Записки научных
семинаров ПОМИ
Том 520, 2023 г.

N. M. Bogoliubov, C. L. Malyshev

SCALAR PRODUCT OF THE FIVE-VERTEX MODEL AND COMPLETE SYMMETRIC POLYNOMIALS

Abstract. The scalar product of the state-vectors of the exactly solvable five-vertex model with the fixed boundary conditions is considered. Various relations including those in terms of complete symmetric polynomials are derived. The limiting forms of the obtained answers may be interpreted in terms of random walks on a square grid.

§1. Introduction

The vertex models with fixed boundary conditions of two-dimensional statistical mechanics play an important role in contemporary studies of integrable systems [1–8]. There are intriguing connections of these models with the problems of enumerative combinatorics [9–12], the symmetric functions [13], and the limit shapes phenomena [14–17].

The five-vertex model is a particular case of the six-vertex model [18], in which one of the vertices is frozen out. The study of the model has been relevant for many years [19, 20].

The Quantum Inverse Scattering Method (QISM) [21, 22] allows to express the scalar product of the state-vectors of the five-vertex model in the determinantal form [5] and reveals its close connection with Grothendieck polynomials [23, 24].

In the present paper we study the limiting forms of the scalar product of the state-vectors. The determinantal representation of the scalar product is obtained under off-shell parameterization and the special limits are considered.

§2. Spin description of five-vertex model

Consider the five-vertex model on a lattice consisting of $2N-1$ columns with M square cells in each. The five-vertex model describes the statistical

Key words and phrases: five-vertex model, integrable models, Quantum Inverse Scattering Method, symmetric polynomials, random walks.

The work was supported by the Russian Science Foundation (Grant 23-11-00311).

physics of configurations of arrows on edges connecting neighbouring sites of the lattice. There are two “incoming” and two “outgoing” arrows at each site. Every admissible arrow configuration at a site constitutes a vertex characterized by a statistical weight w_p, $p = 2, 3, \ldots, 6$ (Fig. 1). Provided that the edges with arrows directed “upwards” or “rightwards” are replaced by thick segments (and the other edges by thin segments), one gets an alternative description of arrow configurations in terms of *nests of lattice paths*. Since each edge of the lattice admits only two states (either a thick line or a thin line), there is a one-to-one correspondence between admissible configurations of arrows on the lattice and nests of lattice paths.

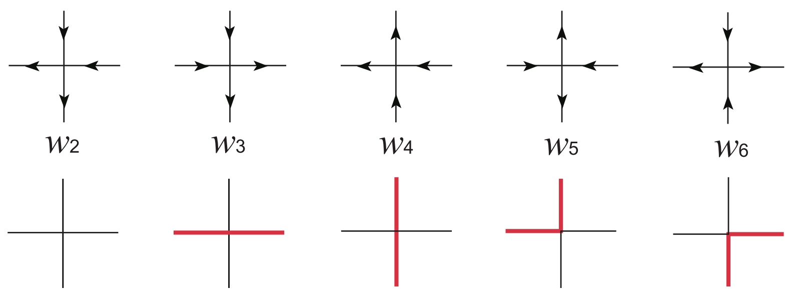

Figure 1. Five allowed vertices represented by arrows or lines.

Allowed arrow configurations depend on the imposed boundary conditions which specify the direction of the arrows on the lattice boundary. The *fixed boundary condition* implies that the boundary arrows on the top and bottom of N vertical lines (counting from the left) are pointing ‘inwards’, and the arrows on the top and bottom of N last ones are pointing ‘outwards’. All arrows on the left and right boundaries of the lattice are pointing to the left.

To enumerate admissible configurations of arrows it is more convenient to use description in terms of nests of lattice paths. Each path connects one of N bottom left sites with a top right one, and it always is directed towards the east or the north. The paths are *self-avoiding* since they cannot touch one another. The length of any path is $N + M$, and N paths constitute a nest of lattice paths (a typical nest is shown in Fig. 2).

The spin description implies that a local space isomorphic to $\mathbb{C}^2$ is associated with each vertical and horizontal edge of the lattice so that spin

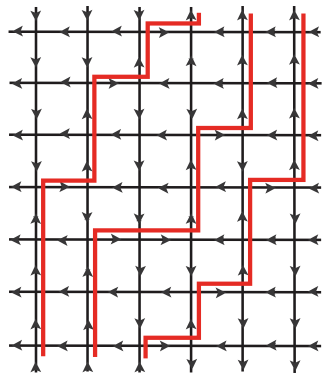

Figure 2. Admissible nest of lattice paths with the fixed boundary conditions.

"up" and "down" states constitute a basis in this space: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. The spin "up"/ "down" state on l^{th} vertical edge is denoted by arrows directed "upwards" or "downwards": $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_l \equiv |\uparrow\rangle_l$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}_l \equiv |\downarrow\rangle_l$. The spin "up"/ "down" state on i^{th} horizontal edge corresponds to the horizontal arrow pointing to the left or to the right: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \equiv |\leftarrow\rangle_i$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}_i \equiv |\rightarrow\rangle_i$. The *auxiliary* space $\mathbb{V}$ is the tensor product of all the local spaces associated with the vertical lines, $\mathbb{V} = (\mathbb{C}^2)^{\otimes 2N}$, and the *quantum* space $\mathbb{H}$ is the tensor product of all local spaces associated with the horizontal lines: $\mathbb{H} = (\mathbb{C}^2)^{\otimes (M+1)}$.

The Quantum Inverse Scattering Method [21,22] prescribes the so-called L-operator on $\mathbb{V} \otimes \mathbb{H}$, which acts non-trivially on a particular site of the lattice, while it acts as identity operator on remaining ones. The L-operator in question is of the form [5]:

$$\begin{aligned} L(n|u;\gamma) &= u\check{e}\check{e}_n + \hat{e}(\gamma u 1_n - u^{-1}\check{e}_n) + \sigma^+\sigma_n^- + \sigma^-\sigma_n^+ \\ &= \begin{pmatrix} u\check{e}_n & \sigma_n^- \\ \sigma_n^+ & \gamma u 1_n - u^{-1}\check{e}_n \end{pmatrix}, \end{aligned} \tag{1}$$

where $u, \gamma \in \mathbb{C}$ are the parameters, $\sigma^{z,\pm}$ are the Pauli matrices, and

$$\check{e} = \frac{1}{2}(1+\sigma^z)\,, \qquad \hat{e} = \frac{1}{2}(1-\sigma^z)$$

are projectors onto spin "up" and "down" states, respectively. The matrix with subindex n acts nontrivially only in n^{th} space:

$$\mathsf{s}_n = 1 \otimes \cdots \otimes 1 \otimes \underbrace{\mathsf{s}}_{n} \otimes 1 \otimes \cdots \otimes 1\,,$$

where $0 \leqslant n \leqslant M$. When $\gamma = 0$, L-operator (1) becomes L-operator of the four-vertex model [4, 7, 12].

The entries of L-operator (1) can be represented graphically as dots with adjacent arrows (Fig. 3). The entry $L_{11}(n|u)$ corresponds to the vertex (i)

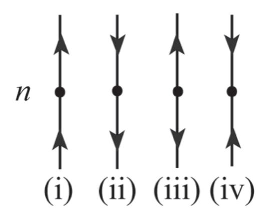

Figure 3. Vertex representation of the matrix elements of L-operator.

(Fig. 3), where the dot denotes the operator $u\check{e}_n$, which acts on the local spin state. The only non-zero expectation of this operator ${}_n\langle\leftarrow|u\check{e}_n|\leftarrow\rangle_n$ determines the vertex with weight $w_4 = u$ (Fig. 1). The entry $L_{22}(n|u)$ corresponds to the vertex (ii), where the dot means the operator $\gamma u 1_n - u^{-1}\check{e}_n$. Its expectation ${}_n\langle\leftarrow|\gamma u 1_n - u^{-1}\check{e}_n|\leftarrow\rangle_n$ is represented by the vertex with weight $w_2 = \gamma u - u^{-1}$ (Fig. 1), while the expectation ${}_n\langle\rightarrow|\gamma u 1_n - u^{-1}\check{e}_n|\rightarrow\rangle_n$ corresponds to the vertex with a weight $w_3 = \gamma u$. The entries $L_{12}(n|u) = \sigma_n^-$ and $L_{21}(n|u) = \sigma_n^+$ correspond to the vertices (iii) and (iv), and their respective expectations ${}_n\langle\rightarrow|\sigma_n^-|\leftarrow\rangle_n$ and ${}_n\langle\leftarrow|\sigma_n^+|\rightarrow\rangle_n$ determine vertices with weights $w_5 = w_6 = 1$ (Fig. 1).

The monodromy matrix $T(u;\gamma)$ is the product of L-operators (1):

$$T(u;\gamma) = L(M|u;\gamma)L(M-1|u;\gamma)\cdots L(0|u;\gamma) = \begin{pmatrix} A(u;\gamma) & B(u;\gamma) \\ C(u;\gamma) & D(u;\gamma) \end{pmatrix}. \tag{2}$$

The operator-valued matrices (1) and (2) are associated with the R-matrix [4]:

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{3}$$

where

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \qquad g(v,u) = \frac{uv}{u^2 - v^2}.$$

The commutation relations of the matrix elements of the monodromy matrix (2) are defined by R-matrix (3). The most important relations are:

$$\begin{aligned} C(u)B(v) &= g(u,v)\big(A(u)D(v) - A(v)D(u)\big), \\ A(u)B(v) &= f(u,v)\, B(v)A(u) + g(v,u)\, B(u)A(v), \\ D(u)B(v) &= f(v,u)\, B(v)D(u) + g(u,v)\, B(u)D(v), \\ \big[B(u), B(v)\big] &= \big[C(u), C(v)\big] = 0. \end{aligned} \tag{4}$$

The entries of the monodromy matrix (2) are expressed as sums over all admissible configurations of arrows with different boundary conditions on a chain of $M+1$ sites (Fig. 4). Namely, the operator $B(u)$ corresponds to the boundary conditions, when arrows on the top and bottom of the chain are pointing "outwards". Operator $C(u)$ corresponds to the boundary conditions, when arrows on the top and bottom of the chain are pointing "inwards". Operators $A(u)$ and $D(u)$ correspond to the boundary conditions, when arrows on the top and bottom of the chain are pointing up and down, respectively.

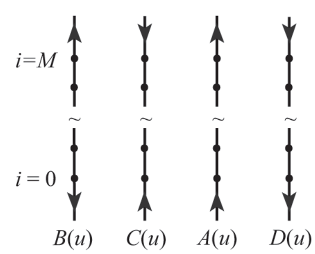

Figure 4. Graphical representation of the entries of the monodromy matrix

§3. The scalar product as determinant and its limiting form

3.1. Scalar product as determinant. Let us consider the scalar product of N-particle state-vectors,

$$W(\mathbf{v},\mathbf{u}) \equiv \langle\Leftarrow| \prod_{i=1}^{N} C(v_i) \prod_{j=1}^{N} B(u_j) |\Leftarrow\rangle\,, \tag{5}$$

where $\mathbf{v} \equiv (v_1, v_2, \ldots, v_N)$ and $\mathbf{u} \equiv (u_1, u_2, \ldots, u_N)$ are the sets of N independent *off-shell* parameters. The reference state is defined in (5),

$$|\Leftarrow\rangle = \bigotimes_{i=0}^{M} |\leftarrow\rangle_i = \bigotimes_{i=0}^{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i ,$$

so that the state-vector

$$|\Psi_N(\mathbf{u})\rangle = \prod_{j=1}^{N} B(u_j) |\Leftarrow\rangle \tag{6}$$

is symmetric in u_j, $1 \leqslant j \leqslant N$, since the operators $B(\cdot)$ are mutually commuting. The state-vector (6) is expressible as the sum over all admissible lattice paths associated with the product of $B(u)$ operators in, so-called, B-grid (Fig. 5). The state-vector

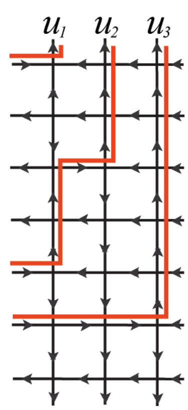

Figure 5. Typical nest of paths contributing into state-vector $B(u_1)B(u_2)B(u_3)|\Leftarrow\rangle$.

$$\langle \Psi_N(\mathbf{v})| = \langle \Leftarrow | \prod_{i=1}^{N} C(v_i) \tag{7}$$

is conjugated and may be expressed as the sum of all admissible lattice paths in C-grid. A typical nest contributing into the scalar product $\langle \Psi_N(\mathbf{v})|\Psi_N(\mathbf{u})\rangle$ (5) in the case of the fixed boundary condition is shown in (Fig. 2) at $N = 3$.

The scalar product (5) is evaluated for arbitrary N and M by means of the commutation relations (4). For the integrable models associated with the R-matrix (3), it acquires the determinantal form [5]:

$$\begin{aligned} W(\mathbf{v},\mathbf{u}) = \prod_{i=1}^{N}(v_i u_i)^{-M} \prod_{1\leqslant k<j\leqslant N} g(v_j, v_k) \prod_{1\leqslant m<l\leqslant N} g(u_m, u_l) \\ \times \det \widetilde{H}(\mathbf{v},\mathbf{u}) = \prod_{i=1}^{N}(v_i u_i)^{-M} \prod_{k=1}^{N} \left(\frac{v_k}{u_k}\right)^{N-1} \frac{\det \widetilde{H}(\mathbf{v},\mathbf{u})}{\Delta_N(\mathbf{v}^2)\Delta_N(\mathbf{u}^{-2})}\,, \end{aligned} \tag{8}$$

where $\Delta_N(\mathbf{x}^2)$ is the Vandermonde determinant,

$$\Delta_N(\mathbf{x}^2) = \prod_{1\leqslant m<k\leqslant N} (x_k^2 - x_m^2)\,, \tag{9}$$

and the matrix $\widetilde{H}(\mathbf{v},\mathbf{u}) \equiv \big(\widetilde{H}_{km}(\mathbf{v},\mathbf{u})\big)_{1\leqslant k,m\leqslant N}$ is given by the entries

$$\widetilde{H}_{km}(\mathbf{v},\mathbf{u}) = \frac{\mathcal{A}_{N,M}(v_m,u_k) - \mathcal{A}_{N,M}(u_k,v_m)}{\dfrac{u_k}{v_m} - \left(\dfrac{u_k}{v_m}\right)^{-1}}\,.$$

Here, the following notation is introduced:

$$\mathcal{A}_{N,M}(v_m,u_k) \equiv \alpha_{M+1}(v_m)\delta_{M+1}(u_k)\Big(\frac{u_k}{v_m}\Big)^{N-1}\,, \tag{10}$$

whereas $\alpha_{M+1}(u)$ and $\delta_{M+1}(u)$ are the eigenvalues of $A(u;\gamma)$ and $D(u;\gamma)$, Eq.(2):

$$\begin{aligned} A(u;\gamma)\,|\Leftarrow\rangle &= \alpha_{M+1}(u)\,|\Leftarrow\rangle \equiv u^{M+1}|\Leftarrow\rangle\,, \\ D(u;\gamma)\,|\Leftarrow\rangle &= \delta_{M+1}(u)\,|\Leftarrow\rangle \equiv \left(\gamma u - \tfrac{1}{u}\right)^{M+1}|\Leftarrow\rangle\,. \end{aligned}$$

For the four-vertex model $\gamma = 0$, the scalar product (8) is simplified [4]:

$$W(\mathbf{v},\mathbf{u}) = \frac{(-1)^{MN} \prod\limits_{k=1}^{N} \Bigl(\dfrac{u_k}{v_k}\Bigr)^{M-2N+2}}{\Delta_N(\mathbf{v}^2)\Delta_N(\mathbf{u}^{-2}) \prod\limits_{k=1}^{N} (v_k u_k)^M} \times \det \left[\frac{1-\Bigl(\dfrac{v_m}{u_k}\Bigr)^{2(M-N+2)}}{1-\Bigl(\dfrac{v_m}{u_k}\Bigr)^{2}} \right]_{1\leqslant k,m\leqslant N} .$$

We introduce rescaled notations

$$\mathbf{V} = (V_1, V_2, \ldots, V_N), \quad \mathbf{U} = (U_1, U_2, \ldots, U_N),$$

where

$$U_k \equiv \gamma u_k^2, \quad V_m \equiv \gamma v_m^2 \,, \tag{11}$$

and express (8) in the form symmetric in $\mathbf{V}$, $\mathbf{U}$:

$$W(\mathbf{V},\mathbf{U}) = \frac{(-1)^{\frac{N}{2}(N-1)}\gamma^{NM}}{\prod\limits_{k=1}^{N} (V_k U_k)^{M-N+1}} \times \frac{\det \widetilde{H}(\mathbf{V},\mathbf{U})}{\Delta_N(\mathbf{V})\Delta_N(\mathbf{U})} \,. \tag{12}$$

The matrix $\widetilde{H}(\mathbf{V},\mathbf{U})$ (12) is defined by the entries

$$\bigl(\widetilde{H}(V_m,U_k)\bigr)_{1\leqslant k,m\leqslant N} = \frac{(U_k-1)^A V_m^B - (V_k-1)^A U_m^B}{U_k - V_m} \,, \tag{13}$$

where

$$A \equiv M+1, \quad B \equiv M-N+2 \,,$$

and $A > B$ at $N > 1$, whereas $A = B$ at $N = 1$.

The polynomial entry $\widetilde{H}(V,U)$ (13) acquires the form suitable for studying the limiting behavior of $W(\mathbf{V},\mathbf{U})$ (12):

$$\begin{aligned} \widetilde{H}(V,U) = {} & \sum_{i=1}^{A-B} \Bigl(\sum_{p+q=A-B-i} (-1)^q C_A^q U^p \Bigr) U^B V^{B+i-1} \\ & - \sum_{i=0}^{B-1} \Bigl(\sum_{j=0}^{i} (-1)^{A+j-i} C_A^{A+j-i} U^{B-j-1} \Bigr) V^i \end{aligned} \tag{14}$$

Extracting $U^B V^{A-1}$ from (14), one re-expresses (12):

$$W(\mathbf{V},\mathbf{U}) = (-1)^{\frac{N}{2}(N-1)}\gamma^{NM}\left(\prod_{k=1}^{N} V_k^{A-B}U_k\right)\frac{\det\widetilde{\mathsf{H}}(\mathbf{V},\mathbf{U})}{\Delta_N(\mathbf{V})\Delta_N(\mathbf{U})}\,, \qquad (15)$$

where the matrix $\widetilde{\mathsf{H}}(\mathbf{V},\mathbf{U})$ is given by the entries

$$\begin{aligned}\big(\widetilde{\mathsf{H}}(V_m,U_k)\big)_{1\leqslant k,m\leqslant N} = {} & \sum_{i=1}^{A-B}\Big(\sum_{p+q=A-B-i}(-1)^q C_A^q U^p\Big)V^{B-A+i} \\ & - \sum_{i=0}^{B-1}\Big(\sum_{j=0}^{i}(-1)^{A+j-i}C_A^{A+j-i}U^{-j-1}\Big)V^{-A+i+1}\,,\end{aligned} \qquad (16)$$

and $A-B=N-1$.

3.2. The asymptotical behavior. Let us study the asymptotical behavior of $W(\mathbf{V},\mathbf{U})$ (5), (15) in the special limiting case, when the elements of $\mathbf{V}$ and $\mathbf{U}$, as off-shell parameters, go to infinity.

In the expression (13) we will go to an infinite limit of elements $\mathbf{V}$ subsequently as follows: first, $V_1\to\infty$, whereas V_2, $V_3\ldots$, V_N are fixed. As a next step, $V_2\to\infty$ at V_3, $V_4\ldots$, V_N fixed, and so on. Eventually, the limiting behavior of $\det\widetilde{\mathsf{H}}(\mathbf{V},\mathbf{U})$ (15) and thus of $\det\widetilde{H}(\mathbf{V},\mathbf{U})$ (12) is characterized by the relation:

$$\begin{aligned}&\lim_{V_N\to\infty} V_N^{N-1}\cdots\lim_{V_2\to\infty}V_2^1\lim_{V_1\to\infty}V_1^0 \\ &\qquad\times\det\widetilde{\mathsf{H}}(\mathbf{V},\mathbf{U})\equiv\det\widetilde{\mathsf{H}}(\mathbf{U})\,,\end{aligned} \qquad (17)$$

where $\widetilde{\mathsf{H}}(\mathbf{V},\mathbf{U})$ is given by (16). At each step, the multiplications of the determinant by V_1^0, $V_2^1,\ldots,$ V_N^{N-1} are with respect to the first, second, $\ldots$, N^{th} row, respectively. In turn, the resulting matrix $\widetilde{\mathsf{H}}(\mathbf{U})$ is expressed as

$$\begin{pmatrix} \sum\limits_{p+q=0}(-1)^q C_A^q U_1^p & \cdots & \sum\limits_{p+q=0}(-1)^q C_A^q U_N^p \\ \sum\limits_{p+q=1}(-1)^q C_A^q U_1^p & \cdots & \sum\limits_{p+q=1}(-1)^q C_A^q U_N^p \\ \cdots & \cdots & \cdots \\ \sum\limits_{p+q=N-2}(-1)^q C_A^q U_1^p & \cdots & \sum\limits_{p+q=N-2}(-1)^q C_A^q U_N^p \\ \sum\limits_{j=0}^{B-1}(-1)^{A+1-j}C_A^{A-j}U_1^{-B+j} & \cdots & \sum\limits_{j=0}^{B-1}(-1)^{A+1-j}C_A^{A-j}U_N^{-B+j} \end{pmatrix},$$

where the entries in the first row are unities, $\sum\limits_{p+q=0}(-1)^q C_A^q U_m^p = 1$, $1 \leqslant m \leqslant N$.

Using the limiting relation

$$\lim_{V_N\to\infty}\cdots\lim_{V_2\to\infty}\lim_{V_1\to\infty}\frac{\prod\limits_{k=1}^{N}V_k^{A-B+1-k}}{\Delta_N(\mathbf{V})} = (-1)^{\frac{N}{2}(N-1)}\,,$$

we obtain from (15) and (16):

$$W(\mathbf{V}_\infty,\mathbf{U}) = \frac{\gamma^{NM}\prod\limits_{k=1}^{N}U_k}{\Delta_N(\mathbf{U})}\det\widetilde{\mathsf{H}}(\mathbf{U})\,, \tag{18}$$

where $\mathbf{V}_\infty$ is a formal notation indicating a result of the limit prescribed. The properties of determinant enable to transform $\det\widetilde{\mathsf{H}}(\mathbf{U})$ into

$$\det H(\mathbf{U}) \equiv \det\begin{pmatrix} 1 & 1 & \dots & 1 \\ U_1 & U_2 & \dots & U_N \\ \dots & \dots & \dots & \dots \\ U_1^{N-2} & U_2^{N-2} & \dots & U_N^{N-2} \\ -g(U_1) & -g(U_2) & \dots & -g(U_N)\end{pmatrix}, \tag{19}$$

where

$$g(U_i) = \sum_{j=0}^{B-1}(-1)^{A-j}C_A^{A-j}U_i^{-B+j}\,. \tag{20}$$

The determinant (19) is transformed further:

$$\det H(\mathbf{U}) = (-1)^N\sum_{i=1}^{N}(-1)^{i-1}g(U_i)\Delta_{N-1}(\widehat{U}_i)\,, \tag{21}$$

where $\Delta_{N-1}(\widehat{U}_i) \equiv \Delta_{N-1}(U_1, U_2, \dots, \widehat{U}_i, \dots, U_N)$, while $\widehat{U}_i$ implies that U_i is omitted.

The following properties are valid:

$$\lim_{U_i\to\infty}\lim_{U_{i-1}\to\infty}\cdots\lim_{U_1\to\infty}\frac{\Delta_{N-1}(\widehat{U}_i)}{\Delta_N(\mathbf{U})}\prod_{k=1}^{N}U_k=0\,,\quad 1\leqslant i\leqslant N-2\,,\tag{22}$$

$$\lim_{U_{N-1}\to\infty}\lim_{U_{N-2}\to\infty}\cdots\lim_{U_1\to\infty}\frac{\Delta_{N-1}(\widehat{U}_{N-1})}{\Delta_N(\mathbf{U})}\prod_{k=1}^{N}U_k=(-1)^{N-1}U_N,\tag{23}$$

$$\lim_{U_{N-1}\to\infty}\lim_{U_{N-2}\to\infty}\cdots\lim_{U_1\to\infty}\frac{\Delta_{N-1}(\widehat{U}_{N})}{\Delta_N(\mathbf{U})}\prod_{k=1}^{N}U_k=(-1)^{N-1}U_N\,.\tag{24}$$

The order of calculation of limits in formulas (22), (23), and (24) is defined in (17).

The asymptotical relation at large U reads:

$$g(U)=C^N_{M+1}(-1)^NU^{-1}+o(U^{-1})\,,\tag{25}$$

where $C^N_{M+1}\equiv\binom{M+1}{N}$ is the binomial coefficient.

From (18) with respect to (25) one obtains :

$$W(\mathbf{V}_\infty,\mathbf{U}_\infty)=\gamma^{NM}C^N_{M+1}\,.\tag{26}$$

When the parameters of the scalar product (5), (15) tend to infinity, the obtained answer (26) enables interpretation as the sum over the sets of lattice paths of the special type. Really, from the definitions (1) and (2) one obtains the following representation [5]:

$$\begin{aligned}\lim_{u\to\infty}u^{-M}B(u)&=\sum_{k=0}^{M}\gamma^k\check{e}_M\ldots\check{e}_{k+1}\sigma^-_M,\\ \lim_{v\to\infty}v^{-M}C(v)&=\sum_{k=0}^{M}\gamma^{M-k}\sigma^+_k\check{e}_{k-1}\ldots\check{e}_0.\end{aligned}\tag{27}$$

It means that each lattice path of the nest has only one turn (w_5-vertex) in the B-grid and the only one turn (w_6-vertex) in the C-grid. The number of the allowed nests of lattice paths is equal to the number of possible ways to cross the border between the B and C grids (the dashed line (Fig. 6)) and is equal to C^N_{M+1}.

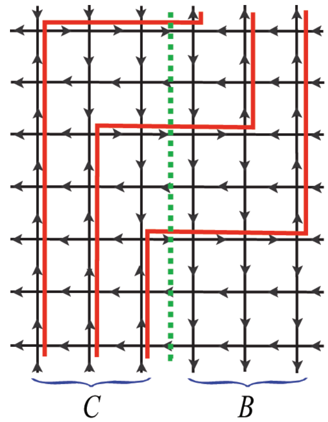

Figure 6. A nest of lattice paths in the limit $(\mathbf{V}_\infty, \mathbf{U}_\infty)$.

3.3. The scalar product and complete symmetric polynomials. Let us remind that complete symmetric polynomial of degree k in n variables $x_1, x_2, \dots, x_n$ is defined as [13]:

$$h_k(x_1, x_2 \dots, x_n) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_k \leqslant n} x_{i_1}, x_{i_2} \dots, x_{i_k} \,.$$

In this section we will present the following

Statement 1. *The limiting form of* $W(\mathbf{V}_\infty, \mathbf{U})$ (18) *is expressed in terms of complete symmetric polynomials* h_i*,* $1 \leqslant i \leqslant N$:

$$W(\mathbf{V}_\infty, \mathbf{U}) = -\gamma^{NM} \sum_{i=0}^{M-N+1} C_{M+1}^{N+i} (-1)^i \, h_i\Big(\frac{1}{\mathbf{U}}\Big) \,. \tag{28}$$

The representation (28) agrees with the asymptotics (26), since only $i = 0$ contributes.

The representation (28) results from (18) considered together with the following statement.

Statement 2. *The determinant* (19) *is expressed by means of complete symmetric plynomials*,

$$\det H(\mathbf{U}) = \frac{\Delta_N(\mathbf{U})}{\prod\limits_{k=1}^{N} U_k} \sum_{i=1}^{M-N+2} C_{M+1}^{N+i-1} (-1)^i \, h_{i-1}\Big(\frac{1}{\mathbf{U}}\Big) \,. \tag{29}$$

Indeed, we obtain from (19) and (20):

$$\det H(\mathbf{U}) = \sum_{i=0}^{M-N+1} C_{M+1}^{N+i} (-1)^{N+i} \times \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ U_1 & U_2 & \dots & U_N \\ \dots & \dots & \dots & \dots \\ U_1^{N-2} & U_2^{N-2} & \dots & U_N^{N-2} \\ U_1^{-(i+1)} & U_2^{-(i+1)} & \dots & U_N^{-(i+1)} \end{pmatrix} . \tag{30}$$

At the next step, we will re-express Eq. (30):

$$\det H(\mathbf{U}) = \sum_{i=1}^{M-N+2} \frac{(-1)^i C_{M+1}^{N+i-1}}{(\prod\limits_{k=1}^{N} U_k)^i} \, \Delta_N(i|\mathbf{U}) \,. \tag{31}$$

Here, $\Delta_N(i|\mathbf{U})$ is the determinant of the form:

$$\Delta_N(i|\mathbf{U}) \equiv \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ U_1^i & U_2^i & \dots & U_N^i \\ U_1^{i+1} & U_2^{i+1} & \dots & U_N^{i+1} \\ \dots & \dots & \dots & \dots \\ U_1^{i+N-2} & U_2^{i+N-2} & \dots & U_N^{i+N-2} \end{pmatrix} , \tag{32}$$

so that $\Delta_N(1|\mathbf{U})$ is the Vandermonde determinant $\Delta_N(\mathbf{U})$ (9). In its turn, the determinant (32) is expressed in terms of complete symmetric polynomials [13]:

$$\frac{\Delta_N(i|\mathbf{U})}{\Delta_N(\mathbf{U})} \equiv S_{\boldsymbol{\lambda}}(\mathbf{U}) = \Big(\prod_{k=1}^{N} U_k\Big)^{i-1} h_{i-1}\Big(\frac{1}{\mathbf{U}}\Big) , \tag{33}$$

where $S_{\boldsymbol{\lambda}}(\mathbf{U})$ is a Schur polynomial [13] indexed by partition

$$\boldsymbol{\lambda} = \Big((i-1)^{(N-1)}, 0\Big).$$

Using (33) in (31) one immediately arrives at (29).

§4. Conclusion

The non-Hermitian Hamiltonian of the totally asymmetric zero-range process (TAZRP) commutes with the transfer matrix of the five-vertex model [25, 26]. The approach developed makes it hopeful to proceed with similar representations in terms of complete symmetric polynomials and limiting relations for the correlation functions both of the five-vertex, and TAZRP models.

§5. Acknowledgments

The authors would like to thank A. G. Pronko for valuable discussions.

References

1. V. E. Korepin, *Calculation of norms of Bethe wave functions.* — Comm. Math. Phys. **86** (1982), 391–418.
2. A. G. Izergin, *Partition function of the six-vertex model in the finite volume.* — Sov. Phys. Dokl. **32** (1987), 878–879.
3. N. Bogoliubov, A. Pronko, M. Zvonarev, *Boundary correlation functions of the six-vertex model.* — J. Phys. A: Math. Gen. **35** (2002), 5525–5541.
4. N. M. Bogoliubov, *Four-vertex model and random tilings.* — Theor. Math. Phys. **155** (2008), 523–535.
5. N. M. Bogolyubov, *Five-vertex model with fixed boundary conditions.* — St. Petersburg Math. J. **21**(2010), 407–421.
6. A. V. Kitaev, A. G. Pronko, *Emptiness formation probability of the six-vertex model and the sixth Painlevé equation.* — Comm. Math. Phys. **345** (2016), 305–354.
7. N. M. Bogoliubov, *Four-Vertex Model in the Linearly Growing External Field under the Fixed and Periodic Boundary Conditions.* — Physics of Particles and Nuclei, **51** (2020), 429–433.
8. N. M. Bogolyubov, A. G. Pronko, *One-point function of the four-vertex model.* — Zap. Nauchn. Semin. POMI, **509** (2021), 39–53.
9. R. Stanley, *Enumerative combinatorics*, Vols. 1, 2, (Cambridge University Press, Cambridge, 1996, 1999).
10. G. Kuperberg, *Another proof of the alternating-sign matrix conjecture.* — Int. Math. Res. Not. **1996** (1996), 139–150.
11. D. M. Bressoud, *Proofs and Confirmations. The Story of the Alternating Sign Matrix Conjecture.* Cambridge University Press, Cambridge, 1999.
12. N. M. Bogoliubov, C. Malyshev, *Integrable models and combinatorics.* — Russian Math. Surveys **70** (2015), 789–856.
13. I. G. Macdonald, *Symmetric functions and Hall polynomials.* Clarendon Press, 1995.
14. F. Colomo, A. G. Pronko, *The arctic curve of the domain-wall six-vertex model.* — J. Stat. Phys. **138** (2010), 662–700.

15. N. Reshetikhin, A. Sridhar, *Integrability of limit shapes of the six vertex model.* — Comm. Math. Phys. **56** (2017), 535–565.
16. I. Lyberg, V. Korepin, G. Ribeiro, J. Viti, *Phase separation in the six-vertex model with a variety of boundary conditions.* — J. Math. Phys. **59** (2018), 053301.
17. F. Colomo, A. G. Pronko, A. Sportiello, *Arctic curve of the free-fermion six-vertex model in an L-shaped domain.* — J. Stat. Phys. **174** (2019), 1–27.
18. R. G. Baxter, *Exactly Solved Models in Statistical Mechanics.* San Diego, Academic press, 1982.
19. J. Noh, D. Kim, *Interacting domain walls and the five-vertex model.* — Phys. Rev. E **49** (1994), 1943–1961.
20. J. de Gier, R. Kenyon, S. S. Watson, *Limit shapes for the asymmetric five vertex model.* — Commun. Math. Phys. **385** (2021), 793–836.
21. L. D. Faddeev, *Quantum Inverse Scattering Method.* — Sov. Sci. Rev. Math. **C1** (1980), 107–160.
22. V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions.* Cambridge University Press, Cambridge, 1993.
23. K. Motegi, K. Sakai, *Vertex models, TASEP and Grothendieck polynomials.* — J. Phys. A: Math. Theor. **46** (2013), 355201.
24. I. N. Burenev, A. G. Pronko *Determinant formulas for the five-vertex model.* — J. Phys. A: Math. Theor. **54** (2021), 055008.
25. N. M. Bogoliubov, *Determinantal Representation of the Time-Dependent Stationary Correlation Function for the Totally Asymmetric Simple Exclusion Model.* — SIGMA **5** (2009), 05200.
26. N. M. Bogoliubov, C. Malyshev, *Zero range process and multi-dimensional random walks.* — SIGMA **13** (2017), 056.

St.Petersburg Department
of Steklov Institute of Mathematics, RAS
Fontanka 27, St.-Petersburg, Russia
E-mail: `bogoliub@pdmi.ras.ru`
E-mail: `malyshev@pdmi.ras.ru`

Поступило 3 июля, 2023 г.