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POLYNOMIAL STRUCTURE IN DETERMINANTS FOR IZERGIN–KOREPIN PARTITION FUNCTION

Abstract. We discuss determinant formulas for the partition function of the six-vertex model with domain wall boundary conditions, which are parametrized by an arbitrary basis of polynomials. In this note we show that our recent result on this problem admits a one-parameter extension.

§1. Introduction

The six-vertex model with domain wall boundary conditions was introduced by Korepin in [1] where he established a list of conditions which uniquely fix the partition function as function of external spectral parameters. In [2] Izergin showed that these conditions can be solved in terms of a determinant. The method was exposed in detail in [3].

Subsequently, alternative determinant representations were found. In [4, 5], Kostov showed that in the rational case a determinant formula with a different structure is valid. A generalization of this formula to the trigonometric case was given by Foda and Wheeler in [6], who also showed that these representations are equivalent to the original result of Izergin–Korepin.

In a recent paper [7] Minin and the present authors showed that the original approach of Korepin and Izergin can be slightly modified by replacing the Korepin's recursion relation with a system of algebraic equations with respect to one (out of two) sets of spectral parameters. This makes possible to construct determinant representations for the model on an $N \times N$ lattice that depend on an arbitrary basis of polynomials of degree $N-1$. The basis can be defined in terms of the remaining set of spectral parameters. For example, choosing the basis of Lagrange interpolation polynomials one obtains the original Izergin's determinant. The monomial basis leads to the Kostov and Foda–Wheeler representations.

Key words and phrases: Bethe ansatz, vertex models, domain wall boundary conditions, determinant representations.

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In the present paper we show that the whole construction involves polynomials of degree N rather than $N-1$ that provides a one-parameter extension of our previous result.

We organize our paper as follows. In Sec. 2 we recall definition of the six-vertex model with domain wall boundary conditions. In Sec. 3 we list the properties of the partition function involving the set of algebraic equation which fix the partition function in unique way. In Sec. 4 we state our main result (see Thm. 1) which provides a determinant formula involving polynomials of degree N. In Sec. 5 we discuss particular cases of this formula.

§2. Izergin–Korepin partition function

We consider the six-vertex model on an $N \times N$ square lattice obtained by intersection of N vertical and N horizontal lines. Each vertical and horizontal line carries a spectral parameter: parameters λ_j, $j = 1, \dots, N$, are assigned to the vertical lines (enumerated from right to left) and parameters ν_k, $k = 1, \dots, N$, are assigned to the horizontal lines (enumerated from top to bottom). Furthermore, with each vertical and horizontal line a vector space $\mathbb{C}^2$ is associated, with basis vectors

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We denote by $\mathcal{H}$ the total vector space of the N horizontal lines ($\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$) and by $\mathcal{V}$ the same for the N vertical ones ($\mathcal{V} = (\mathbb{C}^2)^{\otimes N}$). The vectors $|\Uparrow_{\mathrm{H}}\rangle, |\Uparrow_{\mathrm{V}}\rangle = |\uparrow\rangle^{\otimes N}$ and $|\Downarrow_{\mathrm{H}}\rangle, |\Downarrow_{\mathrm{V}}\rangle = |\downarrow\rangle^{\otimes N}$ play an important role in the model; the subscripts indicate to which space, $\mathcal{H}$ or $\mathcal{V}$, these vectors correspond.

Next we introduce an L-operator – a matrix of the Boltzmann weights. The operator $L_{jk}(\lambda_j, \nu_k)$ correspond to the vertex obtained by intersection of the jth horizontal and kth vertical lines; it acts non-trivially only in the two vector spaces associated to these lines and identically in the remaining $\mathbb{C}^2$ vector spaces. It has the form

$$L_{jk}(\lambda_j, \nu_k) = a(\lambda_j, \nu_k) \frac{1 + \sigma_j^z \tau_k^z}{2} + b(\lambda_j, \nu_k) \frac{1 - \sigma_j^z \tau_k^z}{2} \\ + c(\lambda_j, \nu_k)(\sigma_j^+ \tau_k^- + \sigma_j^- \tau_k^+),$$

where

$$
\begin{aligned}
a(\lambda_j, \nu_k) &= \sin\gamma(\lambda_j - \nu_k + 1), \\
b(\lambda_j, \nu_k) &= \sin\gamma(\lambda_j - \nu_k), \\
c(\lambda_j, \nu_k) &= \sin\gamma,
\end{aligned}
\tag{2.1}
$$

and $\sigma_j^{\pm,z}$ (respectively, $\tau_k^{\pm,z}$) stand for operators acting as the Pauli spin matrices in the jth copy of $\mathbb{C}$ in $\mathcal{H}$ (kth copy of $\mathbb{C}$ in $\mathcal{V}$).

The partition function of the six-vertex model with domain wall boundary conditions can be defined as

$$
Z_N(\{\lambda\}, \{\nu\}) = \langle \Downarrow_{\mathrm{H}} \Uparrow_{\mathrm{V}} | \left(\prod_{j,k=1}^{N} L_{jk}(\lambda_j, \nu_k) \right) | \Uparrow_{\mathrm{H}} \Downarrow_{\mathrm{V}} \rangle. \tag{2.2}
$$

Here, $\{\lambda\}$ and $\{\nu\}$ stand for the ordered sets $\{\lambda\} \equiv \lambda_1, \ldots, \lambda_N$ and $\{\nu\} \equiv \nu_1, \ldots, \nu_N$. The vectors are defined as $|\Uparrow_{\mathrm{H}} \Downarrow_{\mathrm{V}}\rangle \equiv |\Uparrow_{\mathrm{H}}\rangle \otimes |\Downarrow_{\mathrm{V}}\rangle$ and $\langle \Downarrow_{\mathrm{H}} \Uparrow_{\mathrm{V}} | \equiv |\Downarrow_{\mathrm{H}} \Uparrow_{\mathrm{V}}\rangle^{\mathsf{T}}$ where T denotes the matrix transposition. The product in (2.2) is doubly ordered from right to left as each of the indices increases (recall that we label the lines from right to left and from top to bottom).

The celebrated determinant formula due to Izergin [2] expresses the partition function (2.2) via the determinant of an $N \times N$ matrix:

$$
\begin{aligned}
Z_N(\{\lambda\}, \{\nu\}) = (-1)^{\frac{N(N-1)}{2}} & \frac{\prod_{j,k=1}^{N} a(\lambda_j, \nu_k) b(\lambda_j, \nu_k)}{v_N(\{\lambda\})\, v_N(\{\nu\})} \\
& \times \det \left[\frac{c(\lambda_j, \nu_k)}{a(\lambda_j, \nu_k) b(\lambda_j, \nu_k)} \right]_{j,k=1,\ldots,N}.
\end{aligned}
\tag{2.3}
$$

Here, $v_N(\{\lambda\})$ and $v_N(\{\nu\})$ denote the Vandermonde factors,

$$
v_N(\{\lambda\}) = \prod_{1 \leqslant j < k \leqslant N} \sin\gamma(\lambda_k - \lambda_j).
$$

The rational case corresponds to $\gamma \to 0$ limit under an overall rescaling of the weights.

As we show below, representation (2.3) is a particular case of some determinant formula involving polynomials of degree N.

§3. Defining properties of the partition function

An important role in our construction is played by a polynomial dependence of the partition function on the inhomogeneity parameters. In the trigonometric case it can be introduced by the "change of variables"

$$x_j = q^{2\lambda_j}, \qquad y_k = q^{2\nu_k}, \qquad q = \mathrm{e}^{\mathrm{i}\gamma}, \tag{3.1}$$

so that the Boltzmann weights (2.1) become

$$\begin{aligned} a(\lambda_j,\nu_k) &= \frac{qx_j - y_k q^{-1}}{2\mathrm{i}\,(x_j y_k)^{1/2}},\\ b(\lambda_j,\nu_k) &= \frac{x_j - y_k}{2\mathrm{i}\,(x_j y_k)^{1/2}},\\ c(\lambda_j,\nu_k) &= \frac{q - q^{-1}}{2\mathrm{i}}. \end{aligned}$$

It can be shown (see discussion in [7, Sec. 5]) that the partition function of the six-vertex model with DWBC has the form

$$Z_N(\{\lambda\};\{\nu\}) = \frac{\widetilde{Z}_N(\{x\};\{y\})}{(2\mathrm{i})^{N^2}\prod\limits_{j=1}^{N}(x_j y_j)^{(N-1)/2}}, \tag{3.2}$$

where $\widetilde{Z}_N(\{x\};\{y\})$ is a polynomial in the variables $\{x\}\equiv x_1,\dots,x_N$ and $\{y\}\equiv y_1,\dots,y_N$.

The partition function $Z_N(\{\lambda\};\{\nu\})$ has the following properties, which can be established due to the Quantum Inverse Scatting method [8, 9]; for details and the proof, see [7, Secs. 2 and 5].

Proposition 1. *The partition function $\widetilde{Z}_N(\{x\};\{y\})$ as a function of the variables $x_1,\dots,x_N$ has the following properties:*

(1) *It is a symmetric polynomial in $x_1,\dots,x_N$;*
(2) *The degree in each of the variables $x_1,\dots,x_N$ equals $N-1$;*
(3) *For each pair of variables, say x_1, x_2, one has*

$$\widetilde{Z}_N(y_j, q^{-2}y_j, x_3,\dots,x_N;\{y\}) = 0, \qquad j=1,\dots,N; \tag{3.3}$$

(4) *As $\{x\}=\{y\}$, the following holds:*

$$\widetilde{Z}_N(\{y\};\{y\}) = (q-q^{-1})^N \prod_{\substack{j,k=1\\ j\neq k}}^{N} (qy_j - q^{-1}y_k). \tag{3.4}$$

The system of equations (3.3), (3.4) has a unique solution.

Proposition 2. *Let $P_N(\{x\})$ be a polynomial in the variables $x_1, \dots, x_N$, depending on the parameters $\{y\} = y_1, \dots, y_N$, that has the following properties:*

(1) *It is a symmetric polynomial in $x_1, \dots, x_N$;*
(2) *The degree in each of the variables $x_1, \dots, x_N$ equals $N-1$;*
(3) *For each pair of variables, say x_1, x_2, one has*

$$P_N(y_j, q^{-2}y_j, x_3, \dots, x_N) = 0, \qquad j = 1, \dots, N;$$

(4) *As $\{x\} = \{y\}$, the following holds:*

$$P_N(\{y\}) = (q - q^{-1})^N \prod_{\substack{j,k=1 \\ j \neq k}}^{N} (qy_j - q^{-1}y_k).$$

Then, $P_N(\{x\}) = \widetilde{Z}_N(\{x\}; \{y\})$.

The proof can be found in [7], see Proposition 5.

§4. Determinant representation

We begin with defining the polynomials that will enter the determinant representation for the partition function.

Consider linearly independent polynomials of degree N,

$$p_k(x) = \sum_{i=0}^{N} p_{k,i} x^i, \quad k = 1, \dots, N,$$

such that

$$p_{k,N} = \alpha p_{k,0}, \qquad k = 1, \dots, N, \tag{4.1}$$

where α is some parameter. To emphasize the dependence on α, we will write $p_k(x; \alpha)$ instead of $p_k(x)$.

If $\alpha = 0$, then the polynomials $p_k(x; 0)$ are of degree $N - 1$. We also consider the case of $\alpha = \infty$ assuming that $p_k(x; \infty) = x\, \tilde{p}_k(x)$, where the polynomials $\tilde{p}_k(x)$ are also of degree $N - 1$.

The following simple result is useful below.

Lemma 1. *If $\alpha \neq \infty$, then*

$$\det[p_k(x_j;\alpha)]_{j,k=1,\dots,N} = \det[p_{k,j-1}]_{j,k=1,\dots,N} \times \left(1+(-1)^{N-1}\alpha\prod_{j=1}^{N} x_j\right) \prod_{1\leqslant j<k\leqslant N}(x_k-x_j). \quad (4.2)$$

If $\alpha = \infty$, then

$$\det[p_k(x_j;\alpha)]_{j,k=1,\dots,N} = \det[p_{k,j}]_{j,k=1,\dots,N} \times \prod_{j=1}^{N} x_j \prod_{1\leqslant j<k\leqslant N}(x_k-x_j). \quad (4.3)$$

Proof. Using (4.1), we can write

$$p_k(x;\alpha) = \sum_{i=1}^{N} p_{k,i-1}\left(x^{i-1}+\alpha\delta_{i,1}x^N\right).$$

Therefore,

$$\det[p_k(x_j;\alpha)]_{j,k=1,\dots,N} = \det\left[p_{k,i-1}\right]_{k,i=1,\dots,N} \times \det\left[x_j^{i-1}+\alpha\delta_{i,1}x_j^N\right]_{i,j=1,\dots,N}.$$

For the second factor we get

$$\begin{aligned}\det\left[x_j^{i-1}+\alpha\delta_{i,1}x_j^N\right]_{i,j=1,\dots,N} &= \det\left[x_j^{i-1}\right]_{i,j=1,\dots,N} + \alpha(-1)^{N-1}\det\left[x_j^{i}\right]_{i,j=1,\dots,N} \\ &= \left(1+(-1)^{N-1}\alpha\prod_{j=1}^{N} x_j\right) \prod_{1\leqslant j<k\leqslant N}(x_k-x_j),\end{aligned}$$

and formula (4.2) follows.

Formula (4.3) is straightforward from $p_k(x;\infty) = x\tilde{p}_k(x)$. $\square$

Now we are ready to present our main formula.

Theorem 1. *If $\alpha \neq \infty$, then the partition function $\widetilde{Z}_N(\{x\};\{y\})$ can be written in the form*

$$\widetilde{Z}_N(\{x\};\{y\}) = \frac{1+(-1)^{N-1}\alpha \prod\limits_{j=1}^{N} x_j}{1+(-1)^{N-1}\alpha \prod\limits_{k=1}^{N} y_k} \times \frac{\det\big[p_k(x_j;\alpha)\Phi(x_j) - p_k(q^2x_j;\alpha)\Phi(q^{-2}x_j)\big]_{j,k=1,\dots,N}}{\left(\prod\limits_{j=1}^{N} x_j\right)\det[p_k(x_j;\alpha)]_{j,k=1,\dots,N}}, \tag{4.4}$$

where the function $\Phi(x)$ is

$$\Phi(x) = \prod_{k=1}^{N}\big(qx - q^{-1}y_k\big) \tag{4.5}$$

and $p_k(x;\alpha)$, $k=1,\dots,N$, are linearly independent polynomials in x of degree N such that their highest and lowest coefficients satisfy the condition (4.1)*. If $\alpha=\infty$, then*

$$\widetilde{Z}_N(\{x\};\{y\}) = \frac{\det\big[p_k(x_j;\infty)\Phi(x_j) - p_k(q^2x_j;\infty)\Phi(q^{-2}x_j)\big]_{j,k=1,\dots,N}}{\left(\prod\limits_{j=1}^{N} y_j\right)\det[p_k(x_j;\infty)]_{j,k=1,\dots,N}}.$$

Proof. The proof is given by verifying that the expression in the right-hand side of (4.4) obeys properties (1)–(4) stated in Proposition 1. □

§5. Particular cases

We first consider two special cases where the polynomials $p_k(x;\alpha)$ turn into polynomials of degree $N-1$. The first one is $\alpha=0$ and (4.4) readily gives

$$\widetilde{Z}_N(\{x\};\{y\}) = \frac{\det\big[p_k(x_j;0)\Phi(x_j) - p_k(q^2x_j;0)\Phi(q^{-2}x_j)\big]_{j,k=1,\dots,N}}{\left(\prod\limits_{j=1}^{N} x_j\right)\det[p_k(x_j;0)]_{j,k=1,\dots,N}}. \tag{5.1}$$

The second case is $\alpha = \infty$ in which $p_k(x;\infty) = x\,\tilde{p}_k(x)$ where $\tilde{p}_k(x)$ are polynomials of degree $N-1$. Then,

$$\widetilde{Z}_N(\{x\};\{y\}) = \frac{\det\left[\tilde{p}_k(x_j)\Phi(x_j) - q^2\tilde{p}_k(q^2x_j)\Phi(q^{-2}x_j)\right]_{j,k=1,\dots,N}}{\left(\prod_{k=1}^{N} y_k\right)\det[\tilde{p}_k(x_j)]_{j,k=1,\dots,N}}. \tag{5.2}$$

Representations (5.1) and (5.2) have been given in [7], see formulas (5.10) and (5.9) therein, respectively. Note the functions $\tilde{a}(x)$ and $\tilde{d}(x)$ of [7] are related to the function $\Phi(x)$ used here as $\tilde{a}(x) = \Phi(x)$ and $\tilde{d}(x) = q^N\Phi(q^{-2}x)$.

Next we discuss possible explicit expressions for the polynomials which lead to the original determinant formula (2.3). To this end, it is useful to recall that as it has been shown in [7], both (5.1) and (5.2) turn into the required form when the involved polynomials of degree $N-1$ are chosen as Lagrange interpolating polynomials,

$$p_k(x,0) = \tilde{p}_k(x) = \prod_{\substack{j=1\\ j\neq k}}^{N}(x-y_j), \qquad k=1,\dots,N. \tag{5.3}$$

For an arbitrary α the following polynomials play a similar role:

$$p_k(x;\alpha) = \frac{1}{C_k(\alpha)}\left\{1 + (-1)^{N-1}\alpha\left(\prod_{\substack{j=1\\ j\neq k}}^{N} y_j\right)x\right\}\prod_{\substack{j=1\\ j\neq k}}^{N}(x-y_j). \tag{5.4}$$

Here, $C_k(\alpha)$ is a normalization constant which can be chosen to have a suitable $\alpha\to\infty$ limit for the polynomials. For example, to reproduce (5.3),

$$C_k(\alpha) = 1 + (-1)^{N-1}\alpha\prod_{\substack{j=1\\ j\neq k}}^{N} y_j.$$

Below we also discuss two other choices of polynomials $p_k(x;\alpha)$ which are not Lagrange interpolation polynomials but which also lead to (2.3).

Let us now show that the determinant representation in (4.4) with the polynomials $p_k(x;\alpha)$ defined in (5.4) reproduces formula (2.3). We first consider the determinant in the denominator in (4.4). The result is given by (4.2) where we have to compute a value of the factor given by

$\det[p_{k,j-1}]_{j,k=1,\dots,N}$. To minimize calculations, we use fact that this factor is independent of the variables $x_1,\dots,x_N$ and hence it can found, e.g., by evaluating both sides of (4.2) at $\{x\}=\{y\}$. Setting $x_j=y_j$, for the polynomials (5.4) one gets

$$p_k(y_j;\alpha)=\delta_{jk}\frac{1+(-1)^{N-1}\alpha\prod\limits_{j=1}^{N}y_j}{C_k(\alpha)}\prod_{\substack{j=1\\ j\neq k}}^{N}(y_k-y_j).$$

Hence, the left-hand side of (4.2) at $\{x\}=\{y\}$ reads

$$\det[p_k(y_j;\alpha)]_{j,k=1,\dots,N}=\frac{\left(1+(-1)^{N-1}\alpha\prod\limits_{j=1}^{N}y_j\right)^N}{\prod\limits_{k=1}^{N}C_k(\alpha)}\prod_{\substack{j,k=1\\ j\neq k}}^{N}(y_k-y_j).$$

Evaluating the right-hand side of (4.2) at $\{x\}=\{y\}$, we find that relation (4.2) holds provided

$$\det[p_{k,j-1}]_{j,k=1,\dots,N}=\frac{\left(1+(-1)^{N-1}\alpha\prod\limits_{j=1}^{N}y_j\right)^{N-1}}{\prod\limits_{k=1}^{N}C_k(\alpha)}\prod_{1\leqslant j<k\leqslant N}(y_j-y_k).$$

Consider now the determinant in the numerator of (4.4). Using the function $\Phi(x)$, see (4.5), we can rewrite (5.4) as follows:

$$p_k(x;\alpha)=\frac{1}{C_k(\alpha)}\left(1-\alpha\frac{q^N\Phi(0)}{y_k}x\right)\frac{q^N\Phi(q^{-2}x)}{x-y_k}.$$

Using this formula, for the entries of the matrix standing in the determinant, we find

$$\begin{aligned}p_k(x_j;\alpha)\Phi(x_j)-p_k(q^2x_j;\alpha)\Phi(q^{-2}x_j)&=\frac{q^N\Phi(x_j)\Phi(q^{-2}x_j)}{C_k(\alpha)}\\ \times\left\{\frac{1}{x_j-y_k}-\frac{1}{q^2x_j-y_k}-\alpha\frac{q^N\Phi(0)}{y_k}\left(\frac{x_j}{x_j-y_k}-\frac{q^2x_j}{q^2x_j-y_k}\right)\right\}&\\ =\frac{q^N\Phi(x_j)\Phi(q^{-2}x_j)}{C_k(\alpha)}\left(1-\alpha q^N\Phi(0)\right)&\frac{(q^2-1)x_j}{(x_j-y_k)(q^2x_j-y_k)}.\end{aligned}$$

As a result, plugging all the ingredients into (4.4), we obtain

$$\widetilde{Z}_N(\{x\};\{y\}) = \frac{q^{N^2}\prod\limits_{j=1}^{N}\Phi(x_j)\Phi(q^{-2}x_j)}{\prod\limits_{1\leqslant j<k\leqslant N}(x_k-x_j)(y_j-y_k)} \times \det\left[\frac{(q^2-1)}{(x_j-y_k)(q^2x_j-y_k)}\right]_{j,k=1,\dots,N}. \tag{5.5}$$

Formula (2.3) now readily follows from (5.5), via change of variables (3.1) and relation (3.2).

Let us now discuss other cases of the polynomials $p_k(x;\alpha)$ which lead to (2.3). We find that for an arbitrary α there exist at least two more possible choices. The first choice is given by the polynomials

$$p_k(x;\alpha) = \prod_{\substack{j=1\\ j\neq k}}^{N}(x-y_j) + (-1)^{N-1}\alpha\left(\prod_{\substack{j=1\\ j\neq k}}^{N} y_j\right)x^N, \tag{5.6}$$

and the second choice is given by the polynomials

$$p_k(x;\alpha) = x\prod_{\substack{j=1\\ j\neq k}}^{N}(x-y_j) + \frac{(-1)^{N-1}}{\alpha}\prod_{\substack{j=1\\ j\neq k}}^{N} y_j. \tag{5.7}$$

Calculations that yield (2.3) are essentially the same for both cases. We will sketch the calculations for the polynomials (5.6).

We start with rewriting (5.6) in terms of the function $\Phi(x)$ defined in (4.5),

$$p_k(x;\alpha) = q^N\frac{\Phi(q^{-2}x)}{x-y_k} - \alpha q^N\frac{\Phi(0)}{y_k}x^N.$$

The entries of the matrix standing the determinant in the numerator in (4.4) then read

$$p_k(x_j;\alpha)\Phi(x_j) - p_k(q^2x_j;\alpha)\Phi(q^{-2}x_j) = q^N\Phi(x_j)\Phi(q^{-2}x_j) \times\left\{\frac{1}{x_j-y_k} - \frac{1}{q^2x_j-y_k} - \alpha\frac{\Phi(0)}{y_k}\left[\frac{x_j^N}{\Phi(q^{-2}x_j)} - \frac{q^{2N}x_j^N}{\Phi(x_j)}\right]\right\}. \tag{5.8}$$

Clearly, the factor $q^N\Phi(x_j)\Phi(q^{-2}x_j)$ can be moved out of the determinant, so we need to focus on the expression standing in the braces. For the first two terms we have

$$\frac{1}{x_j - y_k} - \frac{1}{q^2 x_j - y_k} = \frac{(q^2-1)x_j}{(x_j - y_k)(q^2 x_j - y_k)}, \tag{5.9}$$

which is fact what we need to reproduce (2.3), via (3.2) and under change of the variables (3.1). To treat the term proportional to α, consider the identity

$$\prod_{l=1}^{N} \frac{1}{x - y_l} = \sum_{l=1}^{N} \left(\prod_{\substack{i=1\\ i\neq l}}^{N} \frac{1}{y_l - y_i} \right) \frac{1}{x - y_l}.$$

Making here the change $x \mapsto x^{-1}$, $y_l \mapsto y_l^{-1}$, we get

$$\frac{\Phi(0) x^N}{\Phi(q^{-2}x)} = \sum_{l=1}^{N} \left(\prod_{\substack{i=1\\ i\neq l}}^{N} \frac{y_i y_l}{y_i - y_l} \right) \frac{x y_l}{y_l - x}. \tag{5.10}$$

We also have

$$\frac{\Phi(0) q^{2N} x^N}{\Phi(x)} = \frac{\Phi(0) x^N}{\Phi(q^{-2}x)}\bigg|_{x\mapsto q^2 x},$$

and hence

$$\frac{\Phi(0) x^N}{\Phi(q^{-2}x)} - \frac{\Phi(0) q^{2N} x^N}{\Phi(x)} = \sum_{l=1}^{N} \left(\prod_{\substack{i=1\\ i\neq l}}^{N} \frac{y_i y_l}{y_i - y_l} \right) \frac{(1-q^2) y_l^2 x}{(y_l - x)(y_l - q^2 x)}. \tag{5.11}$$

Comparing (5.11) with (5.9), and denoting the expression in (5.9) by φ_{jk}, we conclude that the determinant of the expression standing in the braces in (5.8) reads

$$\begin{aligned} \det \left[\varphi_{jk} + \alpha \sum_{l=1}^{N} \varphi_{jl} \left(\prod_{\substack{i=1\\ i\neq l}}^{N} \frac{y_i y_l}{y_i - y_l} \right) \frac{y_l^2}{y_k} \right]_{j,k=1,\dots,N} \\ = \left\{ 1 + \alpha \sum_{l=1}^{N} \left(\prod_{\substack{i=1\\ i\neq l}}^{N} \frac{y_i y_l}{y_i - y_l} \right) y_l \right\} \det \left[\varphi_{jk} \right]_{j,k=1,\dots,N}. \end{aligned} \tag{5.12}$$

Taking the limit $x \to \infty$ in (5.10) yields

$$\sum_{l=1}^{N} \left(\prod_{\substack{i=1 \\ i \neq l}}^{N} \frac{y_i y_l}{y_i - y_l} \right) y_l = -q^N \Phi(0) = (-1)^{N-1} \prod_{k=1}^{N} y_k.$$

Plugging this in (5.12), in view of (4.4) and (3.2), we thus arrive at (2.3) with our choice of the polynomials (5.6).

Note that the polynomials (5.6) and (5.7) admit the limits $\alpha \to 0$ and $\alpha \to \infty$, respectively. The limiting polynomials in both cases are the Lagrange interpolating polynomials (5.3).

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