldsymbol{\omega}$, whereas the direction of the axis does depend upon the sign of $\boldsymbol{\omega}$, as well as, it depends upon the orientation (that is, the handedness) of the coordinate system.

[7] Note that the two antisymmetric matrices $\dot{T}S$ and $T\dot{S}$ (which sum to zero) are simultaneously orthogonally similar to the two matrices $S\dot{T}$ and $\dot{S}T$.

[8] This Klein 4-group arises upon factoring the group $\mathbb{Z}_2^3$ by its (2-element) subgroup, which fixes the principal axis, corresponding to the intermediate moment of inertia.

elements of this factor group are orientation reversing: the element which reverses the direction of the axis, corresponding to the major moment of inertia and the element which reverses the direction of the axis, corresponding to the minor moment of inertia. The (commutative) product of these two (generating) elements is the non-trivial, orientation preserving element which reverses the direction of both axes, corresponding to the two extreme moments of inertia. The action of the Klein 4-group upon the angular velocity (which we have described) might also be viewed as either orientation preserving or orientation reversing. The orientation reversing elements σ_{01} and σ_{10} might be further distinguished as "spatial" and "temporal", respectively. The product of σ_{01} and σ_{10} is the orientation preserving element σ_{11}.

The Klein 4-group acts as well on the (directed) Galois axis, which was introduced in [1] and further described in [2–5]. It "naturally" generalizes the concept of the "axis of symmetry" without requiring the moments of intertia of a rigid body to adhere to any equality. A reflection across an (undirected) principal axis, corresponding to an extreme moment of inertia, is an element of this group. We might choose the two reflections across the principal axes, corresponding to the two extreme moments of inertia (major and minor), as two generating elements. Then the composition of these two reflections becomes the element flipping the direction of the Galois axis. The latter non-trivial element, along with the trivial element, preserves the orientation (whether clockwise or counterclockwise) of the (dual) rotary motion of the Dzhanibekov flipping wingnut, as exhibited via a 3D-animation, by E. A. Mityushov and N. E. Misura, available at `https://www.youtube.com/watch?v=e9wGPh-iiRw&list=PLvKkSzWgY7KUUSYDqHVdv0mst4nBnLe64`. The two (coplanar) herpolhodes,[9] corresponding to two opposing directions of a single Galois axis are "mirror-images" of each other.

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[9] A herpolhode is the curve traced out, in the absolute frame, by the endpoint of the angular velocity (which initial point coincides with body's fixed point).

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