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CONVEX HULLS OF RANDOM WALKS: CONIC INTRINSIC VOLUMES APPROACH

Abstract. Sparre Andersen discovered a celebrated distribution-free formula for the probability of a random walk remaining positive up to a moment n. Kabluchko et al. expanded on this result by calculating the absorption probability for the convex hull of multidimensional random walks. They approached this by transforming the problem into a geometric one, which they then solved using Zaslavsky's theorem. We propose a completely different approach that allows us to directly derive the generating function for the absorption probability. The cornerstone of our method is the Gauss–Bonnet formula for polyhedral cones.

§1. Introduction

The beautiful and groundbreaking result of Sparre Andersen [6, 7] states that for the random walk

$$S_k = X_1 + \cdots + X_k, \quad k = 1, \dots, n, \tag{1}$$

with the symmetric absolutely continuous i.i.d. increments, the probability of staying positive equals

$$\mathbb{P}[S_1 > 0, \dots, S_n > 0] = \frac{(2n-1)!!}{(2n)!!}. \tag{2}$$

In terms of generating functions, we have

$$\sum_{n=1}^{\infty} \mathbb{P}[S_1 > 0, \dots, S_n > 0]\, t^n = \frac{1}{\sqrt{1-t}}, \quad |t| < 1. \tag{3}$$

Since the probability in (2) is *distribution-free*, it seems natural that at the heart of this formula lies a deterministic combinatorial statement. Indeed, Sparre Andersen, in fact, proved the following.

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Lemma 1.1 ([7, Lemmas 1, 2]). *Let* $\mathsf{S}(n)$ *denote the symmetric group of order* n. *For* $\sigma \in \mathsf{S}(n)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$, *and arbitrary real* $x_1, \dots, x_n$, *define*

$$s_k(\sigma, \varepsilon) = \varepsilon(1)\, x_{\sigma(1)} + \cdots + \varepsilon(k)\, x_{\sigma(k)}, \quad k = 1, \dots, n.$$

If for any $\sigma \in \mathsf{S}(n)$, $\varepsilon \in \{-1,1\}^n$ *and* $k = 1, \dots, n$ *we have* $s_k(\sigma, \varepsilon) \neq 0$, *then*[1]

$$\sum_{\substack{\sigma \in \mathsf{S}(n) \\ \varepsilon \in \{-1,1\}^n}} \mathbf{1}\big[s_1(\sigma, \varepsilon) > 0, \dots, s_n(\sigma, \varepsilon) > 0\big] = (2n-1)!!. \tag{4}$$

This statement has been generalized to higher dimensions in [2]. In probabilistic language, it states the following. First of all, to avoid trivialities, we always assume that $n \geqslant d+1$. Next, suppose that a sequence of partial sums $\{S_1, \dots, S_n\}$ defined as in (1) forms a *d-dimensional symmetrically exchangeable random walk in general position.* It means that the increments $X_1, \dots, X_n$ are random vectors in $\mathbb{R}^d$ satisfying the following two properties:

(SE) for any $\sigma \in \mathsf{S}(n)$, $\varepsilon \in \{-1,1\}^n$,

$$\big(\varepsilon_1 X_{\sigma(1)}, \dots, \varepsilon_n X_{\sigma(n)}\big) \overset{\mathrm{d}}{=} \big(X_1, \dots, X_n\big);$$

(GP) for any indices $1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n$,

$$\mathbb{P}[X_{i_1}, \dots, X_{i_d} \text{ are linearly dependent}] = 0.$$

Then

$$\mathbb{P}[0 \notin \operatorname{conv}(S_1, S_2, \dots, S_n)] = 2\frac{P_{d-1}^{(n)} + P_{d-3}^{(n)} + \cdots}{2^n n!}, \tag{5}$$

where $\operatorname{conv}(\,\cdot\,)$ denotes the convex hull and $P_i^{(n)}$'s are the coefficients of the polynomial

$$(t+1)(t+3)\cdots(t+2n-1) = \sum_{i=0}^{n} P_i^{(n)} t^i. \tag{6}$$

The idea of the proof is as follows. First, the original problem was reduced to counting the number of the Weil chambers of type B_n in $\mathbb{R}^n$, which are non-trivially intersected by a generic linear subspace of codimension d.

[1] In fact, Sparre Andersen did more: he proved that the number of (σ, ε) for which $\sum_{k=1}^n \mathbf{1}[s_k(\sigma, \varepsilon) > 0] = m$ equals $(2n)!! \frac{(2m-1)!!}{(2m)!!} \frac{(2n-2m-1)!!}{(2n-2m)!!}$. Now this result is known as the *Discrete arcsine law* of Sparre Andersen. For its multidimensional version, see [3].

Then, by means of the Zaslavsky theorem [9], this number was expressed in terms of the coefficients of the characteristic polynomial of the hyperplane arrangement induced by the boundaries of the chambers which happened to be the left-hand side of (6).

In this short note, we give an alternative solution of the problem. Our main result is the following multidimensional version of (3).

Theorem 1.1. *For any d-dimensional random walk $\{S_1, \dots, S_n\}$ satisfying* (SE) *and* (GP) *properties,*

$$\sum_{n=1}^{\infty} \mathbb{P}[0 \notin \operatorname{conv}(S_1, S_2, \dots, S_n)]\, t^n \tag{7}$$
$$= \frac{2}{\sqrt{1-t}} \left(\frac{(-\log(1-t))^{d-1}}{(2d-2)!!} + \frac{(-\log(1-t))^{d-3}}{(2d-6)!!} + \cdots \right),$$

where $|t| < 1$ and the right-hand side has $\lfloor \frac{d+1}{2} \rfloor$ summands.

Although this result can be directly derived from [2], we present a completely different straightforward approach, where (7) turns out to be a corollary from the Gauss–Bonnet formula for polyhedral cones. Applied to the random walk, this formula readily gives

$$\mathbb{P}[0 \notin \operatorname{conv}(S_1, \dots, S_n)] = 2\, \mathbb{E}\, [\upsilon_{d-1}(C_n) + \upsilon_{d-3}(C_n) + \cdots],$$

where υ_k's are the so-called *conic intrinsic volumes* (introduced in the next section) of the *conic* hull C_n of the random walk. The most technical part of the proof of the theorem is the derivation of the generating function for $\mathbb{E}\, \upsilon_k(C_n)$:

$$\sum_{n=0}^{\infty} \mathbb{E}\, \upsilon_k(C_n)\, t^n = \frac{1}{(2k)!!} \frac{(-\log(1-t))^k}{\sqrt{1-t}}, \quad |t| < 1.$$

In particular, it has been done with the help of Lemma 1.1 along with its following *bridge* version, also due to Sparre Andersen.

Lemma 1.2 ([8, Corollary 2])**.** *Denote by $\mathsf{C}(n)$ a subgroup of the symmetric group $\mathsf{S}(n)$ consisting of all cyclic shifts. For $\tau \in \mathsf{C}(n)$ and arbitrary real $x_1, \dots, x_n$, let*

$$s_k(\tau) = x_{\tau(1)} + \cdots + x_{\tau(k)}, \quad k = 1, \dots, n.$$

If

$$x_1 + \cdots + x_n = 0 \quad (\textit{the bridge property})$$

and for any $\tau \in \mathsf{C}(n)$ *and* $k = 1, \ldots, n-1$ *we have* $s_k(\tau) \neq 0$*, then*

$$\sum_{\tau \in \mathsf{C}(n)} \mathbf{1}[s_1(\tau) > 0, \ldots, s_n(\tau) > 0] = 1.$$

While Lemma 1.1 is fairly complicated, this one is straightforward. Among n cyclic shifts τ there is exactly one for which all partial sums are positive: it corresponds to the moment when the walk

$$x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_{n-1}$$

achieves its minimum.

Our main result will be proved in Section 3. The following section will acquaint the reader with key concepts from convex geometry, essential for understanding the proof.

§2. Conic intrinsic volumes

In this section, we recall the definition of the conic intrinsic volumes and discuss their basic properties. They are defined for the arbitrary convex cones; however, it will be convenient for us to use an alternative definition which makes sense only for the *polyhedral* cones. For a more detailed and comprehensive understanding of conic intrinsic volumes and their applications, the reader is referred to [1] or [5, Chap. 6.5], which offer an in-depth exploration of the topic.

Let $C \subset \mathbb{R}^d$ be a polyhedral cone, that is, an intersection of finitely many closed half-spaces in $\mathbb{R}^d$ with boundaries passing through the origin. By definition, the dimension of C coincides with the dimension of its linear span denoted by $\operatorname{span} C$. Denote by $\operatorname{relint} C$ its relative interior, that is, the interior with respect to $\operatorname{span} C$.

A linear hyperplane H dividing $\mathbb{R}^d$ into two half-spaces such that C lies entirely in one of them is called a *supporting hyperplane* of C and $C \cap H$ is called a *face* of C. Additionally, C itself belongs to the set of its faces, which we denote by $\mathcal{F}(C)$. Every face of C is a polyhedral cone. A face of dimension k is also called a *k-face*. Denote by $\mathcal{F}_k(C)$ the set of k-faces of C.

Denote by $\alpha(C)$ the *solid angle* of C defined as

$$\alpha(C) := \mathbb{P}[U \in C], \tag{8}$$

where U is a random vector uniformly distributed in $\mathbb{S}^{d-1} \cap \operatorname{span} C$. It follows from definition that $\alpha(C)$ does not depend on the ambient space and is always positive. In particular, $\alpha(\{0\}) = 1$.

The polyhedral cone *polar* to C is defined as

$$C^\circ = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leqslant 0 \text{ for all } \mathbf{y} \in C\}. \tag{9}$$

For a k-face $F \in \mathcal{F}_k(C)$ consider some $\mathbf{x}_0 \in \operatorname{relint} F$ and denote by $T_F(C)$ the *tangent cone* to C at F defined as

$$T_F(C) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_0 + \varepsilon \mathbf{x} \in C \text{ for some } \varepsilon > 0\}.$$

Clearly, $T_F(C)$ does not depend on the choice of $\mathbf{x}_0 \in \operatorname{relint} F$. The cone polar to $T_F(C)$ is called the *normal cone* to C at F and denoted by $N_F(C)$:

$$N_F(C) = (T_F(C))^\circ. \tag{10}$$

Let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be some linearly independent vectors in $\mathbb{R}^d$. We have

$$\operatorname{span}(\mathbf{x}_1, \dots, \mathbf{x}_k) \cong \mathbb{R}^k.$$

Denote by

$$h_{\mathbf{x}_1,\dots,\mathbf{x}_k} : \mathbb{R}^k \to \operatorname{span}(\mathbf{x}_1, \dots, \mathbf{x}_k) \tag{11}$$

an isometry that aligns the standard orthonormal basis of $\mathbb{R}^k$ with the basis in $\operatorname{span}(\mathbf{x}_1, \dots, \mathbf{x}_k)$ obtained by applying the Gram–Schmidt process to $\mathbf{x}_1, \dots, \mathbf{x}_k$.

Let C be a *conic hull* of $\mathbf{x}_1, \dots, \mathbf{x}_n$, that is, an intersection of all convex cones containing $\mathbf{x}_1, \dots, \mathbf{x}_n$:

$$C = \operatorname{cone}(\mathbf{x}_1, \dots, \mathbf{x}_n).$$

Let us also observe that it readily follows from (8), (9), and (10) that for any face $F \in \mathcal{F}(C)$ we have

$$\begin{aligned} \alpha(N_F(C)) &= \mathbb{P}[\langle \mathbf{x}_i, U \rangle \leqslant 0, i = 1, \dots, n] \\ &= \mathbb{P}[\langle \mathbf{x}_i, U \rangle \geqslant 0, i = 1, \dots, n], \\ &= \mathbb{P}[\langle \mathbf{x}_i, U \rangle > 0, i = 1, \dots, n], \end{aligned} \tag{12}$$

where U is a random vector uniformly distributed in $(\operatorname{span} F)^\perp \cap \mathbb{S}^{d-1}$.

The k-th *conic intrinsic volume* of C can be defined as

$$\upsilon_k(C) = \sum_{F \in \mathcal{F}_k(C)} \alpha(N_F(C))\,\alpha(F). \tag{13}$$

In particular, if $\dim C = k$, then, by definition,

$$\upsilon_k(C) = \alpha(C).$$

The conic intrinsic volumes form a probability distribution on $\{0, 1, \dots, d\}$ for a fixed cone C:

$$\sum_{k=0}^{d} \upsilon_k(C) = 1. \tag{14}$$

In particular, if C is a linear subspace of dimension j, then $\upsilon_j(C) = 1$ and $\upsilon_k(C) = 0$ for $k \neq j$. The conic intrinsic volumes satisfy the following version of the Gauss–Bonnet theorem (see [5, Section 6.5]):

$$\sum_{k=0}^{d} (-1)^k \upsilon_k(C) = \begin{cases} (-1)^{\dim C} & \text{if } C \text{ is a linear subspace,} \\ 0 & \text{otherwise.} \end{cases}$$

If $\dim C = d$, then together with (14) this implies

$$2\left(\upsilon_{d-1}(C) + \upsilon_{d-3}(C) + \cdots\right) = \begin{cases} 0 & \text{if } C = \mathbb{R}^d, \\ 1 & \text{otherwise.} \end{cases} \tag{15}$$

Let us finish this section by a simple observation, which we will need in the proof of Theorem 1.1: for an arbitrary set K in $\mathbb{R}^d$ we have

$$\operatorname{cone} K = \mathbb{R}^d \quad \text{if and only if} \quad 0 \in \operatorname{int} \operatorname{conv} K. \tag{16}$$

§3. Proof of Theorem 1.1

It follows from the (GP) property that

$$\mathbb{P}[0 \in \operatorname{conv}(S_1, \dots, S_n)] = \mathbb{P}[0 \in \operatorname{int} \operatorname{conv}(S_1, \dots, S_n)],$$

which together with (16) leads to

$$\mathbb{P}[0 \notin \operatorname{conv}(S_1, \dots, S_n)] = \mathbb{P}[C_n \neq \mathbb{R}^d], \tag{17}$$

where $C_n = \operatorname{cone}(S_1, \dots, S_n)$. Since $n \geqslant d$, and due to the (GP) property we have that $\dim C_n = d$ almost surely. Therefore (15) implies

$$\mathbb{P}[C_n \neq \mathbb{R}^d] = 2\mathbb{E}\left[\upsilon_{d-1}(C_n) + \upsilon_{d-3}(C_n) + \cdots\right]. \tag{18}$$

Now let us fix some $k \leqslant d-1$ and calculate $\mathbb{E}\, \upsilon_k(C_n)$. Consider some indices $1 \leqslant i_1 < \dots < i_k \leqslant n$. The simplicial cone

$$C_k = C_k(i_1, \dots, i_k) = \operatorname{cone}(S_{i_1}, \dots S_{i_k})$$

may or may not be a k-face of C_n. Moreover, with probability one, any k-face of C_n has this form for some $1 \leqslant i_1 < \dots < i_k \leqslant n$. Therefore, according to (13),

$$\mathbb{E}\, \upsilon_k(C_n) = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} \mathbb{E}\left[\alpha(N_{C_k}(C_n))\, \alpha(C_k)\, \mathbf{1}[C_k \in \mathcal{F}_k(C_n)]\right]. \tag{19}$$

Fix some $1 \leqslant i_1 < \dots < i_k \leqslant n$. Let $V \in \mathbb{R}^k$ be a random vector uniformly distributed in $\mathbb{S}^k$, independent of the random walk. Define

$$U = h_{S_{i_1}, \dots, S_{i_k}}(V),$$

where $h_{S_{i_1}, \dots, S_{i_k}}$ is the isometry between $\mathbb{R}^k$ and $\operatorname{span}^{\perp}(S_{i_1}, \dots, S_{i_k})$ defined in (11).[2]

Applying (12) to $\alpha(N_{C_k}(C_n))$ leads to

$$\begin{aligned} &\mathbb{E}\left[\alpha(N_{C_k}(C_n))\, \alpha(C_k)\, \mathbf{1}[C_k \in \mathcal{F}_k(C_n)]\right] \\ &\quad = \mathbb{E}\left[\mathbf{1}[\langle S_i, U\rangle > 0, i = 1, \dots, n] \cdot \alpha(C_k)\right], \end{aligned} \tag{20}$$

where we also used the observation that

$$\mathbf{1}[\langle S_i, U\rangle > 0, i = 1, \dots, n] = 0 \quad \text{for} \quad C_k \notin \mathcal{F}_k(C_n).$$

Now we aim to calculate the right-hand side of (20). Consider

$$Y_1 = \langle X_1, U\rangle, \dots, Y_n = \langle X_n, U\rangle, \tag{21}$$

which are random variables in $\mathbb{R}^1$ such that

$$Y_{i_l+1} + \dots + Y_{i_{l+1}} = 0, \quad l = 1, \dots, k. \tag{22}$$

It follows from the fact that

$$U \in \operatorname{span}^{\perp}(S_{i_1}, \dots, S_{i_k}).$$

[2]This is well-defined only if $S_{i_1}, \dots, S_{i_k}$ are linearly independent, which occurs with probability one. Otherwise, we define $h_{S_{i_1}, \dots, S_{i_k}}$ as the zero function.

Now for $l = 1, \dots, k+1$ denote

$$\begin{aligned} S_1^{(l)} &= Y_{i_{l-1}+1}, \\ S_2^{(l)} &= Y_{i_{l-1}+1} + Y_{i_{l-1}+2}, \\ &\dots, \\ S_{i_l - i_{l-1}}^{(l)} &= Y_{i_{l-1}+1} + \dots + Y_{i_l}, \end{aligned} \tag{23}$$

where we assumed $i_0 = 0, i_{k+1} = n$.

Given the (GP) property of the original random walk, the increments of

$$S_1^{k+1}, S_2^{k+1}, \dots, S_{n-i_k}^{k+1}$$

with probability one satisfy the assumptions of Lemma 1.1. Similarly, owing to (22), for all $l = 1, \dots, k$ the increments of

$$S_1^l, S_2^l, \dots, S_{i_k - i_{k-1}}^l.$$

with probability one satisfy the assumptions of Lemma 1.2. Thus, we obtain two important relations: with probability one,

$$\sum_{\substack{\sigma \in \mathsf{S}(n-i_k) \\ \varepsilon \in \{-1,1\}^n}} \mathbf{1}[S_1^{(k+1))}(\sigma, \varepsilon), \dots, S_{n-i_k}^{(k+1)}(\sigma, \varepsilon) > 0] = (2n - 2i_k - 1)!! \tag{24}$$

and for $l = 1, \dots, k$,

$$\sum_{\tau_l \in \mathsf{C}(i_l - i_{l-1})} \mathbf{1}[S_1^{(l)}(\tau_l), \dots, S_{i_l - i_{l-1} - 1}^{(l)}(\tau_l) > 0] = 1. \tag{25}$$

Recalling (21), (22), and (23), we have

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}[\langle S_1, U\rangle, \dots, \langle S_n, U\rangle > 0] \cdot \alpha(C_k)\right] \\ &\quad = \mathbb{E}\left[\mathbf{1}[S_1^{(k+1)}, \dots, S_{n-i_k}^{(k+1)} > 0] \prod_{l=1}^{k+1} \mathbf{1}[S_1^{(l)}, \dots, S_{i_l - i_{l-1} - 1}^{(l)} > 0] \cdot \alpha(C_k)\right]. \end{aligned}$$

Now, by applying the (SE) property and the notation from Lemmas 1.1 and 1.2, we obtain

$$(2n - 2i_k)!! \prod_{l=1}^{k} (i_k - i_{k-1}) \left[\mathbf{1}[\langle S_1, U\rangle, \dots, \langle S_n, U\rangle > 0] \cdot \alpha(C_k)\right]$$

$$= \sum_{\substack{\sigma\in\mathsf{S}(n-i_k)\\ \varepsilon\in\{-1,1\}^n}} \sum_{\tau_1\in\mathsf{C}(i_1)} \dots \sum_{\tau_k\in\mathsf{C}(i_k-i_{k-1})} \mathbb{E}\Bigg[\mathbf{1}[S_1^{(k+1))}(\sigma,\varepsilon),\dots,S_{n-i_k}^{(l)}(\sigma,\varepsilon)>0] \times \prod_{l=1}^{k} \mathbf{1}[S_1^{(l)}(\tau_l),\dots,S_{i_l-i_{l-1}-1}^{(l)}(\tau_l)>0]\cdot\alpha(C_k)\Bigg]$$

$$= \mathbb{E}\Bigg(\sum_{\substack{\sigma\in\mathsf{S}(n-i_k)\\ \varepsilon\in\{-1,1\}^n}} \mathbf{1}[S_1^{(k+1)}(\sigma,\varepsilon),\dots,S_{n-i_k}^{(k+1)}(\sigma,\varepsilon)>0]\Bigg) \times \prod_{l=1}^{k}\Bigg(\sum_{\tau_l\in\mathsf{C}(i_l-i_{l-1})} \mathbf{1}[S_1^{(l)}(\tau_l),\dots,S_{i_l-i_{l-1}-1}^{(l)}(\tau_l)>0]\Bigg)\cdot\alpha(C_k)\Bigg]$$

Applying (24) and (25) immediately leads to

$$\mathbb{E}\left[\mathbf{1}[\langle S_1,U\rangle,\dots,\langle S_n,U\rangle>0]\cdot\alpha(C_k)\right] = p_{n-i_k}\prod_{l=1}^{k}\frac{1}{i_k-i_{k-1}}\mathbb{E}\left[\alpha(C_k)\right],$$

where

$$p_m = \frac{(2n-2m-1)!!}{(2n-2m)!!}.$$

Recalling (19) and (20), we arrive at

$$\mathbb{E}\,\upsilon_k(C_n) = \sum_{1\leqslant i_1<\dots<i_k\leqslant n} p_{n-i_k}\prod_{l=1}^{k}\frac{1}{i_k-i_{k-1}}\mathbb{E}\,\alpha(\mathrm{cone}(S_{i_1},\dots S_{i_k})) \tag{26}$$

$$= \sum_{j=0}^{n-k} p_j \sum_{\substack{j_1+\dots+j_k=n-j\\ j_1,\dots,j_k>0}} \frac{1}{j_1 j_2\cdots j_k}\mathbb{E}\,\alpha(\mathrm{cone}(S_{j_1},\dots S_{j_1+\dots+j_k})),$$

where in the last step we changed the variables to

$$j_1=i_1, j_2=i_2-i_1,\dots,j_k=i_k-i_{k-1}, j=n-i_k.$$

Let us calculate the inner sum. We have:

$$\sum_{\substack{j_1+\dots+j_k=n-j\\ j_1,\dots,j_k>0}} \frac{1}{j_1 j_2 \cdots j_k} \mathbb{E}\,\alpha(\mathrm{cone}(S_{j_1},\dots S_{j_1+\dots+j_k})) \tag{27}$$

$$= \sum_{\substack{j'_1+\dots+j'_k=n-j\\ 0<j'_1\leqslant\dots\leqslant j'_k}} \sum_{J(j'_1,\dots,j'_k)} \frac{1}{j_1 j_2 \cdots j_k} \mathbb{E}\,\alpha(\mathrm{cone}(S_{j_1},\dots S_{j_1+\dots+j_k})),$$

where $J(j'_1,\dots,j'_k)$ is a set of all k-tuples $(j_1,\dots,j_k)$ such that

$$(j_{\sigma(1)},\dots,j_{\sigma(k)}) = (j'_1,\dots,j'_k)$$

for some permutation $\sigma \in \mathsf{S}(k)$. Fix some $\mathbf{j}' = (j'_1,\dots,j'_k)$ such that $j'_1 + \dots + j'_k = n-j$ and $0 < j'_1 \leqslant \ldots \leqslant j'_k$. It is notable that $J(j'_1,\dots,j'_k)$ can be parametrized by elements of the quotient group

$$\mathsf{S}'(k) = \mathsf{S}(k)/\mathsf{S}_0(k),$$

where $\mathsf{S}_0(k)$ is a subgroup of $\mathsf{S}(k)$ consisting of permutations that leave $\mathbf{j}'$ unchanged. Therefore, we may think of $\mathsf{S}'(k)$ as a group whose actions on $\mathbf{j}'$ generate the set $J(j'_1,\dots,j'_k)$. This approach simplifies our analysis by reducing the permutations to only those that result in distinct tuples. In particular, we have

$$\sigma\mathsf{S}'(k) = \mathsf{S}'(k) \quad \text{for any} \quad \sigma \in \mathsf{S}(k), \tag{28}$$

indicating the equivalence of all cosets in $\mathsf{S}'(k)$ irrespective of the permutation σ applied.

Let ρ be uniformly chosen from $\mathsf{S}'(k)$, independently with the random walk. Then, we have

$$\sum_{J(j'_1,\dots,j'_k)} \frac{1}{j_1 j_2 \cdots j_k} \mathbb{E}\,\alpha(\mathrm{cone}(S_{j_1},\dots S_{j_1+\dots+j_k})) \tag{29}$$

$$= \frac{|J(j'_1,\dots,j'_k)|}{j_1 j_2 \cdots j_k} \mathbb{E}\,\alpha(\mathrm{cone}(S_{\rho(j'_1)},\dots S_{\rho(j'_1)+\dots+\rho(j'_k)})),$$

where $|\cdot|$ denotes the cardinality of the set. Next, we aim to demonstrate that

$$\mathbb{E}\,\alpha(\mathrm{cone}(S_{\rho(j'_1)},\dots,S_{\rho(j'_1)+\dots+\rho(j'_k)})) = \frac{1}{(2k)!!}. \tag{30}$$

To achieve this, let us first establish that the increments of $S_{\rho(j'_1)},\dots,S_{\rho(j'_1)+\dots+\rho(j'_k)}$ are symmetrically exchangeable. This property is crucial as exchangeability implies certain symmetries in the random walk, which are essential for the validity of equation (30).

For a given k-tuple $\mathbf{j} = (j_1, \dots, j_k) \in J(j_1', \dots, j_k')$, define the variables

$$\begin{aligned} R_1(\mathbf{j}) &= X_1 + \cdots + X_{j_1}, \\ R_2(\mathbf{j}) &= X_{j_1+1} + \cdots + X_{j_2}, \\ &\dots \\ R_k(\mathbf{j}) &= X_{j_{k-1}+1} + \cdots + X_{j_k}, \end{aligned}$$

which represent the sums of the increments over specific intervals. To demonstrate that $R_1(\rho(\mathbf{j}'), \dots, R_k(\rho(\mathbf{j}')$ are symmetrically exchangeable, consider a Borel set $B \subset \mathbb{R}^k$ and some $\sigma \in \mathsf{S}(k), \varepsilon \in \{-1, +1\}^k$. We obtain

$$\begin{aligned} &\mathbb{P}\big[\big(\varepsilon_1 R_1(\sigma\rho(\mathbf{j}')), \dots, \varepsilon_k R_k(\sigma\rho(\mathbf{j}'))\big) \in B\big] \\ &\quad = \mathbb{P}\big[\big(R_1(\sigma\rho(\mathbf{j}')), \dots, R_k(\sigma\rho(\mathbf{j}'))\big) \in B\big] \\ &\quad = |\mathsf{S}'(k)|^{-1} \sum_{\sigma' \in \mathsf{S}'(k)} \mathbb{P}\big[\big(R_1(\sigma\sigma'(\mathbf{j}')), \dots, R_k(\sigma\sigma'(\mathbf{j}'))\big) \in B\big], \end{aligned} \tag{31}$$

where in the first step we used (SE) property. It follows from (28) that

$$\sigma J(j_1', \dots, j_k') = J(j_1', \dots, j_k'),$$

implying that permuting the indices within the set $J(j_1', \dots, j_k')$ preserves its structure. This supports the exchangeability of $R_1(\rho(\mathbf{j}')), \dots, R_k(\rho(\mathbf{j}'))$:

$$\begin{aligned} &|\mathsf{S}'(k)|^{-1} \sum_{\sigma' \in \mathsf{S}'(k)} \mathbb{P}\big[\big(R_1(\sigma\sigma'(\mathbf{j}')), \dots, R_k(\sigma\sigma'(\mathbf{j}'))\big) \in B\big] \\ &\qquad = |\mathsf{S}'(k)|^{-1} \sum_{\sigma' \in \mathsf{S}'(k)} \mathbb{P}\big[\big(R_1(\sigma'(\mathbf{j}')), \dots, R_k(\sigma'(\mathbf{j}'))\big) \in B\big] \\ &\qquad = \mathbb{P}\big[\big(R_1(\rho(\mathbf{j}')), \dots, R_k(\rho(\mathbf{j}'))\big) \in B\big], \end{aligned}$$

confirming together with (31) that the increments

$$Z_1 = R_1(\rho(\mathbf{j}')), \dots, Z_k = R_k(\rho(\mathbf{j}'))$$

are symmetrically exchangeable. Introducing $Z_1, \dots, Z_k$ at this stage simplifies our notation and will facilitate future calculations.

Now we are in a position to prove (30). Symmetric exchangeability implies that

$$\begin{aligned}&\mathbb{E}\,\alpha(\mathrm{cone}(S_{\rho(j_1')},\dots,S_{\rho(j_1')+\dots+\rho(j_k')}))\\&\quad=\mathbb{E}\,\alpha(\mathrm{cone}(Z_1,Z_1+Z_2,\dots,Z_1+\dots+Z_k))\\&\quad=\frac{1}{(2k)!!}\sum_{\substack{\sigma\in\mathsf{S}(n-i_k)\\ \varepsilon\in\{-1,1\}^n}}\mathbb{E}\,\alpha(\mathrm{cone}(\varepsilon_1 Z_{\sigma(1)},\dots,\varepsilon_1 Z_{\sigma(1)}+\dots+\varepsilon_k Z_{\sigma(k)})).\end{aligned}$$

Note that the sum in the right-hand side equals 1 with probability one. This follows from the fact that the $(2k)!!$ cones in this sum do not intersect by their interiors, and their union is $\mathbb{R}^k$. To see this, consider a linear transformation mapping $Z_1,\dots,Z_k$ (which are a.s. linearly independent) to the standard orthonormal basis $\mathbf{e}_1,\dots,\mathbf{e}_k$ in $\mathbb{R}^k$. This transformation will correspondingly map the cones into the Weil chambers:

$$\begin{aligned}&\mathrm{cone}(\varepsilon_1\mathbf{e}_{\sigma(1)},\varepsilon_1\mathbf{e}_{\sigma(1)}+\varepsilon_2\mathbf{e}_{\sigma(2)},\dots,\varepsilon_1\mathbf{e}_{\sigma(1)}+\dots+\varepsilon_k\mathbf{e}_{\sigma(k)})\\&\quad=\{\mathbf{x}=(x_1,\dots,x_k)\in\mathbb{R}^k:\varepsilon_1x_{\sigma(1)}\geqslant\varepsilon_2x_{\sigma(2)}\geqslant\dots\geqslant\varepsilon_kx_{\sigma(k)}\}.\end{aligned}$$

Combining just proved (30) with (29), (27), and (26) gives

$$\mathbb{E}\,\upsilon_k(C_n)=\frac{1}{(2k)!!}\sum_{1\leqslant i_1<\dots<i_k\leqslant n}\frac{p_{n-i_k}}{i_1(i_2-i_1)\cdots(i_k-i_{k-1})},$$

from which summing over n from zero to infinity we obtain

$$\sum_{n=0}^{\infty}\mathbb{E}\,\upsilon_k(C_n)t^n=\frac{1}{(2k)!!}\frac{(-\log(1-t))^k}{\sqrt{1-t}},\quad |t|<1.$$

Summing this up over $k=d-1,d-3,\dots$ and recalling (17) and (18) finishes the proof.

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