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BOUNDED GENERATION OF RELATIVE SUBGROUPS IN CHEVALLEY GROUPS

Abstract. The problem of bounded elementary generation is now completely settled for all Chevalley groups of rank $\geqslant 2$ over arbitrary Dedekind rings R of arithmetic type with the fraction field K, with uniform bounds. Namely, for every reduced irreducible root system Φ of rank $\geqslant 2$ there exists a *uniform* bound $L = L(\Phi)$ such that the simply connected Chevalley groups $\mathrm{G}(\Phi, R)$ have elementary width $\leqslant L$ for all Dedekind rings of arithmetic type, [18]. It is natural to ask, whether similar result holds for the relative elementary groups $E(\Phi, R, I)$, where $I \trianglelefteq R$. Mating the usual rewriting argument, already invoked in this context by Tavgen [28], with the universal localisation by Stepanov [25], we can give a very short proof that this is indeed the case. In other words, the width of $E(\Phi, R, I)$ in *elementary conjugates* $z_\alpha(\xi, \zeta) = x_{-\alpha}(\zeta)x_\alpha(\xi)x_{-\alpha}(-\zeta)$, where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in R$, is indeed bounded by some constant $M = M(\Phi, R, I)$. However, the resulting bounds M are not uniform, they depend on the pair (R, I).

In the present paper we discuss the problem of bounded generation of the *relative* elementary subgroups $E(\Phi, R, I)$ of Chevalley groups $G(\Phi, R)$, where $\mathrm{rk}(\Phi) \geqslant 2$, whereas $I \trianglelefteq R$ is an ideal of a Dedekind ring R of arithmetic type. We prove that these groups are boundedly generated in terms of elementary conjugates.

§1. Absolute bounded elementary generation

Let X be a symmetric generating set of a group G, $G = \langle X \rangle$, $X = X^{-1}$. The **length** $l_X(g)$ of an element $g \in G$ with respect to X is the length l of a *shortest* expression of g as a product $g = x_1 \dots x_l$, where $x_i \in X$. The **width** $w_X(G)$ of G with respect to X is now defined as the supremum of lengths $l_X(g)$ over all $g \in G$. In other words, $w_X(G)$ is the diameter of the Cayley graph of G with respect to X. If $w_X(G)$ is finite, we say that G is **boundedly generated** by X.

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Ultimately, we are interested in Chevalley groups $G = G(\Phi, R)$ and their elementary subgroups $E(\Phi, R)$, over various classes of rings, mostly over Dedekind rings of arithmetic type (we refer to [24, 19, 36, 37], for notation and further references pertaining to Chevalley groups, and to [3] for the number theory background).

Recall that a Dedekind ring of arithmetic type $R = \mathcal{O}_{K,S}$ is essentially a [principal] localisation $\mathcal{O}_K\left[\frac{1}{s}\right]$ of the ring of integers $\mathcal{O}_K$ in a global field K, which is a finite extension of the rational numbers $F = \mathbb{Q}$ in characteristic 0, or of the field of rational functions $F = \mathbb{F}_p(t)$ over the prime field $\mathbb{F}_p$ in characteristic $p > 0$. The degree $d = |K : F|$ is called the degree of K. In characteristic 0 we call such rings **number rings** and in positive characteristic – **function rings**.

One of the outstanding classical problems is to estimate the width $w_\Omega\big(E(\Phi, R)\big)$ of the [absolute] elementary subgroup

$$E(\Phi, R) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R\big\rangle$$

with respect to the **elementary generators**

$$\Omega = \big\{x_\alpha(\xi) \mid \alpha \in \Phi,\ \xi \in R\big\}.$$

Usually, we will refer to this width as the **elementary width** of the Chevalley group itself, and denote it by $w_E(G)$. [In the cases we consider here, for the simply connected groups over arithmetic rings one has $G_{\mathrm{sc}}(\Phi, R) = E_{\mathrm{sc}}(\Phi, R)$, see [3, 19], so that no confusion can possibly arise.]

Denote by $E^L(\Phi, R)$ the subset (not necessarily a subgroup!) of $E(\Phi, R)$ consisting of products of $\leqslant L$ elementary generators. Then the elementary width $w_E(G)$ is the *smallest* such L that $E(\Phi, R) = E^L(\Phi, R)$, any such L for which this equality holds, is an upper bound for $w_E(G)$.

That the following result *may* hold was first suggested by Cooke and Weinberger back in 1974, see [7]. As a result of about half a century effort by many authors, including ["but not limited to"[1]] D. Carter, G. Keller, E. Paige, O. Tavgen, D. Morris, A. Rapinchuk, B. Nica, A. Trost, B. Kunyavskii, E. Plotkin, A. Morgan, B. Sury, and the author, [5, 6, 28, 29, 21, 22, 20, 16, 32, 33, 17], we now have the following definitive answer, see [15, 18] and references there.

Theorem A. *Let Φ be a reduced irreducible root system of rank $l \geqslant 2$. Then there exists a constant $L = L(\Phi)$, depending on Φ alone, such that*

[1] "The President can recognise many countries on the map, including Canada and Mexico".

for any Dedekind ring of arithmetic type R, any element in $G_{\rm sc}(\Phi, R)$ is a product of at most L elementary root unipotents,

$$G_{\rm sc}(\Phi,\, R) = E^L(\Phi, R).$$

Let us make a few observations concerning this result. What is truly remarkable here, is that the upper bounds for the elementary width in this theorem are:

• **unconditional** – the first partial proofs 50 years ago [7], and many partial proofs thereafter depended on very strong arithmetic assumptions such as [strong forms of] **GRH** = Generalised Riemann Hypothesis.

• **uniform**, in the sense that they do not depend on a specific R. Again many partial proofs proposed over the last 40 years provided bounds that depended on some arithmetic invariants of K, such as its discriminant, or, at least, its degree.

• In the **function case**, these bounds are **explicit**. Like, for instance, one has $\mathrm{SL}(3, R) = E^{44}(n, R)$ or $\mathrm{Sp}(4, R) = \mathrm{Ep}^{90}(4, R)$, see [18], and one could produce similar bounds for all types.

• In the **number case**, there are *excellent* explicit bounds when the multiplicative group R^* is infinite, from [20, 14, 16] one can derive that $\mathrm{SL}(3, R) = E^{14}(n, R)$ or $\mathrm{Sp}(4, R) = \mathrm{Ep}^{17}(4, R)$, etc.

• However, for the remaining case of the rings of integers in the **imaginary quadratic** fields, the only proof known today is a pure model theoretic *existence* proof, which cannot afford any specific value of L.

§2. Relative bounded elementary generation

Now, once we have this result, it is natural to try to generalise it to the finite index subgroups of $G(\Phi, R)$. Let us recall the notation necessary to state it more precisely.

• Let $I \trianglelefteq R$ be a non-zero ideal of R. It determines the [ring] reduction homomorphism $\rho_I : R \longrightarrow R/I$. Since $G(\Phi, \_)$ is a functor from rings to groups, this homomorphism induces the [group] reduction homomorphism $\rho_I : G(\Phi, R) \longrightarrow G(\Phi, R/I)$. The kernel of the reduction homomorphism ρ_I modulo I is called the **principal congruence subgroup** of level I and is denoted by $G(\Phi, R, I)$. It is clear that R here is a pure decoration, $G(\Phi, R, I)$ does not depend on R in which I is an ideal, and could be denoted by $G(\Phi, I)$.

However, with the elementary subgroup, and its generators it is not that immediate. There are *several* natural candidates for the appellation of the elementary subgroup of level I, here are two most[2] obvious ones:

• Let, as above, $I \trianglelefteq R$ be a non-zero ideal of R. Then the **true** [= unrelative] **elementary subgroup** $E(\Phi, I)$ of level I is generated by all elementary root unipotents of level I:

$$E(\Phi, I) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in I \big\rangle$$

with the obvious **elementary generators**

$$\Omega(I) = \big\{ x_\alpha(\xi) \mid \alpha \in \Phi,\ \xi \in I \big\}.$$

This group does not depend on the choice of R, but it has very little chance to be normal in the absolute elementary group $E(\Phi, R)$.

• The **relative elementary subgroup** $E(\Phi, I)$ of level I is defined as the *normal closure* of $E(\Phi, I)$ in $E(\Phi, R)$:

$$E(\Phi, R, I) = E(\Phi, I)^{E(\Phi, R)}.$$

This group is usually strictly larger than $E(\Phi, I)$. In fact, it only coincides with $E(\Phi, I)$ for *idempotent* ideals $I = I^2$, and there are very few such ideals in integral domains.

• In other words, $E(\Phi, R, I)$ is generated by elementary generators $\Omega(I)$ as a *normal* subgroup of $E(\Phi, R)$. This means that as an *obvious* generating set for $E(\Phi, R, I)$ one can take

$$\Xi(R, I) = \big\{ x_\alpha(\xi)^h \mid \alpha \in \Phi,\ \xi \in I,\ h \in E(\Phi, R) \big\}$$

and many early authors do exactly this. However, from our prospective in general $\Xi(R, I)$ is too large, to be considered a genuine analogue of the elementary generators Ω of $E(n, R)$.

[2]The larger ones coincide with $E(\Phi, R, I)$ in our context.

• For $\alpha \in \Phi$, $\xi \in I$ and $\zeta \in R$ define the **elementary conjugate**[3] $z_\alpha(\xi,\zeta)$ as

$$z_\alpha(\xi,\zeta) = x_{-\alpha}(\zeta)x_\alpha(\xi)x_{-\alpha}(-\zeta).$$

Next, we set

$$\Theta(R,I) = \big\{z_\alpha(\xi,\zeta) \mid \alpha \in \Phi,\ \xi \in I,\ \zeta \in R\big\}.$$

Clearly, $\Omega(I) \leqslant \Theta(R,I) \leqslant \Xi(R,I)$. A classical result, first stated in this form by Leonid Vaserstein [35], but see also [24, 31], asserts that for rank $\geqslant 2$ the relative elementary group $E(\Phi,R,I)$ is generated by $\Theta(R,I)$ *as a group*.

Now it is natural to ask whether the width of $E(\Phi,I)$ with respect to $\Omega(I)$, and the width of $E(\Phi,R,I)$ with respect to $\Theta(R,I)$ or $\Xi(R,I)$ are bounded. As above, to state our results precisely, we introduce the following notation. For a natural number L we denote by $E^L(\Phi,I)$ the subset of $E(\Phi,I)$ consisting of all products of $\leqslant L$ elements of $\Omega(I)$. Similarly, $E^L(\Phi,R,I)$ denotes the subset of $E(\Phi,R,I)$ consisting of all products of $\leqslant L$ elements of $\Theta(R,I)$. Finally, $\widetilde{E}^L(\Phi,R,I)$ denotes the subset of $E(\Phi,R,I)$ consisting of all products of $\leqslant L$ elements of $\Xi(R,I)$.

In the *number* case some results in this direction were proven for $E(n,R,I)$ in terms of $\Xi(R,I)$ by Carter and Keller [6], and for Chevalley groups $E(\Phi,R,I)$ in terms of $\Xi(R,I)$ and $E(\Phi,I)$ in terms of $\Omega(I)$ by Tavgen [28, 30]. Recently, Sinchuk, Smolensky [23] and Gvozdevsky [8] made first steps towards replacing $\Xi(R,I)$ by $\Theta(R,I)$ and obtaining bounds uniform with respect to I.

Here we observe that the usual Schreier's rewriting procedure, first applied in this context by Tavgen, see [28], and especially [30], allows to derive from Theorem A the following result in this direction valid for all Chevalley groups, and all Dedekind rings of arithmetic type. The proof will be given in the last section.

[3]We use this name in our "yoga of conjugation" and "yoga of commutators" papers [10, 9, 11, 12, 13], by analogy with the **elementary commutators** $y_\alpha(\xi,\zeta) = [x_\alpha(\xi), x_{-\alpha}(\zeta)]$. Here it is in conflict with the colloquial name for the elements of $\Xi(R,I)$ which are also oftentimes called "elementary conjugates" by other authors. Otherwise, $z_\alpha(\xi,\zeta)$ are called "Tits–Vaserstein generators". "Stein–Tits–Vaserstein generators" would be even more appropriate historically, but far too long for such a basic notion. As a compromise we could propose the name **STV-generators**, by analogy with "ESD-transvections".

Theorem B. *Let Φ be a reduced irreducible root system of rank $l \geqslant 2$, R be a Dedekind ring of arithmetic type, and $I \trianglelefteq R$ be an ideal of R. Then there exists a constant $M = M(\Phi, R, I)$ such that any element in $E_{\mathrm{sc}}(\Phi, R, I)$ is a product of at most M elementary conjugates,*

$$E_{\mathrm{sc}}(\Phi, R, I) = E^M(\Phi, R, I).$$

This result seems to be conclusive, but it is not the ultimate ambition of the project we are currently working on. Namely:

• The bounds in this result are **not uniform**, and at that not just with respect to R, but even with respect to I. [The latter dependence can be easily lifted by model theoretic methods, the former one much less so.]

• The bounds in this result are **not explicit**. [We believe though the non-uniform bounds *can* be made explicit even with this naive approach by somewhat more work.]

§3. Width in elementary conjugates

What is pleasing, though, bounded relative generation with respect to a larger set of generators, $\Xi(R, I)$, is not *much*[4] different from the bounded generation with respect to the elementary conjugates $\Theta(R, I)$. In other words, up to a constant that *can* be explicitly calculated – which we do not attempt here! – bounded generation in arbitrary elementary conjugates $x_\alpha(\xi)^h$, $h \in E(\Phi, I)$, is tantamount to the bounded generation in the elementary conjugates proper.

Here, I evade any work by invoking an overriding result by Alexei Stepanov [25] on the commutator width for *arbitrary* commutative rings. This result is based on his powerful **universal localisation** method and is an absolute overkill. Actually, in our setting, for Dedekind rings, any of the previous generation results on the elementary width of commutators would do, valid for *finite-dimensional* rings [27, 9], that can be proven via more familiar methods such as **localisation-completion** [1, 10], or **relative localisation** [11, 12], see [9, 13] for the context and many further references.

The following result is a special case of [25, Theorem 9.1].

Lemma 1. *Let Φ be a reduced irreducible root system of rank $l \geqslant 2$. Then there exists a bound $P = P(\Phi)$ such that for an arbitrary commutative ring*

[4] I am even tempted to say not *any* different, only that there are scores of pesky technical details to take care of, before you can produce *sharp* bounds in all cases.

R and any ideal $I \trianglelefteq R$, any $g \in G(\Phi, R)$ and any $h \in E(\Phi, R, I)$ one has

$$[g, h] \in E^P(\Phi, R, I).$$

This is, of course, *terribly* much more, than we need here. In this result g is not a product of elementaries at all (in our context it is a *bounded* such product), while h is a product of elementary conjugates, but its length is not specified (in our context this length is 1).

In [25] the constant P is not specified, but if you backtrace the proof, it can be expressed in terms of partitions of 1 in the affine algebra $\mathbb{Z}[G]$ of the Demazure–Chevalley group scheme $G(\Phi, \_)$, etc. Eventually, P can be bounded by some power of the order of the Weyl group $W(\Phi)$, etc. (but, to the best of my knowledge, no such explicit estimates are published).

Since $h^g = g^{-1}hg = [g^{-1}, h]h^{-1}$, we get the following corollary, where the constant Q should be much smaller than the constant P described above.

Lemma 2. *Let Φ be a reduced irreducible root system of rank $l \geqslant 2$, R be an arbitrary commutative ring, and $I \trianglelefteq R$ be an ideal of R. Then there exists a bound $Q = Q(\Phi)$ such that*

$$x_\alpha(\xi)^g \in E^Q(\Phi, R, I), \qquad \text{for all} \quad \alpha \in \Phi,\ \xi \in I,\ g \in G(\Phi, R),$$

which implies, in particular, that for any M one has

$$\widetilde{E}^M(\Phi, R, I) \subseteq E^{MQ}(\Phi, R, I).$$

Remark 1. Actually, here we are only interested in the *elementary* conjugates $x_\alpha(\xi)^h$, where $h \in E(\Phi, R)$. Assuming $E(\Phi, R)$ is boundedly elementarily generated with bound L, we *could* in principle give an explicit bound $Q = Q(\Phi, L)$ by tracing the proof of the [Stein]–Tits–Vaserstein theorem [24, 31, 35]. Detailed proofs of this theorem in various situations were performed in many places, including [31, 38, 35, 2, 38, 13].

Since in our case the elementary width of the absolute elementary group $E(\Phi, R)$ is bounded by L, one could estimate Q by induction. Amazingly, none of the above displays *explicit* width bounds in the general case. The only exception known to me is the foundational paper[5] by Tony Bak [1], where he specifies the [worst case] bound $Q = 14^L$ for $\mathrm{SL}(n, R)$, $n \geqslant 3$. Later *some* estimates for other groups were indicated in a similar situation in [27].

[5]At that time, I wondered, why would someone be interested in providing such details: “The stone the builders rejected has become the corner stone”.

Remark 2. Alternatively, I could refer to our papers with Zuhong Zhang, see [39, 40] and references there, where we study the generators of the groups

$$[E(\Phi, I), E(\Phi, J)] = [E(\Phi, R, I), E(\Phi, R, J)],$$

for two ideals $I, J \trianglelefteq R$, with explicit bounds in terms of *elementary commutators*. Plugging inside $J = R$ we get bounds for Q in Lemma 2. However, we cannot directly quote these results here, since in those papers we imposed some (minor) additional restrictions on R, such as the absence of the factor field $\mathbb{F}_2$ of two elements for $\Phi = \mathsf{C}_2$ and G_2, etc., and it would be silly to tender any such proviso in Theorem B.

Remark 3. Finally, polynomial expressions of the conjugates $x_\alpha(\xi)^g$, $g \in G(\Phi, R)$, form the gist of the **decomposition of unipotents** developed by Alexei Stepanov and myself, see [26], etc. However, the details of proofs are published not in all cases, and in the relative case the bounds $M(\Phi)$ are *only* explicitly calculated for $\mathrm{SL}(n, R)$, see [4]. In our joint paper with Kaisar Tulenbaev [34] we follow this trail and get sensible explicit bounds in Theorem B in this case.

§4. Proof of Theorem B: rewriting argument

The following argument closely follows the proof of [28, Proposition 7].

Proof. If $I = 0$ there is nothing to prove. Thus, in the sequel we may assume that $I \neq 0$, so that I has finite index in R and we can choose a complete residue system $Y = Y(I)$ modulo I. By the very definition Y is finite, $|Y| = |R/I|$. [If we are interested in getting explicit bounds, somewhat more caution in the choice of Y is advised, but for a nonconstructive proof any Y will do.] Thus, every $\xi \in R$ can be uniquely written in the form $\xi = \eta + \zeta$, for some $\eta \in I$ and $\zeta \in Y$.

• Take an arbitrary element $g \in E(\Phi, R, I)$. By Theorem A it can be expressed as

$$g = x_{\beta_1}(\xi_1) \dots x_{\beta_L}(\xi_L),$$

for appropriate roots $\beta_1, \dots, \beta_L \in \Phi$, and appropriate parameters $\xi_1, \dots, \xi_L \in R$. We can now express each ξ_i in the above form as $\xi_i = \eta_i + \zeta_i$, for unique $\eta_i \in I$ and $\zeta_i \in Y$. This means that

$$g = x_{\beta_1}(\eta_1 + \zeta_1) \dots x_{\beta_L}(\eta_L + \zeta_L) = x_{\beta_1}(\eta_1) x_{\beta_1}(\zeta_1) \dots x_{\beta_L}(\eta_L) x_{\beta_L}(\zeta_L),$$

where $x_{\beta_i}(\eta_i) \in E(\Phi, R, I)$. For brevity, we denote $y_i = x_{\beta_i}(\eta_i)$ and $z_i = x_{\beta_i}(\zeta_i)$.

Now, we can apply the usual Abel trick and rewrite g in the form

$$g = y_1\big(z_1 y_2 z_1^{-1}\big)\big(z_1 z_2 y_3 z_2^{-1} z_1^{-1}\big)\big(z_1 z_2 z_3 y_4 z_3^{-1} z_2^{-1} z_1^{-1}\big) \cdot \ldots \\ \cdot \big(z_1 \ldots z_{L-1} y_L z_{L-1}^{-1} \ldots z_1^{-1}\big) \cdot z_1 \ldots z_L.$$

• Clearly, each of the first L factors belongs to $\Xi(R, I)$. By Lemma 2 the length of each such factor is $\leqslant Q$ so that the total length of their product is $\leqslant LQ$.

• Thus, we can now concentrate on the last factor $z = z_1 \ldots z_L$. By the very definition $z \in Z$, where

$$Z = \big\{x_{\beta_1}(\zeta_1) \ldots x_{\beta_L}(\zeta_L) \mid \zeta_1, \ldots, \zeta_L \in Y\big\} \subseteq E^L(\Phi, R).$$

Since there are exactly $|Y| = |R/I|$ options for ζ_i, $i = 1, \ldots, L$, it follows that Z is **finite**, viz. $|Z| = |R/I|^L$.

On the other hand, by assumption both g and the first L factors on the right hand side all belong to $E(\Phi, R, I)$, so that one has $z \in E(\Phi, R, I)$ as well.

Now setting

$$H = \max\big(l_{\Theta(R,I)}(z)\big), \qquad \text{where} \quad z \in Z \cap E(\Phi, R, I),$$

we see that the last factor in the above expression of g belongs to $E^H(\Phi, R, I)$.

• Combining this with the above, we can conclude that

$$g \in E^{LQ+H}(\Phi, R, I),$$

as claimed. This finishes the proof of Theorem B. □

Remark 4. Obviously, H in the above construction depends on the size of R/I and maybe on the specific choice of Y. Pavel Gvozdevsky gives essentially the same proof for $\mathrm{SL}(n, R, I)$ in [8, Proposition 5.1] and comments that “this proof does not allow to obtain any explicit estimate”. We believe this might be an exaggeration. First of all, the requirement that $z \in E(\Phi, R, I)$ imposes very very strong restrictions on the subfactors $z_1, \ldots, z_L$. Only a tiny fraction of the products $x_{\beta_1}(\zeta_1) \ldots x_{\beta_L}(\zeta_L)$ actually fall into $E(\Phi, R, I)$. We expect that for a smart choice of Y it might be possible to backtrace the exact structure of z and to obtain a bound on H that is *explicitly* expressed in terms of $|R/I|$ and L alone. Surely that would involve a lot of combinatorial fiddling with the Chevalley commutator formula and the like.

In [34] we plan to come up with explicit *realistic* bounds for the elementary width of $\mathrm{SL}(n, R, I)$ based on [26, 4], and also, using the ideas of [6, 21] lift the dependence on I in the number case (we could not get rid of the dependence on the degree of K, though). Eventually, with some diligence one should be able to produce such similar bounds also for other groups.

In July–September 2019 we started to specifically discuss bounded generation in the arithmetic case with Boris Kunyavskii and Eugene Plotkin. I thank them for our sparkling cooperation over the last four years, which eventually lead to [17, 18] and, with crucial contributions by Andrei Lavrenov, to [15], with a lot of further items currently underway. I am extremely grateful to Alexei Stepanov for his savvy advice on the proof of Theorem B which lead to a *much* shorter existence proof than the one I originally had in mind. Finally, I thank Kaisar Tulenbaev, conversations with whom during the "Algebraic Groups: White Nights Season III" lead to the present paper.

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