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RESTRICTION ON MINIMUM DEGREE IN THE CONTRACTIBLE SETS PROBLEM

Abstract. Let G be a 3-connected graph. A set $W \subset V(G)$ is *contractible* if $G(W)$ is connected and $G-W$ is a 2-connected graph. In 1994, McCuaig and Ota formulated the conjecture that, for any $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that any 3-connected graph G with $v(G) \geqslant m$ has a k-vertex contractible set. In this paper we prove that, for any $k \geqslant 5$, the assertion of the conjecture holds if $\delta(G) \geqslant \left[\frac{2k+1}{3}\right] + 2$.

§1. Basic definitions

We consider undirected graphs without loops and multiple edges and use standard notation. We use notation $v(G)$ for the number of vertices of G and $\delta(G)$ for the minimum degree of G.

Definition 1. Let $R \subset V(G)$.

1) We denote by $G - R$ the graph obtained from G upon deleting all vertices of the set R and all edges incident to vertices of R.

2) We denote by $G(R)$ the induced subgraph of the graph G on the set R.

3) We say that R is *connected* if $G(R)$ is connected.

4) We say that R is a *k-vertex set* if $|R| = k$.

5) We say that R is *contractible* if $G(R)$ is connected and $G - R$ is 2-connected.

6) We say that R is *k-contractible* if R is a k-vertex contractible set.

7) Let $R_1 \subset V(G)$, $R \cap R_1 = \varnothing$. We denote by $E_G(R, R_1)$ the set of such edges $e \in E(G)$ that $e = xy$, $x \in R$, $y \in R_1$. Let $e_G(R, R_1) = |E_G(R, R_1)|$. We say that R_1 is *adjacent* to R if $e_G(R, R_1) \geqslant 1$.

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§2. Introduction

Consider a 2-connected graph G on n vertices and let n_1 and n_2 be positive integers with $n_1 + n_2 = n$. It is a well-known fact that $V(G)$ can be partitioned into 2 connected sets V_1 and V_2 such that $|V_1| = n_1$ and $|V_2| = n_2$.

In 1994, McCuaig and Ota [7] formulated the following conjecture for 3-connected graphs. This conjecture was mentioned in Mader's survey on connectivity [6].

Conjecture. *Let $k \in \mathbb{N}$. Then there exists an integer n such that every 3-connected graph G on at least n vertices has a k-contractible set.*

It follows from the Mader's paper [5] that the answer to the analogous problem is negative for n-connected graphs with $n \geqslant 4$. More precisely, for any $k \geqslant 2$ there exists an arbitrarily large n-connected graph G such that G does not contain a connected set W such that $|W| = k$ and $G - W$ is $n-1$-connected. So, the question remains open only for 3-connected graphs.

Statement of the conjecture is clear for $k = 1$. The conjecture is proved for $k = 2$ in [8], for $k = 3$ in [7], for $k = 4$ in [4] and for $k = 5$ in [9]. The author claims that the conjecture is proved for $k = 6$ in [1].

The following theorem of [2] establishes existence of large contractible sets in 3-connected graphs.

Theorem 1. *Let $m \geqslant 5$ be an integer and let G be a 3-connected graph with $v(G) \geqslant 2m + 1$. Then G has a contractible set W such that $m \leqslant |W| \leqslant 2m - 4$.*

We need to formulate clearly the main result of [4].

Theorem 2. *Let G be a 3-connected graph such that $v(G) \geqslant 7$, let G be not isomorphic to $K_{3,4}$. Then G has a 4-contractible set.*

Our main result is the following.

Theorem 3. *Let G be a 3-connected graph, let $k \geqslant 5$ be a natural number and let $v(G) \geqslant k+3$, $\delta(G) \geqslant \left[\frac{2k+1}{3}\right] + 2$. Then there exists a k-contractible set in the graph G.*

§3. Necessary tools

We formulate several definitions and facts on the structure of n-connected graphs and after that, with the help of them, we prove Theorem 3.

Definition 2. A contractible set $W \subset V(G)$ of a 3-connected graph G is *maximal* if there exists no vertex $x \in V(G) \setminus W$ such that the set $W \cup \{x\}$ is contractible.

Definition 3. Let G be a n-connected graph.

1) Let $R \subset V(G)$. R is a *cutset* if $G - R$ is disconnected.

2) We denote by $\mathfrak{R}_n(G)$ the set of all n-vertex cutsets of G.

3) Let $R \subset V(G)$ be a cutset. We say that R *splits* a set $X \subset V(G)$ if a set $X \setminus R$ is not contained in one connected component of the graph $G - R$.

4) Two cutsets $S, T \in \mathfrak{R}_n(G)$ are *independent* if S does not split T and T does not split S. Otherwise, these cutsets are *dependent*.

Definition 4. Let $\mathfrak{S} \subset \mathfrak{R}_n(G)$.

1) A set $A \subset V(G)$ is a *part of decomposition* of G by $\mathfrak{S}$ if no cutset of $\mathfrak{S}$ splits A and A is a maximal up to inclusion set with this property. By $\mathrm{Part}(G; \mathfrak{S})$, we denote the set of all parts of decomposition of G by $\mathfrak{S}$.

2) Let $A \in \mathrm{Part}(G; \mathfrak{S})$. A vertex of A is *inner* if it does not belong to any cutset of $\mathfrak{S}$. The set of all inner vertices of the part A is called the *interior* of A, which is denoted by $\mathrm{Int}(A)$.

The *boundary* of A is the set $\mathrm{Bound}(A) = A \setminus \mathrm{Int}(A)$.

Definition 5. Let G be a 2-connected graph.

1) A cutset $S \in \mathfrak{R}_2(G)$ is *single* if S is is independent with all other cutsets of $\mathfrak{R}_2(G)$. We denote by $\mathfrak{O}(G)$ the set of all single cutsets of the graph G.

2) We will write $\mathrm{Part}(G)$ instead of $\mathrm{Part}(G; \mathfrak{O}(G))$. Parts of this decomposition will be called simply *parts* of G.

Definition 6. The *block tree* $\mathrm{BT}(G)$ of a 2-connected graph G is a bipartite graph with bipartition $(\mathfrak{O}(G), \mathrm{Part}(G))$, where a single cutset S and a part A are adjacent if and only if $S \subset A$.

We need the following property of $\mathrm{BT}(G)$.

Lemma 1. [3, Lemma 1] *Let G be a 2-connected graph. Then BT(G) is a tree. Every leaf of BT(G) corresponds to a part of Part(G).*

Definition 7. Let G be a 2-connected graph and let $A \in \mathrm{Part}(G)$. A part A is *pendant* if it corresponds to a leaf of $\mathrm{BT}(G)$.

Definition 8. Let G be a 2-connected graph.

1) We denote by G' the graph obtained from G upon adding all edges of type ab where $\{a, b\} \in \mathfrak{O}(G)$.

2) A part $A \in \mathrm{Part}(G)$ is called a *cycle* if the graph $G'(A)$ is a cycle. If A is a cycle then $|A|$ is the *length* of A.

Lemma 2. [2, Lemma 13] *Let G be a 3-connected graph. Let $W \subset V(G)$ be a maximal contractible set such that the graph $H = G - W$ is not a simple cycle. Then the following statements hold.*

1) *Let $A \in Part(H)$ be a cycle. Then each inner vertex of A is adjacent to W.*

2) *There are at least two pendant parts in $Part(H)$ and all these parts are cycles of length at least 4.*

3) *Let $A \in Part(H)$ be a pendant part. Then $H - Int(A)$ is 2-connected.*

The following lemma is an obvious corollary of Lemma 2. The original version of this Lemma was proved in [4], Lemma 3.

Lemma 3. *Let G be a 3-connected graph. Let $W \subset V(G)$ be a maximal contractible set such that the graph $H = G - W$ is not a simple cycle. Let A_1, A_2 be two pendant parts of $G - W$, $W_1 = Int(A_1), W_2 = Int(A_2)$. Then the following statements hold.*

1) *$G(W_1)$ and $G(W_2)$ are simple paths.*

2) $|W_1| \geqslant 2, |W_2| \geqslant 2$.

3) $W_1 \cap W_2 = \varnothing$.

4) *All vertices in $W_1 \cup W_2$ have degree 2 in $G - W$.*

5) *Both $G - W - W_1$ and $G - W - W_2$ are 2-connected.*

6) $N_G(W_1) \cap W_2 = \varnothing, N_G(W_2) \cap W_1 = \varnothing$.

§4. Proof of Theorem 3

Lemma 4. *Let G be a 3-connected graph such that $v(G) \geqslant k + 3$ and G has a $(k-1)$-contractible set W for some integer $k \geqslant 2$. Suppose that there are 4 distinct vertices $v_1, v_2, v_3, v_4 \in V(G - W)$ such that the following statements hold.*

1) *$v_1v_2 \in E(G)$ and $d_{G-W}(v_i) = 2$ for any $i \in \{1, 2, 3, 4\}$.*

2) *For any vertex $x \in W$ such that $xv_3, xv_4 \in E(G)$, the graph*

$$G - (W \setminus \{x\}) - \{v_1, v_2\}$$

is 2-connected.

3) $|N_G(v_3) \cap N_G(v_4) \cap W| > |W \setminus N_G(\{v_1, v_2\})|$.

Then G has a k-contractible set.

Proof. It follows from $d_{G-W}(v_i) = 2$ for any $i \in \{1,2,3,4\}$ and $\delta(G) \geqslant 3$ that $e_G(v_i, W) \geqslant 1$ for any $i \in \{1,2,3,4\}$. By the condition 3 of Lemma 4, $|N_G(v_3) \cap N_G(v_4) \cap W| > 0$. Let $x \in W$ be a common neighbour of v_3 and v_4. Clearly, there are $|N_G(v_3) \cap N_G(v_4) \cap W|$ candidates on x.

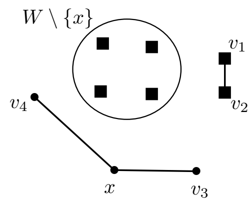

Figure 1. Proof of Lemma 4.

Consider $\{v_1, v_2\} \cup (W \setminus \{x\})$ (see figure 1). Clearly,

$$|\{v_1, v_2\} \cup (W \setminus \{x\})| = k.$$

By the condition of Lemma 4, $G - (\{v_1, v_2\} \cup (W \setminus \{x\}))$ is 2-connected. Therefore, the only chance of the set $\{v_1, v_2\} \cup (W \setminus \{x\})$ to be non-contractible is when it is not connected. Then, since $v_1v_2 \in E(G)$ (the condition of Lemma 4), there is a component in the graph $G(W \setminus \{x\})$ such that all the vertices of this component are not adjacent to $\{v_1, v_2\}$.

Consequently, we need to take x such that there does not exist a connected component of non-adjacent to $\{v_1, v_2\}$ vertices in $G(W \setminus \{x\})$ (let us call such vertices in $G(W)$ *forbidden*). Clearly, there are $|W \setminus N_G(\{v_1, v_2\})|$ forbidden vertices. Recall that there are $|N_G(v_3) \cap N_G(v_4) \cap W|$ candidates on x. Then, by the condition 3 of Lemma 4, there are more candidates on x than forbidden vertices.

For every forbidden vertex we take the shortest path from this forbidden vertex to the set of non-forbidden vertices in W (if it is not the only path, we take one of them). For every such path we take a neighbour of the forbidden endpoint (the second vertex of the path), let P be the union of these vertices. Clearly, $|P|$ is at most the number of forbidden vertices. Recall that there are more candidates on x than forbidden vertices. Consequently, there exists $x \notin P$ such that $xv_3 \in E(G)$, $xv_4 \in E(G)$, we fix this x. Note that this x is suitable. Indeed, assume the converse. Then there is a connected component in $G(W \setminus \{x\})$ consisting only of forbidden vertices. Since $G(W)$ is connected, x has at least one neighbour in this component.

For this neighbour x should be the second vertex of the aforementioned path, and hence $x \in P$, a contradiction. □

We prove this Theorem by induction on k. For the sake of convenience, we take the induction step as a separate statement.

Lemma 5. *Let k be an integer and let c be a non-negative integer such that $k \geqslant 3c + 5$. Let G be a 3-connected graph such that G has a $(k-1)$-contractible set and $v(G) \geqslant k + 3$, $\delta(G) \geqslant k - c$. Then there is a k-contractible set in the graph G.*

Proof. Let W be a $(k-1)$-contractible set in G. Assuming the converse, W is maximal.

Case 1. $G - W$ is a simple cycle.

We enumerate vertices of the cycle $G - W$ in the order of passing the cycle: $r_1, r_2, ..., r_m$, where $m \geqslant 4$. Our purpose is applying Lemma 4, where $v_1 = r_{i+1}$, $v_2 = r_{i+2}$, $v_3 = r_i$, $v_4 = r_{i+3}$ for any i.

Clearly, the condition 1 of Lemma 4 holds.

Let us verify the condition 2 of Lemma 4. Assume that r_i and r_{i+3} have a common neighbour x in W (see figure 2). Then $G - \{r_{i+1}, r_{i+2}\} - (W \setminus \{x\})$ is 2-connected because this graph has a Hamiltonian cycle $r_i x r_{i+3} r_{i+4} .... r_{i-1}$.

Hence, it remains to verify the condition 3 of Lemma 4. It follows from the fact that $G - W$ is a simple cycle and $\delta(G) \geqslant k - c$ that $e_G(r_i, W) \geqslant k - c - 2$, $e_G(r_{i+1}, W) \geqslant k - c - 2$, $e_G(r_{i+3}, W) \geqslant k - c - 2$. Then it follows from $|W| = k - 1$ and $2(k - c - 2) - (k - 1) > c + 1$ that $|N_G(r_i) \cap N_G(r_{i+3}) \cap W| \geqslant c + 2$. It follows from $e_G(r_{i+1}, W) \geqslant k - c - 2$ that $|W \setminus N_G(\{r_{i+1}, r_{i+2}\})| \leqslant c + 1$. Consequently, $|N_G(r_i) \cap N_G(r_{i+3}) \cap W| > |W \setminus N_G(\{r_{i+1}, r_{i+2}\})|$. Hence, the condition 3 of Lemma 4 holds.

Thus, Lemma 4 can be applied. By Lemma 4, G has a k-contractible set.

Case 2. $G - W$ is not a simple cycle.

By Lemma 2, $G - W$ has at least 2 pendant parts and each of them is a cycle of length at least 4. Therefore, they both have at least 2 internal vertices.

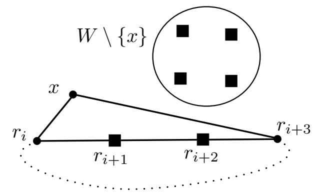

Figure 2. $G - W$ is a simple cycle.

Case 2.1. There exists a pendant part A of $G - W$ such that $|A| \geqslant 5$ (in particular, $|Int(A)| \geqslant 3$).

Let $Bound(A) = \{r, s\}$ and $Int(A) = \{w_1, ..., w_l\}$, these vertices are enumerated in the order of passing the path such that the path is $rw_1...w_ls$. Let u be an internal vertex of the pendant part different from A. By Lemma 3.4, $d_{G-W}(w_i) = 2$ for any $i \in \{1, 2, 3\}$ and $d_{G-W}(u) = 2$.

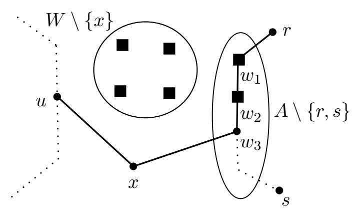

Figure 3. $G - W$ is not a simple cycle, there exists a pendant part A of $G - W$ such that $|A| \geqslant 5$.

Our goal is applying Lemma 4, where $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$, $v_4 = u$. Clearly, the condition 1 of Lemma 4 holds.

Let us verify the condition 2 of Lemma 4. Assume that w_3 and u have a common neighbour $x \in W$ (see figure 3). Then $G - (\{w_1, w_2\} \cup (W \setminus \{x\}))$ is 2-connected. Indeed, this is true because $G - W - Int(A)$ is 2-connected (Lemma 2.3) and there is a path $sw_l....w_3xu$.

Hence, it remains to verify the condition 3 of Lemma 4. Recall that $\delta(G) \geqslant k - c$, $d_{G-W}(w_i) = 2$ for any $i \in \{1, 2, 3\}$ and $d_{G-W}(u) = 2$. Therefore, $e_G(w_i, W) \geqslant k - c - 2$ for any $i \in \{1, 2, 3\}$ and $e_G(u, W) \geqslant k - c - 2$. Then it follows from $|W| = k - 1$ and $2(k - c - 2) - (k - 1) > c + 1$ that $|N_G(w_3) \cap N_G(u) \cap W| \geqslant c + 2$. It follows from $e_G(w_1, W) \geqslant k - c - 2$

that $|W \setminus N_G(\{w_1, w_2\})| \leqslant c+1$. Consequently, $|N_G(w_3) \cap N_G(u) \cap W| > |W \setminus N_G(\{w_1, w_2\})|$. Hence, the condition 3 of Lemma 4 holds.

Therefore, Lemma 4 can be applied. By Lemma 4, G has a k-contractible set.

Case 2.2. Each of the pendant parts of $G - W$ consists of 4 vertices.

Remark 1. In this case, we need $k \geqslant 2c+4$ only instead of $k \geqslant 3c+5$.

Let $N_v = N_G(v) \cap W$ for any vertex $v \in G - W$.

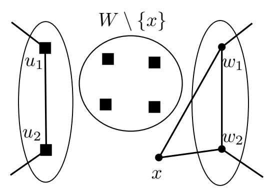

Figure 4. $G-W$ is not a simple cycle, each of the pendant parts of $G - W$ consists of 4 vertices.

Let A, B be two pendant parts of $G - W$, $Int(A) = \{u_1, u_2\}$, $Int(B) = \{w_1, w_2\}$. By Lemma 3.4, $d_{G-W}(u_1) = 2$, $d_{G-W}(u_2) = 2$, $d_{G-W}(w_1) = 2$, $d_{G-W}(w_2) = 2$. Then it follows from $\delta(G) \geqslant k - c$ that $e_G(u_1, W) \geqslant k-c-2$, $e_G(u_2, W) \geqslant k-c-2$, $e_G(w_1, W) \geqslant k-c-2$, $e_G(w_2, W) \geqslant k-c-2$.

Case 2.2.1. $|N_{w_1} \cap N_{w_2}| > |W \setminus (N_{u_1} \cup N_{u_2})|$ or $|N_{u_1} \cap N_{u_2}| > |W \setminus (N_{w_1} \cup N_{w_2})|$.

Without loss of generality, $|N_{w_1} \cap N_{w_2}| > |W \setminus (N_{u_1} \cup N_{u_2})|$. Our purpose is applying Lemma 4, where $v_1 = u_1$, $v_2 = u_2$, $v_3 = w_1$, $v_4 = w_2$. Clearly, the condition 3 of Lemma 4 holds. Recall that $d_{G-W}(u_1) = 2$, $d_{G-W}(u_2) = 2$, $d_{G-W}(w_1) = 2$, $d_{G-W}(w_2) = 2$. Hence, the condition 1 of Lemma 4 holds.

Hence, it remains to verify the condition 2 of Lemma 4. Note that if w_1 and w_2 have a common neighbour x (see figure 4) then $G - \{u_1, u_2\} - (W \setminus \{x\})$ is 2-connected. Indeed, by Lemma 2.3, $G - \{u_1, u_2\} - W$ is 2-connected and $xw_1, xw_2 \in E(G)$.

Consequently, Lemma 4 can be applied. By Lemma 4, G has a k-contractible set.

Case 2.2.2. $|N_{w_1} \cap N_{w_2}| \leqslant |W \setminus (N_{u_1} \cup N_{u_2})|$ and $|N_{u_1} \cap N_{u_2}| \leqslant |W \setminus (N_{w_1} \cup N_{w_2})|$.

Denote

$$f_1 = |N_{u_1} \cap N_{u_2}|, \quad f_2 = |N_{u_1} \setminus N_{u_2}|, \quad f_3 = |N_{u_2} \setminus N_{u_1}|,$$
$$e_1 = |N_{w_1} \cap N_{w_2}|, \quad e_2 = |N_{w_1} \setminus N_{w_2}|, \quad e_3 = |N_{w_2} \setminus N_{w_1}|.$$

It follows from the observation before the case 2.2.1 that, for any $v \in \{u_1, u_2, w_1, w_2\}$, $|N_v| \geqslant k - c - 2$. Therefore,

$$f_1 + f_2 \geqslant k - c - 2, \quad f_1 + f_3 \geqslant k - c - 2, e_1 + e_2 \geqslant k - c - 2, \quad e_1 + e_3 \geqslant k - c - 2.$$

Adding all these inequalities, we obtain that

$$2f_1 + f_2 + f_3 + 2e_1 + e_2 + e_3 \geqslant 4(k - c - 2).$$

By the condition of the case, $|N_{w_1} \cap N_{w_2}| \leqslant |W \setminus (N_{u_1} \cup N_{u_2})|$. Therefore, $e_1 \leqslant |W \setminus (N_{u_1} \cup N_{u_2})|$. Clearly, $N_{u_1} \cup N_{u_2} \subset W$ (by the definition of N_v for any $v \in V(G - W)$). Then it follows from $|W| = k - 1$ that $e_1 \leqslant k - 1 - |N_{u_1} \cup N_{u_2}|$. Hence, $e_1 \leqslant k - 1 - f_1 - f_2 - f_3$. Therefore, we derived from $|N_{w_1} \cap N_{w_2}| \leqslant |W \setminus (N_{u_1} \cup N_{u_2})|$ that $e_1 + f_1 + f_2 + f_3 \leqslant k - 1$. Similarly, if follows from $|N_{u_1} \cap N_{u_2}| \leqslant |W \setminus (N_{w_1} \cup N_{w_2})|$ that $f_1 + e_1 + e_2 + e_3 \leqslant k - 1$. Adding these inequalities, we get that

$$2f_1 + 2e_1 + e_2 + e_3 + f_2 + f_3 \leqslant 2k - 2.$$

Hence, $2k - 2 \geqslant 4(k - c - 2) \Rightarrow 2c + 3 \geqslant k$, contrary to $k \geqslant 2c + 4$. □

Proof of Theorem 3.

We prove this Theorem by induction. A case $k = 4$ serves as a base case of the induction. We do not require the restriction on the $\delta(G)$ in the base case. By Theorem 2, any 3-connected graph on at least 8 vertices contains a 4-contractible set. Hence, there is no problem with the restriction on the vertices number in the induction step from $k = 4$ to $k = 5$.

Induction step.

By the induction hypothesis, there is a $(k - 1)$-contractible set in the graph G. Let $c = \left[\frac{k-5}{3}\right]$. Then we have $k \geqslant 3c + 5, \quad \delta(G) \geqslant k - c$. Therefore, by applying Lemma 5, we get that there is a k-contractible set in the graph G. □

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